ASYMPTOTIC APPROXIMATION OF A MODIFIED COMPRESSIBLE NAVIER-STOKES SYSTEM

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Abstract. We study the effects of localization on the long time asymptotics of a modified compressible Navier-Stokes system (mcNS) inspired by the previous work of Hoff and Zumbrun [4]. We introduce a new decomposition of the momentum field into its irrotational and incompressible parts, and a new method for approximating solutions of jointly hyperbolic-parabolic equations in terms of Hermite functions in which n^{th} order approximations can be computed for solutions with n^{th} order moments. We then obtain existence of solutions to the mcNS system in weighted spaces and, based on the decay rates obtained for the various pieces of the solutions, determine the optimal choice of asymptotic approximation with respect to the various localization assumptions, which in certain cases can be evaluated explicitly in terms of Hermite functions.

1. Introduction

The compressible Navier-Stokes equations are given by

$$\begin{aligned} &\partial_t \rho + \nabla \cdot \vec{m} = 0 \\ &\partial_t \vec{m} + \left[\nabla \cdot \left(\frac{\vec{m}\otimes\vec{m}}{\rho}\right)\right]^T + \nabla P = \epsilon \Delta \left(\frac{\vec{m}}{\rho}\right) + \eta \nabla \left(\nabla \cdot \left(\frac{\vec{m}}{\rho}\right)\right) \end{aligned} \tag{1}$$

These equations model the flow of a fluid with density ρ, momentum $\vec{m}$ and pressure P. We assume that the fluid is barotropic, hence $P = P(\rho)$ is a function only of the density. In the present paper, we are motivated by the question of stability of the constant density, constant momentum solution $(\rho^*, \vec{m}^*)^T$ to the compressible Navier-Stokes system in three dimensions, which without loss of generality we can take $(\rho^*, \vec{m}^*)^T = (1, 0)^T$.

Kawashima appears to have been the first to partially answer this question in the whole space $\mathbb{R}^d$ in dimension $d \geq 1$. In [7], he proves existence of global solutions for a general class of hyperbolic-parabolic systems which include (1) and proves these solutions decay in L^p at a given rate for $p \geq 2$.

Building on this work, Hoff and Zumbrun ([4], [5]) studied the asymptotic behavior of small perturbations from the constant state for the compressible Navier Stokes equations. Given $s \geq [\frac{d}{2}] + 1$, they prove global existence of solutions $u(t) = (\rho(t), m(t))^T$ for initial data $u_0 \in L^1 \cap H^s$ such that $E = \max(\|u_0\|_{H^{s+\ell}}, \|u_0\|_{L^1})$ is sufficiently small, and find that the solutions decay as

$$\|u(\cdot,t)\|_{L^p} \leq CEt^{-\frac{d}{2}(1-\frac{1}{p})}$$

for $p \geq 2$. They go further by obtaining decay rates in L^p for $1 \leq p < 2$, showing that the momentum field can be decomposed into an irrotational and incompressible piece, and that the solutions are asymptotically irrotational as measured in L^p for $1 \leq p < 2$ and asymptotically incompressible for $p > 2$. Furthermore, they show that these solutions are asymptotically well-approximated by the linearization of (1), in the sense that

$$\|u(t) - G(t) * u_0\|_{L^p} \leq CEt^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}} \tag{2}$$

for $2 \leq p \leq \infty$, where $G(t)$ is the Green's matrix for the linearization of (1). This linear evolution, while simpler than the full nonlinear evolution, is nevertheless complicated, and must be studied in Fourier space. Following Kawashima [7], they show there exists a unique linear, artificial-viscosity system associated with (1) given by

$$\begin{aligned} &\partial_t \rho + \nabla \cdot \vec{m} = \frac{1}{2}(\epsilon + \eta)\Delta \rho \\ &\partial_t \vec{m} + c^2 \nabla \rho = \epsilon \Delta \vec{m} + \frac{1}{2}(\eta - \epsilon)\nabla(\nabla \cdot \vec{m}) \end{aligned} \tag{3}$$

2010 *Mathematics Subject Classification.* 76N10.

Acknowledgments: The authors would like to acknowledge partial support by the National Science Foundation through grants DMS-1603416, DMS-2006887 (RG), DMS-1813384 (CEW) and (RW). CEW thanks Th. Gallay for numerous discussions of the use of the Hermite expansion in the analysis of the asymptotic behavior of fluids. The authors would also like to thank the anonymous reviewer for the many helpful suggestions.

which can be used to approximate the linear evolution, in the sense that

$$\|G(t) * u_0 - \tilde{G}(t) * u_0\|_{L^p} \leq C t^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}} \tag{4}$$

where $\tilde{G}$ is the Green's matrix of (3). The matrix $\tilde{G}(t)$ is shown to possess nice analytical properties, and is specified in terms of diffusing Gaussians convected by the fundamental solution of the linearized Euler equations. Furthermore if one additionally assumes some spatial localization in the form of spatial moments, i.e. $(1+|x|)u_0 \in L^1$, then the artificial-viscosity evolution can be approximated by a simple matrix multiplication:

$$\|\tilde{G}(t) * u_0 - \tilde{G}(t)U_0\|_{L^p} \leq C t^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}} \tag{5}$$

where $U_0 = \int u_0 dy$ is the total mass vector. Taken together, these results show that the dominant asymptotic behavior of the compressible Navier-Stokes equations is given by the explicit functions $\tilde{G}(t)U_0$, which they refer to as "diffusion waves".

Recently, Kagei and Okita [6] showed that if one assumed some additional localization of the initial data, one could extend the results of Hoff and Zumbrun by computing a second order approximation to the solutions of (1) in dimension $d \geq 3$. Among their findings, they prove that with the additional assumption that $(1+|x|)u_0 \in L^1$ one has

$$\left\|u(t) - G(t) * u_0 - \sum_{i=1}^{d} \partial_{x_i} G_1(t,\cdot) \int_0^\infty \int_{\mathbb{R}^d} \mathcal{F}_i^0 dy ds\right\|_{L^p} \leq C \log(1+t)(1+t)^{-\frac{d}{2}(1-\frac{1}{p})-\frac{3}{4}}$$

for $p \geq 2$, where $G(t)$ is the Green's matrix for the linearization of (1), $G_1(t)$ is a low frequency cutoff of $G(t)$, and the $\mathcal{F}_i^0$ are quantities which can be computed with knowledge of the solution $\rho(t), m(t)$, as well as knowledge of the pressure P and its derivatives. Furthermore, with the additional assumption that $(1+|x|^2)u_0 \in L^1$ their results also show that the solutions can be explicitly approximated by Gaussian functions

$$\left\|u(t) - G_1(t) \int u_0 dy + \sum_{i=1}^{d} \partial_{x_i} G_1(t,\cdot) \Big[\int y_i u_0(y) dy - \int_0^\infty \int_{\mathbb{R}^d} \mathcal{F}_i^0 dy ds \Big] \right\|_{L^p} \leq C \log(1+t)(1+t)^{-\frac{d}{2}(1-\frac{1}{p})-\frac{3}{4}} \tag{6}$$

if one includes the additional correction factor given by the $\mathcal{F}_i^0$ terms.

On the other hand, Gallay and Wayne ([2], [3]) study the localization properties and asymptotic behavior of solutions of the incompressible equations in two and three dimensions. Previously, Brandolese [1] had shown that there exist solutions of the Navier-Stokes equation which have finite moments at $t = 0$, but which fail to have finite moments on any time interval $[0, T]$ for any $T > 0$. Gallay and Wayne show that this instantaneous loss of localization does not occur if one works with the vorticity equation, obtained by computing the curl of the incompressible Navier-Stokes equations. Specifically, they show that if an initial vorticity $\vec{\omega}_0$ is such that for some $n \geq 0$ one has $(1+|x|)^n \vec{\omega}_0 \in L^2$, then there exists a unique global solution $\vec{\omega}(t)$ of the vorticity equation such that $\vec{\omega}(0) = \vec{\omega}_0$ and $(1+|x|)^n \vec{\omega}_0 \in L^q$ for all $t > 0$, $q \geq 2$. They also show that the localization properties are intimately related to the asymptotic behavior by showing that by increasing the assumptions of spatial locality one can obtain increasingly accurate asymptotic approximations. Namely, if one chooses $\frac{3}{2} < \mu \leq 2$ and $n \in \mathbb{Z}_{\geq 0}$ such that $n > 2\mu + \frac{1}{2}$ then for initial data $(1+|x|)^n \vec{\omega}_0 \in L^2$ there exist approximations $u_{app,k}(t)$ such that

$$\|u(t) - \sum_{k=1}^{n} u_{app,k}(t)\|_{L^p} \leq C t^{-\frac{d}{2}(1-\frac{1}{p})+1-\mu}$$

where $u(t)$ is the velocity recovered from the vorticity field $\vec{\omega}(t)$ and the approximation terms $u_{app,k}(t)$ are also given in terms of diffusing Gaussians and their derivatives. They obtain first and second order approximations, and their analysis points the way toward obtaining approximations of arbitrary order.

We aim to use the tools developed in [2], [3] to extend the asymptotic approximation of solutions to the compressible Navier-Stokes in [4], [6] to a higher order. The first major step in this direction is to study the localization properties of the compressible Navier-Stokes system, which have yet to be systematically studied. To do so we begin with a modified compressible Navier-Stokes system

$$\begin{aligned} \partial_t \rho + \nabla \cdot \vec{m} &= \frac{1}{2}(\epsilon + \eta)\Delta \rho \\ \partial_t \vec{m} + \big[\nabla \cdot (\vec{m} \otimes \vec{m})\big]^T + c^2 \nabla \rho &= \epsilon \Delta \vec{m} + \frac{1}{2}(\eta - \epsilon)\nabla(\nabla \cdot \vec{m}) \end{aligned} \tag{7}$$

obtained from (1) by replacing the linear part by artificial viscosity system (3) and dropping all nonlinear terms aside from the Lagrangian derivative. Furthermore, we will restrict to dimension $d = 3$. We make these modifications since this model is simpler from a technical point of view. However, as Hoff and Zumbrun have shown, we know that the leading order long-time asymptotics of (7) are the same as those of the compressible Navier-Stokes equations, and much of the analysis developed here carries through in higher dimensions with a modest increase in complexity. We defer the consideration of (1) to forthcoming work. While it is not known if the momentum field of the compressible Navier-Stokes equation exhibits the instantaneous loss of localization described by Brandolese, we avoid its possible appearance by working with the curl and divergence of $\vec{m}$. If one lets $a = \nabla \cdot \vec{m}$, $\vec{\omega} = \nabla \times \vec{m}$, and $u(t) = (\rho(t), a(t), \vec{\omega}(t))^T$ and computes the divergence and curl of (7), one arrives at the curl-divergence form of the modified compressible Navier-Stokes system:

$$\partial_t u = \mathcal{L}u - \mathcal{Q}(u,u) \tag{8}$$

where we let $\nu = \frac{1}{2}(\epsilon + \eta)$, I_3 be the 3×3 identity matrix and where

$$\mathcal{L} = \begin{pmatrix} \nu\Delta & -1 & \\ -c^2\Delta & \nu\Delta & \\ & & \epsilon\Delta I_3 \end{pmatrix} \quad , \quad \mathcal{Q}(u,u) = \begin{pmatrix} 0 \\ \nabla \cdot \left[\sum_{j=1}^3 \partial_{x_j}(m_j \vec{m})\right] \\ \nabla \times \left[\sum_{j=1}^3 \partial_{x_j}(m_j \vec{m})\right] \end{pmatrix}$$

We take (8) as our starting point, and address the question of equivalence to the original system (7) in the course of our analysis. Our main results can then be summarized in the following theorem:

Theorem 1. *In dimension $d = 3$, let $\vec{u}_0 = (\rho_0, a_0, \vec{\omega}_0)^T$ where $a_0, \vec{\omega}_0$ have zero total mass (ie $\int_{\mathbb{R}^3} a_0(x)dx = 0$), and suppose $(1+|x|)^n \vec{u}_0 \in W^{1,p} \times L^p \times (L^p)^3$ for some $0 \le n \le 2$ and for all $1 \le p \le \frac{3}{2}$. If $k \ge 1$ is fixed and if*

$$E_n = \sup_{1 \le p \le 3/2} \left(\left\|(1+|\cdot|)^n \rho_0(\cdot)\right\|_{W^{1,p}} + \left\|(1+|\cdot|)^n a_0(\cdot)\right\|_{L^p} + \left\|(1+|\cdot|)^n \vec{\omega}_0(\cdot)\right\|_{(L^p)^3} \right)$$

is chosen sufficiently small, then there exists a unique mild solution $(\rho(t), a(t), \vec{\omega}(t))$ of (8) such that for a small-time blowup rate $r_{\alpha,p}$ and large-time decay rates $\ell_{n,p,\mu}, \tilde{\ell}_{n,p,\mu}$ defined below we have

$$\|(1+|\cdot|)^\mu \partial_x^\alpha \rho(\cdot,t)\|_{L^p} \le CE_n t^{-r_{\alpha,p}}(1+t)^{-\ell_{n,p,\mu}+\frac{1}{2}}$$
$$\|(1+|\cdot|)^\mu \partial_x^\alpha a(\cdot,t)\|_{L^p} \le CE_n t^{-r_{\alpha,p}}(1+t)^{-\ell_{n,p,\mu}}$$
$$\|(1+|\cdot|)^\mu \partial_x^\alpha \vec{\omega}(\cdot,t)\|_{\mathbb{L}^p} \le CE_n t^{-r_{\alpha,p}}(1+t)^{-\tilde{\ell}_{n,p,\mu}}$$

for $|\alpha| \le k-1$ and for all $1 \le p \le \infty$, $0 \le \mu \le n$ where C depends only on n, k, ν, ϵ. Furthermore, for $n \ge 1$ there exist functions $(\rho_{app}, a_{app}, \vec{\omega}_{app})^T$ computable via a convolution with explicit kernels such that

$$\|(1+|\cdot|)^\mu \partial_x^\alpha \big(\rho(\cdot,t) - \rho_{app}(\cdot,t)\big)\|_{L^p} \le CE_n t^{-r_{\alpha,p}}(1+t)^{-\ell_{n,p,\mu}+\frac{1}{2}-\frac{1}{2}}$$
$$\|(1+|\cdot|)^\mu \partial_x^\alpha \big(a(\cdot,t) - a_{app}(\cdot,t)\big)\|_{L^p} \le CE_n t^{-r_{\alpha,p}}(1+t)^{-\ell_{n,p,\mu}-\frac{1}{2}}$$
$$\|(1+|\cdot|)^\mu \partial_x^\alpha \big(\vec{\omega}(\cdot,t) - \vec{\omega}_{app}(\cdot,t)\big)\|_{\mathbb{L}^p} \le CE_n t^{-r_{\alpha,p}}(1+t)^{-\tilde{\ell}_{n,p,\mu}-\frac{1}{2}}$$

and for $n = 2$ one can take these approximations to be explicit Gaussian functions computable with knowledge only of the moments of order $\lfloor n \rfloor$ of the initial data.

