Homework Assignment 6 **Math 776**

Due Wednesday Dec. 11

1. Reconsider problem 1 from Homework 5. Suppose that instead of looking for a solution $u \in H_0^2(U)$, you looked for a solution in $H_0^1(U) \cap H^2(U)$, i.e. you look for a weak solution of

$$\Delta^2 u = f$$

in the space $H_0^1(U) \cap H^2(U)$. We saw that weak solutions of this problem in H_0^2 correspond to requiring u and its normal derivative to be zero on ∂U. What are the boundary conditions that correspond to solutions in $H_0^1(U) \cap H^2(U)$? (Hint: It may help to begin thinking of the one-dimensional analogue of this problem.)

2. Assume that the sequence $\{u_k\}$ converges weakly to u in $L^2(0,T;H_0^1(U))$, $\{\dot{u}_k\}$ converges to v in $L^2(0,T;L^2(U))$, and $\{\ddot{u}_k\}$ converges to w in $L^2(0,T;H^{-1}(U))$. Prove that $v = \dot{u}$ and $w = \ddot{u}$.

3. Consider the beam equation

$$\begin{aligned} u_{tt} + u_{xxxx} &= 0 \quad \text{in } (0,1) \times (0,T) \\ u = u_x &= 0 \quad \text{on } (\{0\} \times [0,T]) \cup (\{1\} \times [0,T]) \\ u = g\ ,\ u_t = h & \qquad \text{on } [0,1] \times \{t = 0\} \end{aligned}$$

Discuss how you might construct a weak solution to this problem. What is the appropriate definition of a weak solution? In what function spaces should the solution lie? You don't have to prove the existence of such a solution.

4. Assume that u is a smooth solution of

$$\begin{aligned} u_t - \Delta u &= 0 \text{ in } U \times (0,\infty) \\ u &= 0 \text{ on } \partial U \times [0,\infty) \\ u &= g \text{ on} U \times \{t = 0\}\ . \end{aligned}$$

Prove the exponential decay estimate

$$\|u(\cdot,t)\|_{L^2(U)} \le e^{-\lambda_1 t} \|g\|_{L^2(U)}$$

for all $t \ge 0$, where λ_1 is the first eigenvalue of $-\Delta$ (with zero boundary conditions on U.) Hint: You may use the fact that the first eigenvalue satisfies the minimization problem

$$\lambda_1 = \min_{u \in H_0^1} \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2}\ .$$