1. Reconsider problem 1 from Homework 5. Suppose that instead of looking for a solution $u \in H^2_0(U)$, you looked for a solution in $H^1_0(U) \cap H^2_0(U)$, i.e. you look for a weak solution of

$$\Delta^2 u = f$$

in the space $H^1_0(U) \cap H^2(U)$. We saw that weak solutions of this problem in $H^2_0$ correspond to requiring $u$ and its normal derivative to be zero on $\partial U$. What are the boundary conditions that correspond to solutions in $H^1_0(U) \cap H^2(U)$? (Hint: It may help to begin thinking of the one-dimensional analogue of this problem.)

2. Assume that the sequence $\{u_k\}$ converges weakly to $u$ in $L^2(0,T;H^1_0(U))$, $\{\dot{u}_k\}$ converges to $v$ in $L^2(0,T;L^2(U))$, and $\{\ddot{u}_k\}$ converges to $w$ in $L^2(0,T;H^{-1}(U))$. Prove that $v = \dot{u}$ and $w = \ddot{u}$.

3. Consider the beam equation

$$u_{tt} + u_{xxxx} = 0 \quad \text{in } (0,1) \times (0,T)$$

$$u = u_x = 0 \quad \text{on } \{0\} \times [0,T] \cup \{1\} \times [0,T]$$

$$u = g, \ u_t = h \quad \text{on } [0,1] \times \{t=0\}.$$ 

Discuss how you might construct a weak solution to this problem. What is the appropriate definition of a weak solution? In what function spaces should the solution lie? You don’t have to prove the existence of such a solution.

4. Assume that $u$ is a smooth solution of

$$u_t - \Delta u = 0 \quad \text{in } U \times (0,\infty)$$

$$u = 0 \quad \text{on } \partial U \times [0,\infty)$$

$$u = g \quad \text{on } U \times \{t=0\}.$$ 

Prove the exponential decay estimate

$$\|u(\cdot, t)\|_{L^2(U)} \leq e^{-\lambda_1 t}\|g\|_{L^2(U)}$$

for all $t \geq 0$, where $\lambda_1$ is the first eigenvalue of $-\Delta$ (with zero boundary conditions on $U$.)

Hint: You may use the fact that the first eigenvalue satisfies the minimization problem

$$\lambda_1 = \min_{u \in H^1_0(U)} \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2}.$$