For $n, \mu \in \mathbb{R}_{\ge 0}$, let $\lfloor n \rfloor_1 = \min(n,1)$ and $\lfloor \mu \rfloor_1 = \min(\mu,1)$, and we define the rates via

$$r_{\alpha,p} = \begin{cases} \frac{|\alpha|}{2} & \text{for } 1 \le p \le \frac{3}{2} \\ \frac{3}{2}(\frac{2}{3} - \frac{1}{p}) + \frac{|\alpha|}{2} & \text{for } p \ge \frac{3}{2} \end{cases} \quad , \quad \tilde{\ell}_{n,p,\mu} = \begin{cases} \frac{3}{2}(1-\frac{1}{p}) + \frac{\lfloor n \rfloor_1 + \lfloor \mu \rfloor_1}{2} - \mu & \text{for } 1 \le p \le \frac{3}{2} \\ \frac{1}{2} + \frac{\lfloor n \rfloor_1 + \lfloor \mu \rfloor_1}{2} - \mu & \text{for } p \ge \frac{3}{2} \end{cases} \tag{9}$$

$$\ell_{n,p,\mu} = \begin{cases} \frac{5}{2}(1-\frac{1}{p}) - \frac{1}{2} + \frac{\lfloor n \rfloor_1}{2} - \mu & \text{for } 1 \le p \le \frac{3}{2} \\ (1-\frac{1}{p}) + \frac{\lfloor n \rfloor_1}{2} - \mu & \text{for } p \ge \frac{3}{2} \end{cases} \tag{10}$$

The reasons underlying the precise form of these rates will become apparent below in Prop. E.2. For now we say only that the small-time blow up rate $r_{\alpha,p}$ ensures boundedness in the spaces to which the functions initially belong, but allows for increasingly fast blow up for larger L^p norms and higher order derivatives. The large-time decay rate $\tilde{\ell}_{n,p,\mu}$ reflects the parabolic nature of the evolution of $\vec{\omega}$, whereas the large-time decay rate $\ell_{n,p,\mu}$ reflects the combined hyperbolic-parabolic evolution of ρ and a. Their dependence on the parameter n

indicates that increased localization of the initial data leads to faster decay of the solution, and their dependence on μ indicate that the solutions' weighted norms decay more slowly for larger weight as they spread out due to parabolic and/or convective effects.

In section 2, we prove a number of inequalities for later use in our existence and asymptotic analysis. We also introduce an expansion for solutions of the heat equation which we call the Hermite expansion, and demonstrate how it works for related systems. In section 3, we prove that (7) has unique solutions, and that these solutions remain in the same weighted Lebesgue spaces as the initial data, and obtain asymptotic decay rates for these solutions in weighted spaces. In section 4, we prove results about the accuracy of the linear approximation, and then show how this approximation can be improved if the initial data is appropriately localized. Finally, in section 5 we discuss the results obtained and compare them to the previous results of Hoff and Zumbrun and Kagei and Okita.

1.1. **Mild formulation.**

The nonlinear term in (8) still depends on $\vec{m}$, and hence we introduce the operators

$$\begin{aligned} \Pi a &= \nabla\big(\Delta^{-1} a\big) \\ B\vec{\omega} &= -\nabla \times \big(\Delta^{-1}\vec{\omega}\big) \end{aligned} \tag{11}$$

which allow us to write $\vec{m} = \Pi a + B\vec{\omega}$, splitting $\vec{m}$ into an irrotational part, Πa, and an incompressible part $B\vec{\omega}$. This is a form of the well-known Helmholtz decomposition. Note that the inverse Laplacian is well-defined only when we make a suitable choice of function spaces for a and $\vec{\omega}$. We will do so below in subsection 2.1, and then obtain estimates for the action of Π and B over these spaces. For notational convenience, we also introduce the nonlinear operator

$$N(a,\vec{\omega}) = \sum_{j=1}^{3} \partial_{x_j}\Big((\Pi a + B\vec{\omega})_j \big(\Pi a + B\vec{\omega}\big)\Big)$$

We can now apply Duhamel's formula to obtain an integral formulation of (8):

$$\begin{aligned} \rho(t) &= \partial_t w(t) * K_\nu(t) * \rho_0 - w(t) * K_\nu(t) * a_0 + \int_0^t w(t-s) * K_\nu(t-s) * \big[\nabla \cdot N\big(a(s),\vec{\omega}(s)\big)\big]\, ds \\ a(t) &= -\partial_t^2 w(t) * K_\nu(t) * \rho_0 + \partial_t w(t) * K_\nu(t) * a_0 - \int_0^t \partial_t w(t-s) * K_\nu(t-s) * \big[\nabla \cdot N\big(a(s),\vec{\omega}(s)\big)\big]\, ds \\ \vec{\omega}(t) &= \mathbb{K}_\epsilon(t) * \vec{\omega}_0 - \int_0^t \mathbb{K}_\epsilon(t-s) * \big[\nabla \times N\big(a(s),\vec{\omega}(s)\big)\big]\, ds \end{aligned} \tag{12}$$

Here we use the fact that the Green's matrix G for the linear part of the hyperbolic-parabolic system for ρ, a above can be decomposed as the composition of the wave evolution with the heat evolution

$$G(t) * \begin{pmatrix} \rho_0 \\ a_0 \end{pmatrix} = G_W(t) * \Big[K_\nu(t) I_2 * \begin{pmatrix} \rho_0 \\ a_0 \end{pmatrix}\Big] \tag{13}$$

in which

$$G_W(t) = \begin{pmatrix} \partial_t w(t) & -w(t) \\ -\partial_t^2 w(t) & \partial_t w(t) \end{pmatrix}$$

is the Green's matrix for the wave evolution, $K_\nu(t) = \frac{1}{(4\pi\nu t)^{3/2}} \exp\Big[-\frac{|x|^2}{4\nu t}\Big]$ is the scalar heat kernel, $\mathbb{K}_\epsilon(t)$ is the diagonal matrix having the heat kernel $K_\epsilon(t)$ for each entry on the diagonal, and I_2 is the 2×2 identity matrix. The wave operator $w(t)$ is the Fourier multiplier defined by

$$\hat{w}(\xi,t) = \frac{\sin(ct|\xi|)}{c|\xi|}$$

which together with its temporal derivatives determine the components of the wave evolution for various initial data. We recall that for sufficiently smooth functions this can be expressed via Kirchhoff's formula (see [9] pg

71-72 for details), which in dimension $d=3$ is as follows:

$$(14)\qquad \begin{aligned}(w*h)(x,t) &= b_{0,0}t\int_{|z|=1} h(x+ctz)dS(z)\\ (\partial_t w*h)(x,t) &= \sum_{0\le|\alpha|\le1} b_{\alpha,1}(ct)^{|\alpha|}\int_{|z|=1} D^\alpha h(x+ctz)z^\alpha dS(z)\\ (\partial_t^2 w*h)(x,t) &= \sum_{1\le|\alpha|\le2} b_{\alpha,2}(ct)^{|\alpha|-1}\int_{|z|=1} D^\alpha h(x+ctz)z^\alpha dS(z)\end{aligned}$$

with S_z the surface element on the unit sphere, and some constants $b_{\alpha,i}$.

We want to prove existence of mild solutions to (8) in some function space and determine the asymptotic behavior of these solutions. We'll see that the natural setting for our analysis is found in the homogeneous, algebraically weighted Lebesgue spaces

$$\mathring{L}^p(n) = \{f \,:\, \|f(x)\|_{\mathring{L}^p(n)} = \Big(\int_{\mathbb{R}^3} |x|^{np}|f(x)|^p dx\Big)^{1/p} < \infty\}$$

and their inhomogeneous counterparts

$$L^p(n) = \{f \,:\, \|f(x)\|_{L^p(n)} = \Big(\int_{\mathbb{R}^3} (1+|x|)^{np}|f(x)|^p dx\Big)^{1/p} < \infty\}$$

We let $W^{k,p}(n)$ be the subspace of the Sobolev space $W^{k,p}$ consisting of algebraically weighted, weakly differentiable functions:

$$W^{k,p}(n) = \{f\in W^{k,p} : \|f\|^p_{W^{k,p}(n)} = \sum_{|\alpha|\le k} \|\partial_x^\alpha f\|^p_{L^p(n)} < \infty\}$$

We also introduce the vector-valued function space $\mathbb{L}^p = \big(L^p\big)^3$ with norm

$$\|\vec{\omega}\|_{\mathbb{L}^p} = \max_{i=1,2,3} \|\omega_i\|_{L^p}$$

as well as the function spaces $\mathring{\mathbb{L}}^p(n) = \big(\mathring{L}^p(n)\big)^3$, $\mathbb{L}^p(n) = \big(L^p(n)\big)^3$ and $\mathbb{W}^{k,p}(n) = (W^{k,p}(n))^3$ with analogous norms. Furthermore let $\mathbb{L}^p_\sigma$ be the closure of the space of divergence free vector fields in the space $\mathbb{L}^p$, and let $\mathring{\mathbb{L}}^p_\sigma(n)$, $\mathbb{L}^p_\sigma(n)$ and $\mathbb{W}^{k,p}_\sigma(n)$ be the closures in the analogous spaces. Finally, we will make use of Schwartz class functions as tools in our analysis, and hence we will write $\mathcal{S}$ for the space of Schwartz class functions and $\mathbb{S}_\sigma$ for the space of Schwartz class divergence free vector fields.

2. Preliminary analysis

2.1. The Π and B operators. We first define the operators Π and B for $(a,\vec{\omega})\in\mathcal{S}\times\mathbb{S}_\sigma$ via (11). Note that the inverse Laplacian is well defined on the space of Schwartz class functions, and for such functions we have

$$\Pi a = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(x-y)}{|x-y|^3}a(y)dy \qquad \text{and} \qquad B\vec{\omega} = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(x-y)\times\vec{\omega}(y)}{|x-y|^3}dy$$

In the following proposition we obtain estimates on the action of Π and B, which then allow us to extend these operators to be defined on all of $L^p(n)\times\mathbb{L}^p_\sigma(n)$, for suitable choices of p and n.

Proposition 2.1. *Let $a\in\mathcal{S}$ and $\vec{\omega}\in\mathbb{S}_\sigma$*

(a) *Suppose that $1<p_1<\infty$. Then there exists a constant C_1 depending only on p_1 such that*

$$(15)\qquad \|\partial_{x_i}\Pi a\|_{L^{p_1}} \le C_1\|a\|_{L^{p_1}} \quad , \quad \|\partial_{x_i}B\vec{\omega}\|_{\mathbb{L}^{p_1}} \le C_1\|\vec{\omega}\|_{\mathbb{L}^{p_1}}$$

(b) *Suppose that $n\in[0,2)$ and $1<p_3<p_2<\infty$ are such that*

$$(16)\qquad \frac{1}{p_2} = \frac{1}{p_3} - \frac{1}{3}$$

and p_3 satisfies the constraint

$$\frac{1-n}{3} < \frac{1}{p_3} < \frac{3-n}{3}$$

Then there exists a constant C_2 depending only on n, p_3 such that

$$\|\Pi a\|_{L^{p_2}(n)} \leq C_2 \|a\|_{L^{p_3}(n)} \quad , \quad \|B\vec{\omega}\|_{\mathbb{L}^{p_2}(n)} \leq C_2 \|\vec{\omega}\|_{\mathbb{L}^{p_3}(n)} \tag{17}$$

(c) *Suppose $n \in [1,3)$, $1 < p_3 < p_2 < \infty$ solve (16) and p_3 satisfies the new constraint*

$$\frac{3-n}{3} < \frac{1}{p_3} < \frac{4-n}{3}$$

If, in addition, a and $\vec{\omega}$ are such that

$$\int_{\mathbb{R}^3} a(x)dx = 0 \quad , \quad \int_{\mathbb{R}^3} \vec{\omega}(x)dx = 0 \tag{18}$$

then there exists a (possibly different) constant C_2 depending only on n, p_3 such that (17) holds.

The proof of these estimates follows closely the strategy used to the study the B operator in Proposition B.1 of [3], but we extend the results to general values of p and n, rather than focusing on the L^2 based spaces in that reference, as well as studying the operator Π. We defer the proof to Appendix A. The following Corollary is immediate from the definition of the Π,B operators for $a, \vec{\omega} \in L^{p_3}(n) \times \mathbb{L}^{p_3}_\sigma(n)$:

Corollary 2.2. ***(a)*** *Suppose p_1, C_1 are as in Prop. 2.1 part* ***(a)****. Then for $a \in L^{p_1}$, $\vec{\omega} \in \mathbb{L}^{p_1}_\sigma$ (15) holds.*
(b) *Suppose that n, p_2, p_3, C_2 are as in Prop. 2.1 part* ***(b)****. Then for $a \in L^{p_3}(n)$ and $\vec{\omega} \in \mathbb{L}^{p_3}_\sigma(n)$ (17) holds.*
(c) *Suppose that n, p_2, p_3, C_2 are as in Prop. 2.1 part* ***(c)****. If $a \in L^{p_3}(n)$ and $\vec{\omega} \in \mathbb{L}^{p_3}_\sigma(n)$ satisfy (18), then (17) holds.*

2.2. **Heat evolution estimate.** The heat evolution tends to dissipate the L^p norms of a function. We have

$$\|\partial_x^\alpha K_\nu(t) * f\|_{L^p} \leq C(\nu t)^{-\frac{|\alpha|}{2} - \frac{3}{2}(\frac{1}{q} - \frac{1}{p})} \|f\|_{L^q}$$

using Young's inequality for $1 \leq q \leq p \leq \infty$ and $f \in L^q$. In weighted spaces, one can obtain faster decay under certain conditions described in the following proposition, which is an extension of Proposition A.3 found in [3]. We defer the proof to Appendix B. Note that while we restrict to dimension $d = 3$ here, the analogous results can be proven in any dimension (see [8] for details).

Proposition 2.3. *Let $1 \leq q \leq p \leq \infty$ be Lebesgue indices, let $n, \mu \in \mathbb{R}_{\geq 0}$ be weight indices such that $n \geq \mu$ and that $\exists\ \tilde{n} \in \mathbb{Z}_{\geq 0}$ such that $3(1 - \frac{1}{q}) + \tilde{n} < n < 3(1 - \frac{1}{q}) + \tilde{n} + 1$, and let $f \in L^q(n)$ be such that its moments up to order $\tilde{n}$ are zero, ie for all multi-indices $\beta \in \mathbb{N}^3$, $|\beta| \leq \tilde{n}$ we have*

$$\int_{\mathbb{R}^3} x^\beta f(x)dx = 0$$

Then there exists a $C > 0$ depending only on p, q, n, μ, α such that

$$\|\partial_x^\alpha K_\nu(t) * f\|_{\mathring{L}^p(\mu)} \leq C(\nu t)^{-\frac{|\alpha|}{2} - \frac{3}{2}(\frac{1}{q} - \frac{1}{p})} (1 + \nu t)^{-\frac{n-\mu}{2}} \|f\|_{L^q(n)} \tag{19}$$

Remark 2.4. *Note that this estimate is sharp with respect to each of its hypotheses. For instance, to see that the localization assumption $f \in L^1(n)$ is necessary to achieve the given asymptotic bound, consider the example*

$$f(x) = |x|^{-3-n} \psi(x_1) \mathsf{sign}(x_1)$$

for $0 < n < 1$ and a smooth cutoff function $\psi(x)$ which is even in x_1 such that

$$\psi(x) = \begin{cases} 1 & \text{for } |x_1| \geq 2 \\ 0 & \text{for } |x_1| \leq 1 \end{cases}$$

and $|\psi(x)| \leq 1$ for all x. Since this function is odd in x_1, it has zero total mass, and it belongs to $L^1(n - \delta)$ for any $0 < \delta \leq n$, but $f \notin L^1(n)$. By plugging in $x = (2, 0, 0)$ for instance, straightforward explicit calculations show that

$$\lim_{t \to \infty} t^{\frac{3}{2} + \frac{n}{2}} \|K_\nu(t) * f\|_{L^\infty} = \infty$$

and similar results hold for the other L^p norms. Similarly the Hermite functions described below can be used to illustrate that Prop. 2.3 is sharp with respect to the zero moment conditions.

2.3. **Heat-wave evolution estimate.** We obtain the following bounds on the heat-wave operators of the linear evolution of the ρ, a system in homogeneous weighted spaces:

Proposition 2.5. *For Lebesgue index $q \geq 1$ and weight $n \geq 0$ there exists a $C > 0$ depending only on c, ν, n such that the following estimates hold:*

$$\begin{aligned} \|w(t) * K_\nu(t)\|_{\mathring{L}^q(n)} &\leq Ct^{1+\frac{n}{2}-\frac{3}{2}(1-\frac{1}{q})}(1+t)^{\frac{n}{2}-(1-\frac{1}{q})} \\ \|\partial_t w(t) * K_\nu(t)\|_{\mathring{L}^q(n)} &\leq Ct^{\frac{n}{2}-\frac{3}{2}(1-\frac{1}{q})}(1+t)^{\frac{n}{2}+\frac{1}{2}-(1-\frac{1}{q})} \\ \|\partial_t^2 w(t) * K_\nu(t)\|_{\mathring{L}^q(n)} &\leq Ct^{\frac{n}{2}-\frac{1}{2}-\frac{3}{2}(1-\frac{1}{q})}(1+t)^{\frac{n}{2}+\frac{1}{2}-(1-\frac{1}{q})} \end{aligned} \tag{20}$$

We defer the proof to Appendix C.1. Again the analogous results can be proven in higher dimensions (see [8] for details).

Note that the term $\partial_t^2 w(t) * K_\nu(t)$ blows up as $t \to 0$ as a result of the fact that $K_\nu(t)$ tends to a delta function, and hence the L^p norms of derivatives of $K_\nu(t)$ become arbitrarily large. However, when the heat-wave operator $\partial_t^2 w(t) * K_\nu(t)$ acts on a function with a little bit of smoothness we can obtain the following improved estimate with milder blow up, the proof of which we defer to Appendix C.2.

Proposition 2.6. *Suppose $\rho_0 \in W^{1,q}(n)$ for some $q \geq 1$. There exists a $C > 0$ such that for $p \geq q$ and $\mu \leq n$ we have*

$$\|\partial_t^2 w(t) * K_\nu(t) * \rho_0\|_{\mathring{L}^p(\mu)} \leq Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}(1+t)^{\mu-\frac{1}{2}+\frac{1}{2}-(\frac{1}{q}-\frac{1}{p})}\|\rho_0\|_{W^{1,q}(n)}$$

2.4. **Hermite expansion.** We aim to study the asymptotic behavior of solutions to (8) by computing an expansion of the solution using Hermite functions. This is the point where we begin to diverge strongly from the approach of [4] or [6]. We illustrate this process first for the heat equation. To do so, we define

$$\phi_0(x) = (4\pi)^{-\frac{3}{2}} \exp\big[-\frac{|x|^2}{4}\big]$$

and let H_α be the αth Hermite polynomial given by

$$H_\alpha(x) = \frac{2^{|\alpha|}}{\alpha!} e^{\frac{|x|^2}{4}} \partial_x^\alpha (e^{-\frac{|x|^2}{4}})$$

Note that these satisfy the orthonormality property:

$$\langle H_\alpha(\cdot), \partial_x^\beta \phi_0(\cdot)\rangle = \delta_{\alpha\beta} \tag{21}$$

Proposition 2.7. *Suppose that $u_0 \in L^1(n)$ for $n \geq 0$. If $u(t) = K_\nu(t) * u_0$ is the solution of the heat equation in $C^0\big[[0,\infty), L^1(n)\big]$, then we can write*

$$u(x,t) = \sum_{|\alpha| \leq \lfloor n \rfloor} \big\langle H_\alpha, u_0 \big\rangle \partial_x^\alpha K_\nu(t) * \phi_0(x) + R(x,t)$$

where for any $\mu \leq n$

$$\|R(\cdot,t)\|_{\mathring{L}^p(\mu)} \leq C\|u_0\|_{L^1(n)}(\nu t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{n-\mu}{2}}$$

Proof. If we write

$$u(x,t) = \sum_{|\alpha| \leq \lfloor n \rfloor} \big\langle H_\alpha, u_0 \big\rangle \partial_x^\alpha K_\nu(t) * \phi_0(x) + R(x,t)$$

then we note that the remainder term $R(x,t)$ is itself a solution of the heat equation. Furthermore, we note that at time $t = 0$ we have

$$\big\langle H_\beta, R(\cdot,0)\big\rangle = 0$$

for all $|\beta| \leq \lfloor n \rfloor$. Therefore R_j satisfies the moment zero condition required in Proposition 2.3, which then gives us our result. □

The Hermite expansion illustrates a few of the features of the heat evolution. We note that orders of this expansion decay sequentially faster, and the remainder at least matches the fastest decay rate. The Hermite functions are self similar under the heat evolution, in the sense that the heat evolution acts on these functions by dilation and scaling. See [2] for details. Importantly, the Hermite expansion illustrates how the heat evolution dissipates the moments of a function. The αth moment evolves according to the αth term in the Hermite expansion. For instance, the zeroth order Hermite function gives an explicit example of an initial condition for which the heat evolution preserves the L^1 norm, yet has any degree of algebraic decay one could ask for, and hence the estimate in (19) is sharp with respect to the zero mass condition. However, the L^∞ norm decays, so here the heat evolution is spreading mass around, but it conserves the total signed mass. The first order Hermite function provides an example where the total signed mass is zero, and we see that its L^1 norm does decay. The Hermite expansion can be used to show that this holds in general, and similar statements can be made about higher order moments.

2.4.1. *Hermite expansion for the hyperbolic-parabolic system.* We need a Hermite expansion for the hyperbolic-parabolic system

$$\begin{aligned} \partial_t \rho_L &= \nu \Delta \rho_L - a_L \\ \partial_t a_L &= -c^2 \Delta \rho_L + \nu \Delta a_L \end{aligned} \tag{22}$$

As in (8) we can write the solution of the linear equation in terms of the heat-wave operators via

$$\begin{aligned} \rho_L(t) &= \partial_t w(t) * K_\nu(t) * \rho_0 - w(t) * K_\nu(t) * a_0 \\ a_L(t) &= -\partial_t^2 w(t) * K_\nu(t) * \rho_0 + \partial_t w(t) * K_\nu(t) * a_0 \end{aligned} \tag{23}$$

Since the heat and wave operators commute, we can apply them sequentially, and since $K_\nu(t) * \rho_0$ and $K_\nu(t) * a_0$ are solutions of the heat equation, we can use the scalar Hermite expansion. We define

$$\begin{pmatrix} \rho_1(t) \\ a_1(t) \end{pmatrix} = \begin{pmatrix} \partial_t w(t) * K_\nu(t) * \phi_0 \\ -\partial_t^2 w(t) * K_\nu(t) * \phi_0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \rho_2(t) \\ a_2(t) \end{pmatrix} = \begin{pmatrix} -w(t) * K_\nu(t) * \phi_0 \\ \partial_t w(t) * K_\nu(t) * \phi_0 \end{pmatrix} \tag{24}$$

We determine these asymptotic profiles explicitly in Appendix D below. We then have the following analogue of the Hermite expansion, where for convenience we assume that ρ has at least one weak derivative:

Proposition 2.8. *Suppose that $\rho_0 \in W^{1,1}(n)$, $a_0 \in L^1(n)$ for $n \geq 0$. If $(\rho_L(t), a_L(t))^T$ is the solution of (22) in $C^0\big[[0,\infty), L^1(n) \times L^1(n)\big]$, then we can write*

$$\begin{pmatrix} \rho_L(x,t) \\ a_L(x,t) \end{pmatrix} = \sum_{\substack{i \leq 2 \\ |\alpha| \leq \lfloor n \rfloor}} \big\langle H_\alpha \hat{e}_i, \begin{pmatrix} \rho_0 \\ a_0 \end{pmatrix} \big\rangle \, \partial_x^\alpha \begin{pmatrix} \rho_i(x,t) \\ a_i(x,t) \end{pmatrix} + \begin{pmatrix} \rho_{LR}(x,t) \\ a_{LR}(x,t) \end{pmatrix}$$

where $\hat{e}_i$ are the standard unit two vectors and where for any $\mu \leq n$

$$\begin{aligned} \|\rho_{LR}(\cdot,t)\|_{\mathring{L}^p(\mu)} &\leq C\big(\|\rho_0\|_{W^{1,1}(n)} + \|a_0\|_{L^1(n)}\big) t^{-\frac{3}{2}(1-\frac{1}{p})} (1+t)^{1-(1-\frac{1}{p})-\frac{n}{2}+\mu} \\ \|a_{LR}(\cdot,t)\|_{\mathring{L}^p(\mu)} &\leq C\big(\|\rho_0\|_{W^{1,1}(n)} + \|a_0\|_{L^1(n)}\big) t^{-\frac{3}{2}(1-\frac{1}{p})} (1+t)^{\frac{1}{2}-(1-\frac{1}{p})-\frac{n}{2}+\mu} \end{aligned} \tag{25}$$

Proof. Setting $t = 0$, one finds

$$\rho_{LR}(x,0) = \rho_0(x) - \sum_{|\alpha| \leq \lfloor n \rfloor} \big\langle H_\alpha, \rho_0 \big\rangle \, \partial_x^\alpha \phi_0(x) \quad \text{and} \quad a_{LR}(x,0) = a_0(x) - \sum_{|\alpha| \leq \lfloor n \rfloor} \big\langle H_\alpha, a_0 \big\rangle \, \partial_x^\alpha \phi_0(x)$$

Thus, $\rho_{LR}(x,0)$ and $a_{LR}(x,0)$ are spatially localized functions with moments out to order $\lfloor n \rfloor$ equal to zero. Since equation (22) is linear, we have the representation

$$\rho(x,t) = \partial_t w(t) * K_\nu(t) * \rho_{LR}(x,0) - w(t) * K_\nu(t) * a_{LR}(x,0)$$

One can use the fact that the heat kernel satisfies

$$K_\nu(t) = K_\nu(t/2) * K_\nu(t/2)$$

to obtain

$$\partial_t w(t) * K_\nu(t) * \rho_{LR}(x,0) = \partial_t w(t) * K_\nu(t/2) * K_\nu(t/2) * \rho_{LR}(x,0)$$

This fact will be used repeatedly thorough out the paper. Using this fact along with Young's inequality and the estimates in Props 2.3 and 2.5 to obtain

$$\begin{aligned} \|\partial_t w(t) * K_\nu(t) * \rho_{LR}(x,0)\|_{\mathring{L}^p(\mu)} &\leq \|\partial_t w(t) * K_\nu(t/2)\|_{\mathring{L}^p(\mu)} \|K_\nu(t/2) * \rho_{LR}(x,0)\|_{L^1} \\ &\quad + \|\partial_t w(t) * K_\nu(t/2)\|_{L^p} \|K_\nu(t/2) * \rho_{LR}(x,0)\|_{\mathring{L}^1(\mu)} \\ &\leq C\|\rho_0\|_{L^1(n)} t^{-\frac{3}{2}(1-\frac{1}{p})} (1+t)^{1-(1-\frac{1}{p})-\frac{n}{2}+\mu} \end{aligned}$$

for $1 \le p \le \infty$, $0 \le \mu \le n$, and $t \ge 0$. The bounds for the other term can be obtained in the same way, and the same methods can be used to obtain bounds on a_{LR}, although there one must make use of Prop 2.6 to control the blowup as $t \to 0$. ☐

2.4.2. *Hermite expansion for divergence free vector fields.* When considering the asymptotics of the vorticity equation, we will need a Hermite expansion for divergence free vector fields. If we write

$$\vec{\omega}_L(t) = \mathbb{K}_\epsilon(t) * \vec{\omega}_0 \tag{26}$$

and naively expand each component of $\vec{\omega}(t)$ using the scalar Hermite expansion, the terms we obtain are not, in general, divergence free. This is because the moments of the components of a vector field are not independent if the vector field is divergence free. For any multi-index $\tilde{\alpha} \in \mathbb{Z}^3_{\ge 0}$, one must have

$$\int_{\mathbb{R}^3} (\nabla x^{\tilde{\alpha}}) \cdot \vec{\omega}(x) dx = \int_{\mathbb{R}^3} x^{\tilde{\alpha}} \nabla \cdot \vec{\omega}(x) dx = 0 \tag{27}$$

Hence, for $\tilde{\alpha}$ with only one non-zero component, we see that these moments must equal zero, for $\tilde{\alpha}$ with only two non-zero components, these moments come in pairs, and for $\tilde{\alpha}$ with all three components non-zero, these moments come in triples. For the purposes of this paper, we will only consider Hermite expansions out to moments of order 2, hence we define these asymptotic profiles explicitly in the following table and let $\vec{p}_{\tilde{\alpha},j} = \vec{f}_{\tilde{\alpha},j} = 0$ for all $|\tilde{\alpha}| \le 3$ not listed below. Higher order Hermite expansions can be defined, but their definition is more complicated (see [8] for details). We determine the action of the Biot-Savart operator on these profiles explicitly in Appendix D below.

$\tilde{\alpha}$	j	$\vec{p}_{\tilde{\alpha},j}$	$\vec{f}_{\tilde{\alpha},j}$
(1,1,0)	1	$(-\frac{1}{2}x_2, \frac{1}{2}x_1, 0)^T$	$\nabla \times (\phi_0 \vec{e}_3)$
(1,0,1)	1	$(\frac{1}{2}x_3, 0, -\frac{1}{2}x_1)^T$	$\nabla \times (\phi_0 \vec{e}_2)$
(0,1,1)	1	$(0, -\frac{1}{2}x_3, \frac{1}{2}x_2)^T$	$\nabla \times (\phi_0 \vec{e}_1)$
(2,1,0)	1	$(\frac{1}{2}x_1x_2, -\frac{1}{4}x_1^2, 0)^T$	$\nabla \times (\partial_{x_1}\phi_0 \vec{e}_3)$
(1,2,0)	1	$(\frac{1}{4}x_2^2, -\frac{1}{2}x_1x_2, 0)^T$	$\nabla \times (\partial_{x_2}\phi_0 \vec{e}_3)$
(2,0,1)	1	$(-\frac{1}{2}x_1x_3, 0, \frac{1}{4}x_1^2)^T$	$\nabla \times (\partial_{x_1}\phi_0 \vec{e}_2)$
(1,0,2)	1	$(-\frac{1}{4}x_3^2, 0, \frac{1}{2}x_1x_3)^T$	$\nabla \times (\partial_{x_3}\phi_0 \vec{e}_2)$
(0,2,1)	1	$(0, \frac{1}{2}x_2x_3, -\frac{1}{4}x_2^2)^T$	$\nabla \times (\partial_{x_2}\phi_0 \vec{e}_1)$
(0,1,2)	1	$(0, \frac{1}{4}x_3^2, -\frac{1}{2}x_2x_3)^T$	$\nabla \times (\partial_{x_3}\phi_0 \vec{e}_1)$
(1,1,1)	1	$(x_2x_3, 0, 0)^T$	$\nabla \times (\partial_{x_3}\phi_0 \vec{e}_3)$
(1,1,1)	2	$(0, 0, -x_1x_2)^T$	$\nabla \times (\partial_{x_1}\phi_0 \vec{e}_1)$

TABLE 1. Asymptotic profiles for the divergence-free vector field Hermite expansion.

Here $\tilde{\alpha}$ specifies which monomial determines this moment via (27). The parameter j specifies which of the independent moments determined by $x^{\tilde{\alpha}}$ is given by the vector $\vec{p}_{\tilde{\alpha},j}$. For $\tilde{\alpha}$ depending on two variables there is only one independent moment, hence $\vec{p}_{\tilde{\alpha},2} = 0$, whereas for $\tilde{\alpha}$ depending on all three there are two independent moments to consider. All of the profiles $\vec{f}_{\tilde{\alpha},j}$ are clearly divergence-free, and straightforward computations show that for $\vec{p}_{\tilde{\alpha},j}, \vec{f}_{\tilde{\alpha},j}$ defined above we have the orthonormality condition

$$\langle \vec{p}_{\tilde{\alpha},j}, \vec{f}_{\tilde{\beta},k} \rangle = \delta_{jk}\delta_{\tilde{\alpha}\tilde{\beta}}$$

We then have an analogue of the Hermite expansion:

Proposition 2.9. *Suppose that $\vec{\omega}_0 \in \mathbb{L}^1_\sigma(n)$ for $0 \le n \le 2$. If $\vec{\omega}_L(t)$ is the solution of the heat equation in $C^0\left[[0,\infty), \mathbb{L}^1_\sigma(n)\right]$ given by (26), then we can write*

$$\vec{\omega}_L(x,t) = \sum_{\substack{j \le 2 \\ |\tilde{\alpha}| \le \lfloor n \rfloor + 1}} \langle \vec{p}_{\tilde{\alpha},j}, \vec{\omega}_0 \rangle \mathbb{K}_\epsilon(t) * \vec{f}_{\tilde{\alpha},j}(x) + \vec{\omega}_{LR}(x,t)$$

where for any $\mu \leq n$

$$\|\vec{\omega}_{LR}(\cdot,t)\|_{\mathring{\mathbb{L}}^p(\mu)} \leq C\|\vec{\omega}_0\|_{\mathbb{L}^1(n)}(\nu t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{n-\mu}{2}} \tag{28}$$

The proof again makes use of the zero moment property of the remainder terms and Prop 2.3. We leave the details to the reader.

3. Existence and uniqueness of solutions to the $(\rho, a, \vec{\omega})^T$ system

Note that from the form of (12), if we can prove the existence of a and $\vec{\omega}$, we can get the solution for ρ by integration. Hence we need to choose a function space for $(a, \vec{\omega})$. In the Hermite expansions above, we saw that we could obtain higher order approximations by increasing the spatial localization of the initial conditions. Hence for a given $n \in \mathbb{R}_{\geq 0}$ we might choose $(\rho_0, a_0, \vec{\omega}_0) \in L^1(n) \times L^1(n) \times \mathbb{L}^1_\sigma(n)$ as a sufficiently general space to start with, and expect to obtain solutions with $\lfloor n \rfloor$ orders of asymptotic profiles. Note however that we expect that a and $\vec{\omega}$ come from a velocity vector field via $a = \nabla \cdot \vec{m}$ and $\vec{\omega} = \nabla \times \vec{m}$, hence we can assume they have zero total mass as in (18). It is for this reason that the expression $\lfloor n \rfloor_1$ enters into the definition of the decay rates $\ell_{n,p,\mu}$ and $\tilde{\ell}_{n,p,\mu}$ in (9), (10). Since $\vec{m}$ is assumed to have at least one derivative, we assume that ρ has at least one as well, hence we assume $(\rho_0, a_0, \vec{\omega}_0) \in W^{1,1}(n) \times L^1(n) \times \mathbb{L}^1_\sigma(n)$.

It will be desirable that the moments be continuous functions of time. To obtain this we will see that we need a slightly stronger assumption: we require that $(\rho_0, a_0, \vec{\omega}_0)$ belong to $W^{1,\tilde{p}}(n) \times L^{\tilde{p}}(n) \times \mathbb{L}^{\tilde{p}}_\sigma(n)$ for all $1 \leq \tilde{p} \leq 3/2$. We therefore define the function space

$$Z^0_n = \bigcap_{1 \leq p < \frac{3}{2}} C^0\big[[0,\infty), L^p(n) \times \mathbb{L}^p_\sigma(n)\big] \tag{29}$$

Due to the smoothing properties of the heat evolution the solutions have more regularity for $t > 0$, so if we fix a degree of smoothness $k \geq 1$ we define

$$Z^+_{n,k} = \bigcap_{1 \leq p \leq \infty} C^0\big[(0,\infty), W^{k,p}(n) \times \mathbb{W}^{k,p}_\sigma(n)\big] \tag{30}$$

Our existence analysis begins by studying the linear part of the evolution in 8. To this end we let $(\rho_L(t), a_L(t), \vec{\omega}_L(t))^T$ be defined by (23) and (26) for $t > 0$ and $(\rho_L(t), a_L(t), \vec{\omega}_L(t))^T = (\rho_0, a_0, \vec{\omega}_0)^T$ for $t = 0$. In Appendix E, we determine the smoothness properties and decay rates of these functions. Based on our findings we look for solutions of (12) in the function space

$$X_{n,k} = \Big\{(a, \vec{\omega}) \in Z^0_n \cap Z^+_{n,k} : \int_{\mathbb{R}^3} a(x,t)dx = 0 \text{ and } \int_{\mathbb{R}^3} \vec{\omega}(x,t)dx = 0\Big\} \tag{31}$$

with norm

$$\|(a,\vec{\omega})\|_{X_{n,k}} = \sup_{|\alpha| \leq k} \sup_{1 \leq p \leq \infty} \sup_{0 \leq \mu \leq n} \sup_{0 < t < \infty} \Big[t^{r_{\alpha,p}}(1+t)^{\ell_{n,p,\mu}+\hat{\ell}_{k,p,\alpha}}\|\partial^\alpha_x a(t)\|_{\mathring{L}^p(\mu)} + t^{r_{\alpha,p}}(1+t)^{\tilde{\ell}_{n,p,\mu}}\|\partial^\alpha_x \vec{\omega}(t)\|_{\mathring{\mathbb{L}}^p(\mu)}\Big]$$

where $r_{\alpha,p}$, $\tilde{\ell}_{n,p,\mu}$ and $\ell_{n,p,\mu}$ are as in (9), (10), and $\hat{\ell}_{k,p,\alpha}$ is defined by

$$\hat{\ell}_{k,p,\alpha} = \begin{cases} 0 & \text{for } |\alpha| < k \text{ and for } |\alpha| = k \text{ , } 1 \leq p \leq 2 \\ -\frac{2}{3}(1-\frac{1}{p}) + \frac{1}{3} & \text{for } |\alpha| = k \text{ , } p \geq 2 \end{cases} \tag{32}$$

The factor $\hat{\ell}_{k,p,\alpha}$ accounts for a slightly slower admissible decay rate for the highest order derivative in L^p, $p > 2$ as compared to the linear evolution. Note that $X_{n,k}$ is a Banach space with this norm. We will also need to define

$$L_{n,\tilde{n}}(t) = \log(1+t) \text{ when } n = \tilde{n} \text{ and } L_{n,\tilde{n}}(t) = 1 \text{ otherwise}$$

Theorem 2. *Fix* $n \in [0,2]$, $k \geq 1$ *and let* $(\rho_0, a_0, \vec{\omega}_0)$ *belong to* $W^{1,p}(n) \times L^p(n) \times \mathbb{L}^p_\sigma(n)$ *for all* $1 \leq p \leq \frac{3}{2}$, *and suppose that* a_0 *and* $\vec{\omega}_0$ *have zero total mass. If*

$$E_n = \sup_{1 \leq p \leq 3/2} \big(\|\rho_0\|_{W^{1,p}(n)} + \|a_0\|_{L^p(n)} + \|\vec{\omega}_0\|_{\mathbb{L}^p(n)}\big) \tag{33}$$

is chosen sufficiently small, then there exists a unique solution $(a(t), \vec{\omega}(t))$ *of* (12) *belonging to* $X_{n,k}$ *such that* $(a(0), \vec{\omega}(0)) = (a_0, \vec{\omega}_0)$.

Proof. Having chosen an initial condition satisfying the above, define the map $F_{(\rho_0,a_0,\vec{\omega}_0)}$ on $X_{n,k}$ sending $(a(s),\vec{\omega}(s))^T$ to a new function of space and time by letting $F_{(\rho_0,a_0,\vec{\omega}_0)}[(a,\vec{\omega})](0) = (a_0,\vec{\omega}_0)^T$ and

$$F_{(\rho_0,a_0,\vec{\omega}_0)}\big[a,\vec{\omega}\big](t) = \begin{pmatrix} -\partial_t^2 w * K_\nu * \rho_0 + \partial_t w * K_\nu * a_0 - \int_0^t \big[\partial_t w * K_\nu\big](t-s) * \Big[\nabla \cdot N\big(a(s),\vec{\omega}(s)\big)\Big] ds \\ \mathbb{K}_\epsilon * \vec{\omega}_0 - \int_0^t \mathbb{K}_\epsilon(t-s) * \Big[\nabla \times N\big(a(s),\vec{\omega}(s)\big)\Big] ds \end{pmatrix}$$

for $t > 0$. For convenience, we'll drop the subscript. We claim that F maps $X_{n,k}$ into itself and has Lipschitz constant equal to $1/2$ on a ball of radius R centered at the origin, which we prove below. Given these two claims, we can conclude our proof as follows. If $(a_L,\vec{\omega}_L)$ are as above, we note that each of the bounds determined in Appendix E depend on the magnitude of the initial condition, hence

$$\big\|(a_L,\vec{\omega}_L)\big\|_{X_{n,k}} \le CE_n$$

Therefore if we choose the initial condition sufficiently small, (ie $E_n \le \frac{R}{2C}$) we then have

$$\begin{aligned} \|F\big(a,\vec{\omega}\big) - (a_L,\vec{\omega}_L)\|_{X_{n,k}} &= \|F\big(a,\vec{\omega}\big) - F\big(0,0\big)\|_{X_{n,k}} \\ &\le \frac{1}{2}\|\big(a,\vec{\omega}\big) - \big(a_L,\vec{\omega}_L\big)\|_{X_{n,k}} + \frac{1}{2}\|\big(a_L,\vec{\omega}_L\big)\|_{X_{n,k}} \le \frac{R}{2} \end{aligned}$$

for $(a,\vec{\omega}) \in B\big((a_L,\vec{\omega}_L),\frac{R}{2}\big)$, the closed ball of radius $\frac{R}{2}$ centered at $(a_L,\vec{\omega}_L)^T$. Therefore F maps $B\big((a_L,\vec{\omega}_L)^T,\frac{R}{2}\big)$ into itself, and since F is a contraction here, the unique solution of (12) is given by the fixed point of F.

Claim One: $F : X_{n,k} \mapsto X_{n,k}$. We begin by proving that for $(a,\vec{\omega}) \in X_{n,k}$ the $X_{n,k}$ norm of $F(a,\vec{\omega})$ is finite and that $F(a,\vec{\omega}) \in Z_n^0 \cap Z_{n,k}^+$. We note again that the decay rates and smoothness requirements to belong to $X_{n,k}$ were found to be more than satisfied by those of the linear terms in Appendix E, so we need only analyze the evolution of the Duhamel terms. Furthermore we note that is sufficient to bound the $\mathring{L}^p(\mu)$ norms for $\mu = 0$ and $\mu = n$ since we can interpolate via

$$\|a\|_{\mathring{L}^p(\mu)} \le \big(\|a\|_{\mathring{L}^p(n)}\big)^{\frac{\mu}{n}} \big(\|a\|_{L^p}\big)^{1-\frac{\mu}{n}}$$

For μ fixed either as $\mu = 0$ or $\mu = n$, we need only bound the $\mathring{L}^p(\mu)$ norms $p = 1,2,\infty$ for times $t > 1$ and L^p norms for $p = 1,3/2,\infty$ for times $t < 1$, and the result then follows from interpolation via

$$\|a\|_{\mathring{L}^r(\mu)} \le \|a\|_{\mathring{L}^p(\mu)}^{\frac{p}{r}\frac{q-r}{q-p}} \|a\|_{\mathring{L}^q(\mu)}^{1-\frac{p}{r}\frac{q-r}{q-p}}$$

for any p,q,r such that $1 \le p \le r \le q \le \infty$.

We begin by bounding the unweighted L^p norms of the Duhamel term corresponding to $a(t)$ using our estimates above. First we use Young's inequality, then split the integral into two parts:

$$\begin{aligned} &\int_0^t \Big\|\partial_t w(t-s) * \partial_x^\alpha K_\nu(t-s) * \big[\nabla \cdot N(a(s),\vec{\omega}(s))\big]\Big\|_{L^p} ds \\ &\quad\le \int_0^t \Big\|\partial_t w(t-s) * K_\nu(\frac{t-s}{2})\Big\|_{L^q} \Big\|\partial_x^\alpha K_\nu(\frac{t-s}{2}) * \big[\nabla \cdot N(a(s),\vec{\omega}(s))\big]\Big\|_{L^{q_1}} ds \\ &\quad\le \left(\int_0^{t/2} + \int_{t/2}^t\right)(t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})}(1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\Big\|\partial_x^\alpha K_\nu(\frac{t-s}{2}) * \big[\nabla \cdot N(a(s),\vec{\omega}(s))\big]\Big\|_{L^{q_1}} ds \\ &\quad=: I_1 + I_2 \end{aligned}$$

Here $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{q_1}$. We can then bound the integrals for $s \in (0,t/2)$ and $s \in (t/2,t)$ separately.

First we handle the I_1 term. We use the heat estimate to pull the divergence and the ∂_x^α derivative off of the nonlinearity:

$$\begin{aligned} I_1 &\le \int_0^{t/2} (t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})-\frac{1+|\alpha|}{2}}(1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\big\|N(a(s),\vec{\omega}(s))\big\|_{L^{q_1}} ds \\ &\le \max_{ijl} \int_0^{t/2} (t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})-\frac{1+|\alpha|}{2}}(1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\big\|\partial_{x_i}(m_j)m_l\big\|_{L^{q_1}} ds \end{aligned}$$

We can then use our above estimates on Π, B in Cor. 2.2 parts **(a)**, **(b)** to bound the nonlinear term:

$$\begin{aligned}\big\|\partial_{x_i}(m_j)m_l\big\|_{L^{q_1}} &\le \big\|\partial_{x_i}m_j\|_{L^{p_1}}\big\|m_l\big\|_{L^{p_2}} \le C\big(\|a\|_{L^{p_1}}+\|\vec\omega\|_{\mathbb{L}^{p_1}}\big)\big(\|a\|_{L^{p_3}}+\|\vec\omega\|_{\mathbb{L}^{p_3}}\big)\\ &\le Cs^{-r_{0,p_1}-r_{0,p_3}}(1+s)^{-\min(\ell_{n,p_1,0},\tilde\ell_{n,p_1,0})-\min(\ell_{n,p_3,0},\tilde\ell_{n,p_3,0})}\|(a,\vec\omega)\|^2_{X_{n,k}}\end{aligned} \tag{34}$$

Note that the use of Young's inequality, Hölder's inequality, (15) and (17) puts the following restrictions on the set of admissible values for p_1, p_3:

$$1<p_1<\infty \quad , \quad 1<p_3<3 \quad , \quad \frac{1}{p}\le\frac{1}{p_1}+\frac{1}{p_3}-\frac{1}{3}\le 1 \tag{35}$$

We choose $q_1=1$ hence we require $\frac{1}{p_1}+\frac{1}{p_3}-\frac{1}{3}=1$. Letting $p_1=p_3=\frac{3}{2}$ (34) becomes

$$\big\|\partial_{x_i}(m_j)m_l\big\|_{L^1}\le C(1+s)^{-\frac{2}{3}-\lfloor n\rfloor_1}\|(a,\vec\omega)\|^2_{X_{n,k}}$$

hence putting this together we have

$$\begin{aligned}I_1 &\le \int_0^{t/2}(t-s)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1+|\alpha|}{2}}(1+t-s)^{\frac{1}{2}-(1-\frac{1}{p})}(1+s)^{-\frac{2}{3}-\lfloor n\rfloor_1}\|(a,\vec\omega)\|^2_{X_{n,k}}ds\\ &\le C\|(a,\vec\omega)\|^2_{X_{n,k}}t^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1+|\alpha|}{2}}(1+t)^{\frac{1}{2}-(1-\frac{1}{p})+\max(\frac{1}{3}-\lfloor n\rfloor_1,0)}L_{n,1/3}(t)\end{aligned} \tag{36}$$

for $t\ge 1$. Thus the L^p norms of I_1 have sufficiently fast decay as $t\to\infty$ for all $1\le p\le\infty$ such that the $X_{n,k}$ norm remains bounded. For $t<1$ we have

$$I_1\le\int_0^{t/2}(t-s)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1+|\alpha|}{2}}(1+t-s)^{\frac{1}{2}-(1-\frac{1}{p})}(1+s)^{-\frac{2}{3}-\lfloor n\rfloor_1}\|(a,\vec\omega)\|^2_{X_{n,k}}ds\le C\|(a,\vec\omega)\|^2_{X_{n,k}}t^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1-|\alpha|}{2}}$$

and hence we see the L^p norms have the right behavior for $1\le p\le\infty$ such that the $X_{n,k}$ norms remain bounded. Furthermore we note that for $1\le p<3/2$ and $|\alpha|=0$ the L^p norms tend to zero, which is consistent with the continuity of $F(a,\vec\omega)$ at $t=0$.

For I_2 we use the heat estimate to pull the divergence off of the nonlinearity:

$$\begin{aligned}I_2 &= \int_{t/2}^{t}(t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})}(1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\Big\|\partial_x^\alpha K_\nu(\frac{t-s}{2})*\big[\nabla\cdot N(a(s),\vec\omega(s))\big]\Big\|_{L^{q_1}}ds\\ &\le \max_{ijl}\int_{t/2}^{t}(t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})-\frac{1}{2}}(1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\big\|\partial_x^\alpha\partial_{x_i}(m_jm_l)\big\|_{L^{q_1}}ds\end{aligned}$$

For an arbitrary multi-index β, we can use the estimates in Cor. 2.2 parts **(a)**, **(b)** to obtain

$$\begin{aligned}\big\|\partial_x^\beta\partial_{x_i}(m_jm_l)\big\|_{L^{q_1}} &\le \sum_{\gamma_1+\gamma_2=\beta}\big\|\partial_x^{\gamma_1}\partial_{x_i}m_j\|_{L^{p_1}}\big\|\partial_x^{\gamma_2}m_l\big\|_{L^{p_2}}\\ &\le C\sum_{\gamma_1+\gamma_2=\beta}\big(\|\partial_x^{\gamma_1}a\|_{L^{p_1}}+\|\partial_x^{\gamma_1}\vec\omega\|_{\mathbb{L}^{p_1}}\big)\big(\|\partial_x^{\gamma_2}a\|_{L^{p_3}}+\|\partial_x^{\gamma_2}\vec\omega\|_{\mathbb{L}^{p_3}}\big)\\ &\le Cs^{-r_{0,p_1}-r_{0,p_3}-\frac{|\beta|}{2}}(1+s)^{-\min(\ell_{n,p_1,0},\tilde\ell_{n,p_1,0})-\min(\ell_{n,p_3,0},\tilde\ell_{n,p_3,0})}\|(a,\vec\omega)\|^2_{X_{n,k}}\end{aligned} \tag{37}$$

provided that the constraints in (35) are met. Here we take $\beta=\alpha$. We must also ensure that the singularity at $s=t$ is integrable. For $1\le p<3/2$ we can choose $p_1=p_3=3/2$ as before, and we obtain

$$\begin{aligned}I_2 &\le \int_{t/2}^{t}(t-s)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2}}(1+t-s)^{\frac{1}{2}-(1-\frac{1}{p})}s^{-\frac{|\alpha|}{2}}(1+s)^{-\frac{2}{3}-\lfloor n\rfloor_1}\|(a,\vec\omega)\|^2_{X_{n,k}}ds\\ &\le Ct^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1-|\alpha|}{2}}(1+t)^{\frac{1}{2}-(1-\frac{1}{p})-\frac{2}{3}-\lfloor n\rfloor_1}\|(a,\vec\omega)\|^2_{X_{n,k}}\end{aligned} \tag{38}$$

for $0<t<\infty$, hence these L^p norms have the right behavior as $t\to 0$ and as $t\to\infty$, and tend to zero for $|\alpha|=0$ which is consistent with continuity at $t=0$. Similarly, for $3/2\le p\le 2$ we can choose $p_1=p_3=2$ in (37) and obtain the pointwise bound

$$\big\|\partial_x^\alpha\partial_{x_i}(m_jm_l)\big\|_{L^{\frac{3}{2}}}\le Cs^{-\frac{1}{2}-\frac{|\alpha|}{2}}(1+s)^{-1-\lfloor n\rfloor_1}\|(a,\vec\omega)\|^2_{X_{n,k}}$$

from which it follows that

$$I_2 \leq \int_{t/2}^{t} (t-s)^{-\frac{3}{2}(\frac{2}{3}-\frac{1}{p})-\frac{1}{2}}(1+t-s)^{\frac{1}{2}-(\frac{2}{3}-\frac{1}{p})}s^{-\frac{1}{2}-\frac{|\alpha|}{2}}(1+s)^{-1-\lfloor n\rfloor_1}\|(a,\vec{\omega})\|_{X_{n,k}}^2 ds$$
$$\leq Ct^{-\frac{3}{2}(\frac{2}{3}-\frac{1}{p})-\frac{|\alpha|}{2}}(1+t)^{\frac{1}{2}-(1-\frac{1}{p})-\frac{2}{3}-\lfloor n\rfloor_1}\|(a,\vec{\omega})\|_{X_{n,k}}^2$$

for $0<t<\infty$, hence these L^p norms also have the right behavior as $t\to 0$ and as $t\to\infty$. Finally, we can obtain bounds on the L^∞ norm by choosing $p_1=8$, $p_3=8/3$ in (37) to obtain the pointwise bound

$$\left\|\partial_x^\alpha \partial_{x_i}(m_j m_l)\right\|_{L^6} \leq Cs^{-\frac{5}{4}-\frac{|\alpha|}{2}}(1+s)^{-1-\lfloor n\rfloor_1}\|(a,\vec{\omega})\|_{X_{n,k}}^2$$

from which we then obtain the following bound on the integral for $0<t<\infty$:

$$I_2 \leq Ct^{-1-\frac{|\alpha|}{2}}(1+t)^{-1+\frac{1}{3}-\lfloor n\rfloor_1}\|(a,\vec{\omega})\|_{X_{n,k}}^2 \tag{39}$$

Note this is slower than the linear evolution rate. For $|\alpha|<k$ we can make an improved estimate to match the linear rate as follows. With $p=\infty$, we keep all derivatives on the nonlinearity when using the heat estimate, and we obtain

$$I_2 \leq \max_{ij}\int_{t/2}^{t}(t-s)^{-\frac{3}{2q_1}}(1+t-s)^{\frac{1}{2}-\frac{1}{q_1}}\left\|\partial_x^\alpha\partial_{x_i}\partial_{x_j}(m_i m_j)\right\|_{L^{q_1}}ds \tag{40}$$

We can then use the estimate in 37 by taking $\beta=\alpha+e_j$, and we choose $p_1=p_3=12/5$ to obtain

$$I_2 \leq C\int_{t/2}^{t}(t-s)^{-\frac{3}{4}}s^{-\frac{3}{4}-\frac{|\alpha|+1}{2}}(1+s)^{-1-\lfloor n\rfloor_1}ds\|(a,\vec{\omega})\|_{X_{n,k}}^2 \leq C\|(a,\vec{\omega})\|_{X_{n,k}}^2 t^{-1-\frac{|\alpha|}{2}}(1+t)^{-1-\lfloor n\rfloor_1}$$

For $n=0$ we are done. For $n>0$ we bound the weighted norms when $\mu=n$ of the Duhamel term corresponding to $a(t)$, and the results then follow by interpolation. We first bound the weighted norm of the convolution in terms of the weighted norms of each of its components using Young's inequality:

$$\begin{aligned}
&\int_0^t \left\|\partial_t w(t-s)*\partial_x^\alpha K_\nu(t-s)*\left[\nabla\cdot N(a(s),\vec{\omega}(s))\right]\right\|_{\mathring{L}^p(n)}ds\\
&\leq \int_0^t \left\|\partial_t w(t-s)*K_\nu(\tfrac{t-s}{2})\right\|_{\mathring{L}^{\tilde{q}}(n)}\left\|\partial_x^\alpha K_\nu(\tfrac{t-s}{2})*\left[\nabla\cdot N(a(s),\vec{\omega}(s))\right]\right\|_{L^{\tilde{q}_1}}ds\\
&\quad+\int_0^t \left\|\partial_t w(t-s)*K_\nu(\tfrac{t-s}{2})\right\|_{L^q}\left\|\partial_x^\alpha K_\nu(\tfrac{t-s}{2})*\left[\nabla\cdot N(a(s),\vec{\omega}(s))\right]\right\|_{\mathring{L}^{q_1}(n)}ds
\end{aligned}\tag{41}$$

For the first term, we can use the weighted estimate of the heat-wave operator in Prop 2.5 and then repeat the analysis used above for the unweighted norm of the nonlinearity line by line to obtain the appropriate bounds for this term. So we need only bound the second term.

For the second term we use the unweighted estimate in Prop 2.5 and split the integral as before:

$$\begin{aligned}
&\int_0^t \left\|\partial_t w(t-s)*K_\nu(\tfrac{t-s}{2})\right\|_{L^q}\left\|\partial_x^\alpha K_\nu(\tfrac{t-s}{2})*\left[\nabla\cdot N(a(s),\vec{\omega}(s))\right]\right\|_{\mathring{L}^{q_1}(n)}ds\\
&\leq \int_0^t (t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})}(1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\left\|\partial_x^\alpha K_\nu(\tfrac{t-s}{2})*\left[\nabla\cdot N(a(s),\vec{\omega}(s))\right]\right\|_{\mathring{L}^{q_1}(n)}ds\\
&\leq \left(\int_0^{t/2}+\int_{t/2}^t\right)(t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})}(1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\left\|\partial_x^\alpha K_\nu(\tfrac{t-s}{2})*\left[\nabla\cdot N(a(s),\vec{\omega}(s))\right]\right\|_{\mathring{L}^{q_1}(n)}ds\\
&= I_1+I_2
\end{aligned}$$

The next step is to use our heat estimate, and then we will need bounds for the weighted norm of the nonlinear term analogous to (34), (37). Note however that these bounds are essentially the same, so here we will derive both at once. The derivation is similar to (37), but one must always place the weight on the term with fewer derivatives in order to use Cor. 2.2 part **(a)**. For $0 < n < 2$ we make the estimate

$$\begin{aligned} \|\partial_x^\beta \partial_{x_i}(m_j m_l)\|_{\mathring{L}^{q_1}(n)} &\le \|\partial_x^\beta \partial_{x_i}(m_j m_l)\|_{L^{q_1}(n)} \\ &\le C \sum_{\gamma_1+\gamma_2=\beta} \left(\|\partial_x^{\gamma_1} a\|_{L^{p_1}} + \|\partial_x^{\gamma_1}\vec{\omega}\|_{\mathbb{L}^{p_1}}\right)\left(\|\partial_x^{\gamma_2} a\|_{L^{p_3}(n)} + \|\partial_x^{\gamma_2}\vec{\omega}\|_{\mathbb{L}^{p_3}(n)}\right) \\ &\le C s^{-r_{0,p_1}-r_{0,p_3}-\frac{|\beta|}{2}}(1+s)^{-\min(\ell_{n,p_1,0},\tilde{\ell}_{n,p_1,0})-\min(\ell_{n,p_3,n},\tilde{\ell}_{n,p_3,n})}\|(a,\vec{\omega})\|^2_{X_{n,k}} \end{aligned} \tag{42}$$

using parts **(a)** and **(b)** of Cor 2.2, which requires the set of constraints

$$1 < p_1 < \infty \quad , \quad \frac{3}{3-n} < p_3 < \frac{3}{1-\lfloor n \rfloor_1} \quad , \quad \frac{1}{p} \le \frac{1}{p_1} + \frac{1}{p_3} - \frac{1}{3} \le 1 \tag{43}$$

or for $1 \le n \le 2$ we can obtain the same bound using parts **(a)** and **(c)** of Cor 2.2, which require

$$1 < p_1 < \infty \quad , \quad \frac{3}{4-n} < p_3 < \frac{3}{3-n} \quad , \quad \frac{1}{p} \le \frac{1}{p_1} + \frac{1}{p_3} - \frac{1}{3} \le 1 \tag{44}$$

Note that in the overlapping region $1 \le n < 2$ we can use either bound, but if we use Cor 2.2 **(a)** and **(c)** by satisfying the constraints in (44), we are allowed to choose smaller p_3 than (43) allow, a fact which we will exploit. The task then becomes obtaining various choices of p_1 and p_3 for I_1, I_2, $1 \le p \le \infty$, $0 < n \le 2$.

For I_1 we use the heat estimate to pull the divergence and the ∂_x^α derivative off of the nonlinearity, and use (42) with $\beta = 0$. For $0 < n < 1$ we can satisfy the constraints in (43) with $q_1 = 1$ by taking $p_1 = p_3 = 3/2$, and we obtain

$$\begin{aligned} I_1 &\le \int_0^{t/2} (t-s)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1+|\alpha|}{2}}(1+t-s)^{\frac{1}{2}-(1-\frac{1}{p})}(1+s)^{-\frac{2}{3}-\lfloor n \rfloor_1+n}\|(a,\vec{\omega})\|^2_{X_{n,k}} ds \\ &\le C\|(a,\vec{\omega})\|^2_{X_{n,k}} t^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1+|\alpha|}{2}}(1+t)^{\frac{1}{2}-(1-\frac{1}{p})+\frac{1}{3}-\lfloor n \rfloor_1+n} \end{aligned}$$

whereas for $1 < n < 2$ precisely the same estimate holds by taking $p_1 = p_3 = 3/2$ in (44). Hence these weighted L^p norms decay sufficiently quickly as $t \to \infty$ for $1 \le p \le \infty$. For $t < 1$ this bound becomes

$$I_1 \le C\|(a,\vec{\omega})\|^2_{X_{n,k}} t^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1-|\alpha|}{2}} \tag{45}$$

hence these norms have the right behavior as $t \to 0$. For $1 \le n < 3/2$ we can use (44) by taking $p_1 = 2$, $p_3 = 6/5$ and for $3/2 < n \le 2$ we can use $p_1 = 6/5$, $p_3 = 2$. In both cases we have

$$\begin{aligned} I_1 &\le \int_0^{t/2} (t-s)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1+|\alpha|}{2}}(1+t-s)^{\frac{1}{2}-(1-\frac{1}{p})}s^{-\frac{1}{4}}(1+s)^{-\frac{5}{6}+n-\frac{7}{12}}\|(a,\vec{\omega})\|^2_{X_{n,k}} ds \\ &\le C\|(a,\vec{\omega})\|^2_{X_{n,k}} t^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{4}-\frac{|\alpha|}{2}}(1+t)^{\frac{1}{2}-(1-\frac{1}{p})+n-\frac{4}{3}} \end{aligned}$$

for $0 < t < \infty$, hence the weighted L^p norms of this term decay sufficiently fast to remain in $X_{n,k}$ for $1 \le p \le \infty$. For $t < 1$ this bound shows that the $\mathring{L}^p(n)$ norms have the right behavior as $t \to 0$ for $1 \le p < 6/5$. Then we need only prove that the $L^p(n)$ norms for $6/5 \le p \le \infty$ have the right behavior as $t \to 0$ for $n = 1$ and $n = 2$. Here we can choose $p_1 = 3/2$, $p_3 = 2$ for $n = 1$ using (43) and using (44) for $n = 2$ and we again obtain (45), so the weighted L^p norms blow up sufficiently slowly for $6/5 \le p \le \infty$ as $t \to 0$, hence I_1 belongs to $X_{n,k}$.

For I_2 we can reuse many of the estimates in the unweighted case, but we have to modify these slightly. We again use the heat estimate to pull the divergence off the nonlinearity, and we again have to worry about the singularity at $s = t$. For $0 < n < 1$ we can make precisely the same choices as in the unweighted case. Namely that we can obtain the appropriate bounds for the $\mathring{L}^p(n)$ norms using (43) by taking $p_1 = p_3 = 3/2$ for $1 \le p < 3/2$ and we obtain the analogous weighted pointwise bound

$$\|\partial_x^\alpha \partial_{x_i}(m_j m_l)\|_{\mathring{L}^1(n)} \le C s^{-\frac{|\alpha|}{2}}(1+s)^{-\frac{5}{3}+n}\|(a,\vec{\omega})\|^2_{X_{n,k}}$$

We can then make the identical estimate in (38) with this analogous pointwise bound to show that these norms have the correct behavior for $0 < t < \infty$. Similarly we can use (43) by taking $p_1 = p_3 = 2$ for $3/2 \le p \le 2$ and taking $p_1 = 8$, $p_3 = 8/3$ for $p = \infty$ and obtain the analogous pointwise bounds, from which it follows in the same way that these norms have the correct behavior for $0 < t < \infty$, except for $p = \infty$, $|\alpha| < k$. We can then match the decay rate for $p = \infty$, $|\alpha| < k$ by keeping all derivatives on the nonlinearity as in (40), taking $\beta = \alpha + e_j$ in (42) and taking $p_1 = p_3 = 12/5$ in (43).

The case $1 < n < 2$ is also similar, and we can show that the $\mathring{L}^p(n)$ norms have the correct behavior for $1 \leq p < 3/2$ by taking $p_1 = p_3 = 3/2$ in (44). For the $\mathring{L}^p(n)$ norms for $3/2 \leq p \leq 2$ we make a slightly different estimate by taking $p_1 = 3$, $p_3 = 3/2$ in (44) and we obtain the pointwise bound

$$\|\partial_x^\alpha \partial_{x_i}(m_j m_l)\|_{\mathring{L}^{\frac{3}{2}}(n)} \leq C s^{-\frac{1}{2}-\frac{|\alpha|}{2}}(1+s)^{-\frac{11}{6}+n}\|(a,\vec{\omega})\|^2_{X_{n,k}}$$

and repeating the above analysis. For $1 < n < 2$ we can set $q_1 = 6$ by choosing $p_1 = 8/3$ and $p_3 = 8$ using (43), and show that the $\mathring{L}^\infty(n)$ norms have the correct behavior for $0 < t < \infty$, except for $p = \infty$, $|\alpha| < k$. We can then match the decay rate for $p = \infty$, $|\alpha| < k$ by keeping the derivatives on the nonlinearity and using $\beta = \alpha + e_j$ in (42) with $p_1 = 2$, $p_3 = 3$ in (43).

It remains to show the $\mathring{L}^p(n)$ norms have the correct behavior for $n = 1$ and $n = 2$. We can choose $p_1 = 2$, $p_3 = 6/5$ for $1 \leq n < 3/2$ and $p_1 = \frac{6}{5}$, $p_3 = 2$ for $3/2 < n \leq 2$ and we find

$$\begin{aligned} I_2 &\leq \int_{t/2}^t (t-s)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2}}(1+t-s)^{\frac{1}{2}-(1-\frac{1}{p})}s^{-\frac{1}{4}-\frac{|\alpha|}{2}}(1+s)^{-\frac{17}{12}+n}\|(a,\vec{\omega})\|^2_{X_{n,k}}ds \\ &\leq Ct^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{4}-\frac{|\alpha|}{2}}(1+t)^{\frac{1}{2}-(1-\frac{1}{p})-\frac{17}{12}+n}\|(a,\vec{\omega})\|^2_{X_{n,k}} \end{aligned}$$

for $1 \leq p < 3/2$, which decays appropriately quickly as $t \to \infty$. Note also that this bound holds for $t < 1$, and hence the weighted L^p norms tend to zero as $t \to 0$ for $1 \leq p < 6/5$. For $3/2 \leq p \leq 2$ we can set $q_1 = 3/2$ by choosing $p_1 = p_3 = 2$ using (43) for $1 \leq n < 3/2$ and (44) for $3/2 < n \leq 2$ and we find

$$I_2 \leq Ct^{-\frac{3}{2}(\frac{2}{3}-\frac{1}{p})-\frac{|\alpha|}{2}}(1+t)^{\frac{1}{2}-(1-\frac{1}{p})-\frac{5}{3}+n}\|(a,\vec{\omega})\|^2_{X_{n,k}}$$

for $0 < t < \infty$. Finally, for $1 \leq n < 3/2$ we choose $p_1 = 8/3$, $p_3 = 8$ using (43) and for $15/8 < n \leq 2$ we choose $p_1 = 8$, $p_3 = 8/3$ using (44) and we see that the $\mathring{L}^\infty(n)$ norm has the right behavior for $t > 1$ $|\alpha| = k$ and $t < 1$ for all α, and we can then match the decay rate for $p = \infty$, $|\alpha| < k$ by keeping the derivatives on the nonlinearity and using $\beta = \alpha + e_j$ in (42) with $p_1 = 2$, $p_3 = 3$ in (43) for $1 \leq n < 3/2$ and $p_1 = 24/11$, $p_3 = 8/3$ in (44) for $15/8 < n \leq 2$.

The bounds on the Duhamel term for $\vec{\omega}(t)$ can be obtained in a very similar manner. The only difference is that one need not make the initial step of using Young's inequality. Namely, we begin by looking at the unweighted norms, and we first split the integral

$$\begin{aligned} \int_0^t \left\|\partial_x^\alpha \mathbb{K}_\epsilon(t-s) * \left[\nabla \times N(a(s),\vec{\omega}(s))\right]\right\|_{\mathbb{L}^p} ds &= \left(\int_0^{t/2} + \int_{t/2}^t\right)\left\|\partial_x^\alpha \mathbb{K}_\epsilon(t-s) * \left[\nabla \times N(a(s),\vec{\omega}(s))\right]\right\|_{\mathbb{L}^p} ds \\ &=: I_1 + I_2 \end{aligned}$$

We can then use the heat estimate directly, and for $s \in (0, t/2)$ we pull the divergence and the ∂_x^α derivative off the nonlinear term using the heat estimate, whereas $s \in (t/2, t)$ we only pull the divergence off. By making the exact same estimates as for the Duhamel term for $a(t)$ with the same choices of p_1 and p_3 we arrive at the analogous bounds. The weighted norms can be obtained in the same way. For brevity we omit this, although this work is carried out in full form in [8].

It remains to obtain continuity for $t > 0$, in which case we would have $F(a, \vec{\omega}) \in Z_n^0 \cap Z_{n,k}^+$. Beginning with the Duhamel term for $a(t)$, we note that this is equivalent to showing that

$$\begin{aligned} &\lim_{h\to 0}\int_t^{t+h}\left\|\partial_t w(t+h-s) * \partial_x^\alpha K_\nu(t+h-s) * \left[\nabla \cdot N(a(s),\vec{\omega}(s))\right]\right\|_{L^p} ds = 0 \\ &\lim_{h\to 0}\int_0^t\left\|\left[\partial_t w(t+h-s) * \partial_x^\alpha K_\nu(t+h-s) - \partial_t w(t-s) * \partial_x^\alpha K_\nu(t-s)\right] * \left[\nabla \cdot N(a(s),\vec{\omega}(s))\right]\right\|_{L^p} ds = 0 \end{aligned} \tag{46}$$

For the first limit we can re-use the methods used to obtain a bound on the I_2 term above to show that this limit is zero. For the second, we can use the estimate

$$\begin{aligned} &\left\|\left[\partial_t w(t+h-s) * \partial_x^\alpha K_\nu(t+h-s) - \partial_t w(t-s) * \partial_x^\alpha K_\nu(t-s)\right] * \left[\nabla \cdot N(a(s),\vec{\omega}(s))\right]\right\|_{L^p} \\ &\quad \leq \left\|\partial_t w(t+h-s) * \partial_x^\alpha K_\nu(\frac{t-s}{2}+h) - \partial_t w(t-s) * \partial_x^\alpha K_\nu(\frac{t-s}{2})\right\|_{L^1}\left\|K_\nu(\frac{t-s}{2}) * \left[\nabla \cdot N(a(s),\vec{\omega}(s))\right]\right\|_{L^p} \end{aligned}$$

and show that this first factor tends to zero uniformly in s as $h \to 0$. The weighted norms can be bounded similarly, and one can obtain continuity for the Duhamel term corresponding to $\vec{\omega}(t)$ by showing that the limits analogous to (46) are zero.

Claim Two: F has Lipschitz constant $K = \frac{1}{2}$ on a ball $B(0,R)$ in $X_{n,k}$. We need to bound $\|F(a,\vec{\omega}) - F(\tilde{a},\tilde{\vec{\omega}})\|_{X_{n,k}}$ for $(a,\vec{\omega}),(\tilde{a},\tilde{\vec{\omega}}) \in B(0,R)$, where R is yet to be chosen. The analysis is similar to the above, but now we use the bilinear property of the nonlinearity to get the analogous unweighted estimates

$$\begin{aligned}
&\left\|\partial_x^\beta N(a(s),\vec{\omega}(s)) - \partial_x^\beta N(\tilde{a}(s),\tilde{\vec{\omega}}(s))\right\|_{L^{q_1}} \\
&\qquad \le \max_{ijl} \left\|\partial_x^\beta\left[\partial_{x_i}(m_j)(m_l - \tilde{m}_l)\right]\right\|_{L^{q_1}} + \left\|\partial_x^\beta\left[\partial_{x_i}(m_j - \tilde{m}_j)\tilde{m}_l\right]\right\|_{L^{q_1}} \qquad (47)\\
&\le C\Big(\|(\tilde{a},\tilde{\vec{\omega}})\|_{X_{n,k}} + \|(a,\vec{\omega})\|_{X_{n,k}}\Big)\|(a-\tilde{a},\vec{\omega}-\tilde{\vec{\omega}})\|_{X_{n,k}} s^{-r_{0,p_1}-r_{0,p_3}-\frac{|\beta|}{2}}(1+s)^{-\min(\ell_{n,p_1,0},\tilde{\ell}_{n,p_1,0})-\min(\ell_{n,p_3,0},\tilde{\ell}_{n,p_3,0})}
\end{aligned}$$

corresponding to (34) and (37), which require the set of constraints (35), as well as the analogous weighted estimate

$$\begin{aligned}
&\left\|\partial_x^\beta\left[N(a(s),\vec{\omega}(s)) - N(\tilde{a}(s),\tilde{\vec{\omega}}(s))\right]\right\|_{\mathring{L}^{q_1}(n)} \qquad (48)\\
&\le C\big(\|(\tilde{a},\tilde{\vec{\omega}})\|_{X_{n,k}} + \|(a,\vec{\omega})\|_{X_{n,k}}\big)\|(a-\tilde{a},\vec{\omega}-\tilde{\vec{\omega}})\|_{X_{n,k}} s^{-r_{0,p_1}-r_{0,p_3}-\frac{|\beta|}{2}}(1+s)^{-\min(\ell_{n,p_1,0},\tilde{\ell}_{n,p_1,0})-\min(\ell_{n,p_3,n},\tilde{\ell}_{n,p_3,n})}
\end{aligned}$$

corresponding to (42) which requires the set of constraints (43) for $0 < n < 2$ and (44) for $1 \le n \le 2$.

The proof then follows exactly the steps used to prove Claim 1 with these analogous estimates. We begin by looking at the norms of the difference between the Duhamel term corresponding to $a(t)$:

$$\begin{aligned}
&\int_0^t \left\|\partial_t w(t-s) * \partial_x^\alpha K_\nu(t-s) * \left[\nabla\cdot\left[N(a(s),\vec{\omega}(s)) - N(\tilde{a}(s),\tilde{\vec{\omega}}(s))\right]\right]\right\|_{L^p} ds \\
&\le \left(\int_0^{t/2} + \int_{t/2}^t\right)(t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})}(1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\left\|\partial_x^\alpha K_\nu(\frac{t-s}{2}) * \left[\nabla\cdot\left[N(a(s),\vec{\omega}(s)) - N(\tilde{a}(s),\tilde{\vec{\omega}}(s))\right]\right]\right\|_{L^{q_1}} ds \\
&=: I_1 + I_2
\end{aligned}$$

For I_1 we can then use the heat estimate and the bilinearity to obtain

$$I_1 \le C\max_{ijk}\int_0^{t/2}(t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})-\frac{1}{2}-\frac{|\alpha|}{2}}(1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\Big(\left\|\partial_{x_i}(m_j)(m_l-\tilde{m}_l)\right\|_{L^{q_1}} + \left\|\partial_{x_i}(m_j-\tilde{m}_j)\tilde{m}_l\right\|_{L^{q_1}}\Big)ds$$

We can then repeat the analysis for the Duhamel term above for $a(t)$ line by line for each of these terms, using (47) with $\alpha = 0$ and then making the same choices for p_1 and p_3 to handle the cases $t \ge 1$ and $t < 1$ separately for different values of p, and we find

$$\sup_{|\alpha|\le k}\sup_{1\le p\le\infty}\sup_{0\le t<\infty} t^{r_{\alpha,p}}(1+t)^{\ell_{n,p,0}+\hat{\ell}_{k,p,\alpha}} I_1 \le C\Big(\|(\tilde{a},\tilde{\vec{\omega}})\|_{X_{n,k}} + \|(a,\vec{\omega})\|_{X_{n,k}}\Big)\|(a-\tilde{a},\vec{\omega}-\tilde{\vec{\omega}})\|_{X_{n,k}} \qquad (49)$$

Similarly for I_2 we can use the heat estimate, (47) and the preceding analysis to obtain

$$\sup_{|\alpha|\le k}\sup_{1\le p\le\infty}\sup_{0\le t<\infty} t^{r_{\alpha,p}}(1+t)^{\ell_{n,p,0}+\hat{\ell}_{k,p,\alpha}} I_2 \le C\Big(\|(\tilde{a},\tilde{\vec{\omega}})\|_{X_{n,k}} + \|(a,\vec{\omega})\|_{X_{n,k}}\Big)\|(a-\tilde{a},\vec{\omega}-\tilde{\vec{\omega}})\|_{X_{n,k}} \qquad (50)$$

The bounds on the weighted norms can be obtained by following the steps used in the proof of Claim 1 with the analogous bound (48), and the bounds on the Duhamel term for $\vec{\omega}(t)$ can be obtained by repeating this procedure. By combining (49), (50), the bounds on the weighted norms and the analogue for the Duhamel term for $\vec{\omega}(t)$, we obtain

$$\left\|F(a,\vec{\omega}) - F(\tilde{a},\tilde{\vec{\omega}})\right\|_{X_{n,k}} \le C\Big(\|(\tilde{a},\tilde{\vec{\omega}})\|_{X_{n,k}} + \|(a,\vec{\omega})\|_{X_{n,k}}\Big)\left\|(a-\tilde{a},\vec{\omega}-\tilde{\vec{\omega}})\right\|_{X_{n,k}}$$

so by letting $R = \frac{1}{4C}$ we have our result.

□

Having proven the existence of solutions $a(t)$ and $\vec{\omega}(t)$, we now complete the proof of existence of solutions to (12) by proving the existence of a solution $\rho(t)$. For $n \in \mathbb{R}_{\ge 0}$ we define the function space

$$Y_{n,k} = \{\rho : \rho \in \bigcap_{1\le p<3/2} C^0\left[[0,\infty), L^p(n)\right] \text{ and } \rho \in \bigcap_{1\le p\le\infty} C^0\left[(0,\infty), W^{k,p}(n)\right]\}$$

equipped with the norm

$$\|\rho\|_{Y_{n,k}} = \sup_{|\alpha|\le k}\sup_{1\le p\le\infty}\sup_{0\le\mu\le n}\sup_{0<t<\infty}\left[t^{r_{\alpha,p}}(1+t)^{\ell_{n,p,\mu}+\hat{\ell}_{k,p,\alpha}-\frac{1}{2}}\|\partial_x^\alpha\rho(t)\|_{\mathring{L}^p(\mu)}\right]$$

where $r_{\alpha,p}, \ell_{n,p,\mu}, \hat{\ell}_{k,p,\alpha}$ are as before.

Corollary 3.1. *Fix $n\in[0,2]$, $k\ge 1$ and let $(\rho_0, a_0, \vec{\omega}_0)$ belong to $W^{1,p}(n)\times L^p(n)\times\mathbb{L}^p_\sigma(n)$ for all $1\le p\le\frac{3}{2}$, where $a_0,\vec{\omega}_0$ have zero total mass and $(\rho_0,a_0,\vec{\omega}_0)$ have sufficiently small norms as in Theorem 2. If $(a(t),\vec{\omega}(t))$ is the solution of (12) from Theorem 2, then the solution $\rho(t)$ defined by (12) belongs to $Y_{n,k}$.*

Proof. As before the decay rates and smoothness properties are chosen to match those of the linear terms hence we need only check the Duhamel term. We first estimate the unweighted norms

$$\int_0^t\left\|w(t-s)*\partial_x^\alpha K_\nu(t-s)*\left[\nabla\cdot N(a(s),\vec{\omega}(s))\right]\right\|_{L^p}ds$$
$$\le\max_{ijk}\left(\int_0^{t/2}+\int_{t/2}^t\right)(t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})+1}(1+t-s)^{-(\frac{1}{q_1}-\frac{1}{p})}\left\|\partial_x^\alpha K_\nu(\frac{t-s}{2})*\left[\nabla\cdot N(a(s),\vec{\omega}(s))\right]\right\|_{L^p}ds =: I_1+I_2$$

For I_1, we pull the divergence and the ∂_x^α derivative off of the nonlinearity using the heat estimate, use estimate (34), let $p_1=p_3=3/2$ and find

$$I_1\le C\|(a,\vec{\omega})\|^2_{X_{n,k}}t^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}-\frac{|\alpha|}{2}}(1+t)^{-(1-\frac{1}{p})+\max(\frac{1}{3}-\lfloor n\rfloor_1,0)}$$

which holds for all $t>0$, hence the L^p norms of this term have sufficiently fast decay for $1\le p\le\infty$ as $t\to\infty$, tend to zero as $t\to 0$ for $1\le p<3/2$, $|\alpha|=0$ and blow up sufficiently slowly for $3/2\le p\le\infty$.

For I_2, we use the heat estimate to pull the divergence off the nonlinearity, use estimate (37) and set $p_1=p_3=3/2$ for $1\le p\le 2$ and find

$$I_2\le C\|(a,\vec{\omega})\|^2_{X_{n,k}}t^{\frac{3}{2p}-\frac{|\alpha|}{2}}(1+t)^{-(1-\frac{1}{p})-\frac{2}{3}-\lfloor n\rfloor_1}$$

which also holds for all t, hence these behave correctly both as $t\to 0$ and as $t\to\infty$ as well. For $p=\infty$, we can choose $p_1=8$, $p_3=8/3$ and we obtain

$$I_2\le C\|(a,\vec{\omega})\|^2_{X_{n,k}}t^{-\frac{|\alpha|}{2}}(1+t)^{-\frac{7}{6}-\lfloor n\rfloor_1}$$

separately for $t>1$ and $t<1$ and hence L^∞ norm has the correct behavior for $t<1$ and $t>1$ if $|\alpha|=k$. We can then match the linear decay rate for $p=\infty$, $|\alpha|<k$ by keeping the derivatives on the nonlinearity and using $\beta=\alpha+e_j$ in (37) with $p_1=p_3=12/5$.

As above, we can bound the weighted norms in terms of the weighted norms of each of the components of the convolution. For the term in which the weight falls on the heat-wave operator we can repeat the estimates on the unweighted norms of the nonlinearity above. For the other term, we split the integral into two pieces:

$$\int_0^t(t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})+1}(1+t-s)^{-(\frac{1}{q_1}-\frac{1}{p})}\left\|\partial_x^\alpha K_\nu(\frac{t-s}{2})*\left[\nabla\cdot N(a(s),\vec{\omega}(s))\right]\right\|_{\mathring{L}^{\tilde{q}_1}(n)}ds$$
$$\le\left(\int_0^{t/2}+\int_{t/2}^t\right)(t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})+\frac{1}{2}}(1+t-s)^{-(\frac{1}{q_1}-\frac{1}{p})}\left\|\partial_x^\alpha K_\nu(\frac{t-s}{2})*\left[\nabla\cdot N(a(s),\vec{\omega}(s))\right]\right\|_{\mathring{L}^{\tilde{q}_1}(n)}ds$$
$$=: I_1+I_2$$

We can then make use of (42) in each to bound the nonlinear term. For I_1 we as usual pull the divergence off of the nonlinearity, and for $0<n<1$ we use (43) to choose $p_1=p_3=3/2$, whereas for $1<n<2$ we use (44) to choose $p_1=p_3=3/2$ and we find

$$I_1\le C\|(a,\vec{\omega})\|^2_{X_{n,k}}t^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}-\frac{|\alpha|}{2}}(1+t)^{-(1-\frac{1}{p})+\frac{1}{3}-\lfloor n\rfloor_1+n}$$

which holds for $0<t<\infty$, $1\le p\le\infty$. Then we use (44) to choose $p_1=2$ and $p_3=6/5$ for $1\le n\le 3/2$ and $p_1=6/5$ and $p_3=2$ for $3/2<n\le 2$ and we obtain

$$I_1\le C\|(a,\vec{\omega})\|^2_{X_{n,k}}t^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1}{2}-\frac{|\alpha|}{2}}(1+t)^{-(1-\frac{1}{p})-\frac{7}{12}+n}$$

for $0 < t < \infty$, $1 \leq p \leq \infty$. Similarly for I_2 we use (43) to choose $p_1 = p_3 = 3/2$ for $0 < n < 1$ and we use (44) to choose $p_1 = p_3 = 3/2$ for $1 < n < 2$ and we obtain

$$I_2 \leq C\|(a,\vec{\omega})\|^2_{X_{n,k}} t^{-\frac{3}{2}(1-\frac{1}{p})+\frac{3}{2}-\frac{|\alpha|}{2}}(1+t)^{-(1-\frac{1}{p})-\frac{2}{3}-\lfloor n\rfloor_1+n}$$

for $1 \leq p \leq 2$ and $0 < t < \infty$. Next we use (44) to choose $p_1 = 2$ and $p_3 = 6/5$ for $1 \leq n \leq 3/2$ and $p_1 = 6/5$ and $p_3 = 2$ for $3/2 < n \leq 2$ and we find

$$I_2 \leq C\|(a,\vec{\omega})\|^2_{X_{n,k}} t^{-\frac{3}{2}(1-\frac{1}{p})+\frac{5}{4}-\frac{|\alpha|}{2}}(1+t)^{-(1-\frac{1}{p})-\frac{17}{12}+n}$$

which holds for $0 < t < \infty$ and $1 \leq p < \infty$. For $p = \infty$ we can set $q_1 = 6$ by choosing $p_1 = 8$ and $p_3 = 8/3$ for $0 < n < 1$ using (43), choosing $p_1 = 8/3$ and $p_3 = 8$ for $1 \leq n < 2$ using (43) and $p_1 = 8$ and $p_3 = 8/3$ for $15/8 < n \leq 2$ using (44) to obtain

$$I_2 \leq C\|(a,\vec{\omega})\|^2_{X_{n,k}} t^{-\frac{|\alpha|}{2}}(1+t)^{-\frac{7}{6}-\lfloor n\rfloor_1+n}$$

We can then match the linear decay rate for $p = \infty$, $|\alpha| < k$ by keeping the derivatives on the nonlinearity and using $\beta = \alpha + e_j$ in (42) and choosing $p_1 = p_3 = 12/5$ for $0 < n < 1$ using (43), choosing $p_1 = 2$ and $p_3 = 3$ for $1 \leq n < 2$ using (43) and $p_1 = 24/11$ and $p_3 = 8/3$ for $15/8 < n \leq 2$ using (44). Continuity for $t > 0$ is proven as before. □

4. Asymptotic approximations for the modified compressible Navier-Stokes

With these solutions in hand, we turn to the task of approximating these solutions efficiently and accurately, especially in the regime $t \to \infty$. If $u(t) = (\rho(t), a(t), \vec{\omega}(t))^T$ is the solution belonging to $Y_{n,k} \times X_{n,k}$ given by Theorem 2 with initial condition $(\rho_0, a_0, \vec{\omega}_0)^T$, $a_0, \vec{\omega}_0$ with zero total mass, then we can write

$$u(t) = u_L(t) + u_N(t) \tag{51}$$

where $u_L(t)$ is the linear evolution defined in (23), (26), and $u_N(t) = u(t) - u_L(t)$. We saw in Prop. 2.8, 2.9 that for initial conditions u_0 belonging to $L^1(n)$ spaces, we can write

$$u_L(t) = u_H(t) + u_{LR}(t) \tag{52}$$

where the Hermite profiles $u_H(t)$ are defined as

$$\begin{pmatrix}\rho_H(x,t)\\ a_H(x,t)\end{pmatrix} = \sum_{\substack{i\leq 2\\ |\alpha|\leq\lfloor n\rfloor}} \langle H_\alpha \vec{e}_i, \begin{pmatrix}\rho_0\\ a_0\end{pmatrix}\rangle\, \partial_x^\alpha \begin{pmatrix}\rho_i(x,t)\\ a_i(x,t)\end{pmatrix} \quad , \ \vec{\omega}_H(x,t) = \sum_{\substack{j\leq 2\\ |\tilde{\alpha}|\leq\lfloor n\rfloor+1}} \langle \vec{p}_{\tilde{\alpha},j}, \vec{\omega}_0\rangle \mathbb{K}_\epsilon(t) * \vec{f}_{\tilde{\alpha},j}(x) \tag{53}$$

where ρ_i, a_i are defined in (24), and where $\vec{p}_{\tilde{\alpha},j}$, $\vec{f}_{\tilde{\alpha},j}$ are defined in Table 1. We obtained the temporal behavior of $u_{LR}(t)$ in Prop 2.8,2.9. In the above existence analysis, we saw that $u_N(t)$ decays faster than $u_L(t)$ in some, but not necessarily all, L^p norms, hence we need to study $u_N(t)$ more closely. We note that $u_N(t)$ can be written as

$$u_N(t) = -\int_0^t e^{\mathcal{L}(t-s)}\mathcal{Q}\big(u(s),u(s)\big)ds$$

so inspired by (51), (52), we define the Hermite-Picard profiles $u_{HP}(t)$ and nonlinear remainder $u_{NR}(t)$:

$$\begin{aligned} u_{HP}(t) &:= -\int_0^t e^{\mathcal{L}(t-s)}\mathcal{Q}\big(u_H(s),u_H(s)\big)ds \\ u_{NR}(t) &:= u_N(t) - u_{HP}(t) \end{aligned} \tag{54}$$

where $u_I(t) = (\rho_I(t), a_I(t), \vec{\omega}_I(t))^T$, $I = L, HP, NR$. We have already obtained upper bounds on the temporal behavior of $u_H(t)$ in Appendix E and $u_{LR}(t)$ in Prop 2.8 and 2.9. In what follows, we will obtain upper bounds for $u_{HP}(t)$ and $u_{NR}(t)$, as well as lower bounds for $u_H(t)$. Our main focus in this section will be to obtain these bounds, and we will discuss the relative decay rates and the implications for understanding the long-time asymptotics of solutions in section 5. Our goal is to emphasize the role that the localization of the initial conditions (and consequently, the localization of the solutions) plays in determining the nature of the asymptotics.

4.1. **Temporal behavior of the Hermite and Hermite-Picard profiles.** We can use the substitution $\tilde{x} = \frac{x}{\sqrt{1+\epsilon t}}$ together with the explicit form of the Hermite profiles $\vec{\omega}_H(t)$ in Table 1 and the explicit form of $B\vec{\omega}_H(t)$ to show that their temporal behavior is given by

$$\|\partial_x^\alpha \mathbb{K}_\epsilon(t) * \vec{f}_{\tilde{\alpha},j}\|_{\mathring{L}^p(\mu)} = C_\alpha(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{1-|\tilde{\alpha}|-|\alpha|}{2}+\frac{\mu}{2}}$$
$$\|\partial_x^\alpha B\mathbb{K}_\epsilon(t) * \vec{f}_{\tilde{\alpha},j}\|_{\mathring{L}^p(\mu)} = \tilde{C}_\alpha(1+t)^{-\frac{3}{2}(1-\frac{1}{p})+\frac{2-|\tilde{\alpha}|-|\alpha|}{2}+\frac{\mu}{2}}$$

The temporal behavior of the Hermite profiles $\rho_H(t), a_H(t)$ are given in the following proposition. These results follow from explicit calculations of the norms involved, as well as the fact that Π commutes with the heat-wave operator, and we leave the proof to the reader. Note that while these estimates might also hold for higher derivatives, we only require derivatives up to the order shown.

Proposition 4.1. *There exist functions $C_{l,\alpha}(t)$, $l = 0, 1, 2$ and constants $m, M \in \mathbb{R}$ such that $0 < m < C_{l,\alpha}(t) < M < \infty$ for all $t > 0$ such that*

$$\|\partial_t^l w(t) * K_\nu(t) * \partial_x^\alpha \phi_0\|_{\mathring{L}^p(\mu)} = C_{l,\alpha}(t)(1+t)^{-\frac{5}{2}(1-\frac{1}{p})+1-\frac{l+|\alpha|}{2}+\mu}$$

for $|\alpha| \leq 2$, $l = 0, 1, 2$, $\mu \in \mathbb{R}_{\geq 0}$ and $1 \leq p \leq \infty$. Furthermore we have

$$\|\Pi\partial_t^l w(t) * K_\nu(t) * \partial_x^\alpha \phi_0\|_{\mathring{\mathbb{L}}^p(\mu)} \leq C(1+t)^{-\frac{5}{2}(1-\frac{1}{p})+\frac{3-l-|\alpha|}{2}+\mu}$$

for any $\alpha \in \mathbb{N}^3$, $l = 1, 2$, $\mu \in \mathbb{R}_{\geq 0}$ and $1 \leq p \leq \infty$, except the case when $(\alpha, l) = (0, 1)$ and $1 \leq p \leq \frac{3}{2+\mu}$.

This implies that the linear Hermite profiles have temporal behavior given by

$$\|\partial_x^\alpha \rho_H(t)\|_{\mathring{L}^p(\mu)} = \tilde{C}_{1,\alpha}(t)E_n(1+t)^{-\frac{5}{2}(1-\frac{1}{p})+\frac{1-|\alpha|}{2}+\mu} \tag{55}$$

$$\|\partial_x^\alpha a_H(t)\|_{\mathring{L}^p(\mu)} = \tilde{C}_{2,\alpha}(t)E_n(1+t)^{-\frac{5}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}+\mu} \quad , \quad \|\partial_x^\alpha \Pi a_H(t)\|_{\mathring{L}^p(\mu)} = \hat{C}_{2,\alpha}(t)E_n(1+t)^{-\frac{5}{2}(1-\frac{1}{p})+\frac{1-|\alpha|}{2}+\mu}$$

$$\|\partial_x^\alpha \vec{\omega}_H(t)\|_{\mathring{\mathbb{L}}^p(\mu)} = \tilde{C}_{3,\alpha}(t)E_n(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1+|\alpha|}{2}+\frac{\mu}{2}} \quad , \quad \|\partial_x^\alpha B\vec{\omega}_H(t)\|_{\mathring{\mathbb{L}}^p(\mu)} = \hat{C}_{3,\alpha}(t)E_n(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}+\frac{\mu}{2}}$$

where E_n is as in (33), $|\alpha| \leq 1$ and $\tilde{C}_{l,\alpha}(t)$, $\hat{C}_{l,\alpha}(t)$, $l = 1, 2, 3$ are independent of $(\rho_0, a_0, \vec{\omega}_0)^T$ and are such that there exist constants $m, M \in \mathbb{R}$ such that $0 < m < \tilde{C}_{l,\alpha}(t), \hat{C}_{l,\alpha}(t) < M < \infty$ for all $t > 0$. We also have the following bounds on the Hermite-Picard profiles:

Proposition 4.2. *There exists a constant C such that we have*

$$\|\rho_{HP}(t)\|_{\mathring{L}^p(\mu)} \leq C(1+t)^{-\frac{5}{2}(1-\frac{1}{p})+\frac{1}{2}+\mu-\frac{1}{2}} \quad , \quad \|a_{HP}(t)\|_{\mathring{L}^p(\mu)} \leq C(1+t)^{-\frac{5}{2}(1-\frac{1}{p})+\mu-\frac{1}{2}}$$
$$\|\vec{\omega}_{HP}(t)\|_{\mathring{\mathbb{L}}^p(\mu)} \leq C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})-\frac{1}{2}+\mu-\frac{1}{2}}$$

for all $t > 0$, $|\alpha| \leq 2$, $0 \leq \mu \leq 2$ and $1 \leq p \leq \infty$.

Proof. We start with the Hermite-Picard profile $a_{HP}(t)$. We look at the weighted norms for an arbitrary weight μ. We first split the convolution:

$$\begin{aligned}
&\int_0^t \left\|\partial_t w(t-s) * K_\nu(t-s) * \left[\nabla \cdot N(a_H(s), \vec{\omega}_H(s))\right]\right\|_{\mathring{L}^p(\mu)} ds \\
&\leq \int_0^t \left\|\partial_t w(t-s) * K_\nu(\frac{t-s}{2})\right\|_{\mathring{L}^{\tilde{q}}(\mu)} \left\|K_\nu(\frac{t-s}{2}) * \left[\nabla \cdot N(a_H(s), \vec{\omega}_H(s))\right]\right\|_{L^{\tilde{q}_1}} ds \\
&\quad + \int_0^t \left\|\partial_t w(t-s) * K_\nu(\frac{t-s}{2})\right\|_{L^q} \left\|K_\nu(\frac{t-s}{2}) * \left[\nabla \cdot N(a_H(s), \vec{\omega}_H(s))\right]\right\|_{\mathring{L}^{q_1}(\mu)} ds
\end{aligned}$$

We'll bound the second term, and then as in the existence proof the bounds on the first term follow by repeating the estimates for the second term line by line after using the weighted estimate on the heat-wave operator in Prop 2.5 and taking $\mu = 0$ on the nonlinear term. We first split the second integral into two:

$$I_1 + I_2 := \left(\int_0^{t/2} + \int_{t/2}^t\right) \left\|\partial_t w(t-s) * K_\nu(\frac{t-s}{2})\right\|_{L^q} \left\|K_\nu(\frac{t-s}{2}) * \left[\nabla \cdot N(a_H(s), \vec{\omega}_H(s))\right]\right\|_{\mathring{L}^{q_1}(\mu)} ds$$

For $t < 1$ we can choose $q = 1$ in both terms, and since our heat estimate and equation (55) can be used to show the resulting integrand is bounded, these remain bounded as $t \to 0$. Hence we need only consider $t > 1$. For I_1 we can use the heat estimate to remove both of the derivatives from the nonlinearity, set $q_1 = 1$, use Cauchy-Schwarz and make use of (55) to bound the norms of $\vec{m}_H$ via

$$
\begin{aligned}
I_1 &\leq C \int_0^{\frac{t}{2}} (t-s)^{-\frac{3}{2}(1-\frac{1}{p})} (1+t-s)^{\frac{1}{2}-(1-\frac{1}{p})} \left\| K_\nu(\frac{t-s}{2}) * \left[\nabla \cdot N(a_H(s), \vec{\omega}_H(s)) \right] \right\|_{\mathring{L}^1(\mu)} ds \\
&\leq C \max_{ij} \int_0^{\frac{t}{2}} (t-s)^{-\frac{3}{2}(1-\frac{1}{p})-1} (1+t-s)^{\frac{1}{2}-(1-\frac{1}{p})} \|m_{H,i}(s)\|_{L^2} \|m_{H,j}(s)\|_{L^2(\mu)} ds \\
&\leq C E_n^2 (1+t)^{-\frac{5}{2}(1-\frac{1}{p})-\frac{1}{2}+\mu}
\end{aligned}
$$

For I_2 we use the heat estimate but keep all of the derivatives on the nonlinearity and we obtain

$$
\begin{aligned}
I_2 \leq & C \max_{ij} \int_{t/2}^t (t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})} (1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})} \left\| \partial_{x_i} \partial_{x_j} \left(m_{H,i}(s) m_{H,j}(s) \right) \right\|_{L^{q_1}(\mu)} ds \\
\leq & C \max_{ijk\ell} \int_{t/2}^t (t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})} (1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})} \Big(\left\| \partial_{x_i} \partial_{x_j} m_{H,i} \right\|_{L^{p_1}} \left\| m_{H,j} \right\|_{L^{p_2}(\mu)} + \left\| \partial_{x_i} m_{H,j} \right\|_{L^{p_1}} \left\| \partial_{x_k} m_{H,\ell} \right\|_{L^{p_2}(\mu)} \Big) ds \\
= & J_1 + J_2
\end{aligned}
$$

For J_1 we use Cor 2.2 part **(a)** to obtain

$$
\left\| \partial_{x_i} \partial_{x_j} m_{H,i} \right\|_{L^{p_1}} \left\| m_{H,j} \right\|_{L^{p_2}(\mu)} \leq C \max_j \left(\left\| \partial_{x_j} a_H \right\|_{L^{p_1}} + \left\| \partial_{x_j} \vec{\omega}_H \right\|_{\mathbb{L}^{p_1}} \right) \left\| m_{H,j} \right\|_{L^{p_2}(\mu)}
$$

so for $1 \leq p \leq 2$ we can set $q_1 = 1$ by choosing $p_1 = p_2 = 2$ and use (55) to obtain

$$
J_1 \leq C E_n^2 t^{-\frac{3}{2}(1-\frac{1}{p})+1} (1+t)^{\frac{1}{2}-(1-\frac{1}{p})-\frac{5}{2}+\mu}
$$

whereas for $p = \infty$ we can let $q_1 = 3/(2-\delta)$ by setting $p_1 = p_2 = 6/(2-\delta)$, where $0 < \delta < 1/5$ is any number and we obtain the following

$$
J_1 \leq C E_n^2 (1+t)^{-\frac{1}{2}(4+\delta)-1+\mu}
$$

For J_2 we can just use (55) directly, and by choosing $p_1 = p_2 = 2$ for $1 \leq p \leq 2$ we obtain

$$
J_2 \leq C \left\| (a, \vec{\omega}) \right\|_{X_n}^2 t^{-\frac{3}{2}(1-\frac{1}{p})+1} (1+t)^{-2-(1-\frac{1}{p})+\mu}
$$

and we can obtain the analogous results for $p = \infty$ by choosing $q_1 = 3/(2-\delta)$ by setting $p_1 = p_2 = 6/(2-\delta)$ for some $0 < \delta < 1/5$.

The bounds for the Hermite-Picard profiles $\rho_{HP}(t)$ and $\vec{\omega}_{HP}(t)$ can be obtained by similar arguments, and are omitted for brevity. However, these calculations are carried out in full form in [8]. □

4.2. **Temporal behavior of the linear and nonlinear remainders.** If one naively uses the estimates in Cor. 2.2 to obtain an asymptotic bound for Πa, then one obtains

$$
\|\Pi a(t)\|_{L^p} \leq C t^{-\frac{5}{2}(1-\frac{1}{p})+\frac{5}{6}}
$$

for $t > 1$ and $3/2 < p < \infty$, hence for these norms the asymptotic bounds for $\Pi a(t)$ differ from those of $a(t)$ by a factor of $t^{5/6}$. This implies that $\Pi a(t)$ might decay more slowly than $\rho(t)$. However, we saw in Prop 4.1 that the asymptotic bounds on $\Pi a_H(t)$ differ from those of $a_H(t)$ by a factor $t^{1/2}$, hence these terms have the same asymptotic bounds as $\rho(t)$. We now prove that the same holds for remainder a_{LR}:

Proposition 4.3. *Let $n \in [0,2]$ and let (ρ_0, a_0) belong to $W^{1,p}(n) \times L^p(n)$ for all $1 \leq p \leq \frac{3}{2}$. Then for $a_{LR}(t)$ defined as in Prop. 2.8, we have*

$$
\|\partial_x^\alpha \Pi a_{LR}(t)\|_{\mathring{L}^p(\mu)} \leq C E_n t^{-\frac{5}{2}(1-\frac{1}{p})+\frac{1}{2}+\mu-\frac{n}{2}-\frac{|\alpha|}{2}}
$$

for $t > 1$, $0 \leq \mu \leq n$ and any nonzero $\alpha \in \mathbb{N}^3$, $1 \leq p \leq \infty$. If $\alpha = 0$ the above estimate holds for $p > \frac{3}{2+\mu}$. On the other hand for $t < 1$, $0 \leq \mu \leq n$ and for $3/(2-\mu) < p < \infty$ if $n < 1$, or $\max(3/2, 3/(3-\mu)) < p < \infty$ if $n \geq 1$ we have

$$
\|\partial_x^\alpha \Pi a_{LR}(t)\|_{\mathring{L}^p(\mu)} \leq C E_n t^{-r_{\alpha,\tilde{p}}}
$$

where $p^{-1} = \tilde{p}^{-1} - 3^{-1}$.

Proof. The estimate for $t < 1$ follows from Cor. 2.2 parts **(c)**, **(c)**, and from interpolation in the case when $n \geq 1$ and $\mu < 1$. For $t > 1$ the interesting case is when $|\alpha| > 0$, and we have

$$\|\partial_x^\alpha \Pi a_{LR}(t)\|_{\mathring{L}^p(\mu)} \leq \|\Pi \partial_t w(t) * \partial_x^\alpha K_\nu(t) * a_{LR}(0)\|_{\mathring{L}^p(\mu)} + \|\Pi \partial_t^2 w(t) * \partial_x^\alpha K_\nu(t) * \rho_{LR}(0)\|_{\mathring{L}^p(\mu)}$$

so if $\alpha = e_i + \beta$ for some i, β, we can use Young's inequality to obtain

$$\begin{aligned}\|\Pi \partial_t w(t) * \partial_x^\alpha K_\nu(t) * a_{LR}(0)\|_{\mathring{L}^p(\mu)} &= \|\pi * \partial_t w(t) * \partial_{x_i} K_\nu(\frac{t}{2}) * \partial_x^\beta K_\nu(\frac{t}{2}) * a_{LR}(0)\|_{\mathring{L}^p(\mu)} \\ &\leq \|\Pi \partial_t w(t) * \partial_{x_i} K_\nu(\frac{t}{2})\|_{\mathring{L}^p(\mu)} \|\partial_x^\beta K_\nu(\frac{t}{2}) * a_{LR}(0)\|_{L^1} + \|\Pi \partial_t w(t) * \partial_{x_i} K_\nu(\frac{t}{2})\|_{L^p} \|\partial_x^\beta K_\nu(\frac{t}{2}) * a_{LR}(0)\|_{\mathring{L}^1(\mu)}\end{aligned}$$

where

$$\pi(x) = -\frac{1}{4\pi} \frac{x}{|x|^3}$$

is the integral kernel of the Π operator. The result then follows from our estimates of the Π operator acting on the Hermite term in Prop 4.1, since the same result applies to the heat-wave operator. However, for $\alpha = 0$ the heat-wave operator only belongs to L^p for $p > 3/2$. We leave the remainder of the proof to the reader. □

In the following lemma, we collect the bounds for $u_N(t)$ obtained during the contraction mapping argument in the existence proof and sharpen one of them. For this purpose we define the rate $b_{n,p}$ to measure the excess decay of $u_N(t)$ above the linear rate as follows, using interpolation for $2 < p < \infty$:

$$b_{n,p} = \begin{cases} \min(\frac{1}{6} + \frac{\lfloor n \rfloor_1}{2}, \frac{3}{10} + \frac{\lfloor n \rfloor_1}{10}) & \text{for } 1 \leq p \leq 2 \\ \min(\frac{\lfloor n \rfloor_1}{2}, \frac{3}{10} + \frac{\lfloor n \rfloor_1}{10}) & \text{for } p = \infty \\ (b_{n,\infty} - b_{n,2})(1 - \frac{2}{p}) + b_{n,2} & \text{for } 2 < p < \infty \end{cases} \tag{56}$$

Lemma 4.4. *Let* $n \in [0, 2]$, $k \geq 1$ *and let* $u_0 = (\rho_0, a_0, \vec{\omega}_0)^T \in \cap_{1 \leq p \leq \frac{3}{2}} W^{1,p}(n) \times L^p(n) \times \mathbb{L}^p_\sigma(n)$. *If* $u(t) = (\rho(t), a(t), \vec{\omega}(t))^T$ *is the solution in* $Y_{n,k} \times X_{n,k}$ *given by Theorem 2 and Corollary 3.1 with initial condition* u_0, *then the nonlinear term* $u_N(t)$ *in (51) satisfies*

$$\begin{aligned}\|\partial_x^\alpha \rho_N(t)\|_{\mathring{L}^p(\mu)} &\leq C E_n^2 t^{-r_{\alpha,p}} (1+t)^{-\ell_{n,p,\mu} + \frac{1}{2} - b_{n,p}} \\ \|\partial_x^\alpha a_N(t)\|_{\mathring{L}^p(\mu)} &\leq C E_n^2 t^{-r_{\alpha,p}} (1+t)^{-\ell_{n,p,\mu} - b_{n,p}} \\ \|\partial_x^\alpha \vec{\omega}_N(t)\|_{\mathring{\mathbb{L}}^p(\mu)} &\leq C E_n^2 t^{-r_{\alpha,p}} (1+t)^{-\tilde{\ell}_{n,p,\mu} - b_{n,p}}\end{aligned}$$

for $1 \leq p \leq \infty$, $0 \leq \mu \leq n$ *and* $|\alpha| < k$.

Proof. The estimates for $t < 1$ are the same as those obtained in the existence proof, hence we need only consider $t > 1$. By inspecting the estimates in the existence proof, we see that all of the bounds obtained already exhibit the extra decay listed in the first argument of the minimum in (56), with one important exception. The estimate of the unweighted norm of I_1 in (36) stops improving relative to the linear rate for $n > 1/3$. The $|\alpha| = k$ derivative also may decay slower, but we don't estimate this here.

So we need only improve on the bound in (36) for $n > 1/3$. We can split I_1 into two pieces:

$$\begin{aligned} I_1 &= \left(\int_0^{t^{3/5}} + \int_{t^{3/5}}^{t/2} \right) (t-s)^{-\frac{3}{2}(\frac{1}{q_1} - \frac{1}{p})} (1+t-s)^{\frac{1}{2} - (\frac{1}{q_1} - \frac{1}{p})} \left\| \partial_x^\alpha K_\nu(\frac{t-s}{2}) * \left[\nabla \cdot N(a(s), \vec{\omega}(s)) \right] \right\|_{L^{q_1}} ds \\ &=: J_1 + J_2 \end{aligned} \tag{57}$$

Since we are interested in the limit $t \to \infty$ we assume $t/2 > t^{3/5}$ here, but for $1 < t^{2/5} \leq 2$ we can obtain the analogous result. For J_1 we make a modified estimate by taking all of the derivatives off of the nonlinearity and onto the heat-wave propagator by using our heat estimate. We can then set $q_1 = 1$, use Cauchy-Schwarz and use Cor 2.2 part **(b)** to obtain

$$\begin{aligned} J_1 &\leq \int_0^{t^{3/5}} (t-s)^{-\frac{3}{2}(\frac{1}{q_1} - \frac{1}{p}) - 1 - \frac{|\alpha|}{2}} (1+t-s)^{\frac{1}{2} - (\frac{1}{q_1} - \frac{1}{p})} \|m_i(s) m_j(s)\|_{L^{q_1}} ds \\ &\leq C E_n^2 t^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2} - \frac{|\alpha|}{2}} \int_0^{t^{3/5}} \left(\|a(s)\|_{L^{6/5}} + \|\vec{\omega}(s)\|_{\mathbb{L}^{6/5}} \right)^2 ds \leq C E_n^2 t^{-\frac{5}{2}(1 - \frac{1}{p}) - \frac{1}{2} - \frac{|\alpha|}{2} + \frac{7}{10} - \frac{3\lfloor n \rfloor_1}{5}} \end{aligned}$$

For J_2, we can use the same estimate as before. Taking the divergence and ∂_x^α off of the nonlinearity by using our heat estimate, setting $q_1 = 1$, using Hölder's inequality and Cor 2.2 parts **(a)** and **(c)** we obtain the following for $n > 1/3$:

$$J_2 \leq \int_{t^{3/5}}^{t/2} (t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})-\frac{1}{2}-\frac{|\alpha|}{2}}(1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\|\partial_{x_i}(m_i)m_k\|_{L^{q_1}}ds$$
$$\leq CE_n^2 t^{-\frac{5}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}\int_{t^{3/5}}^{t/2}(1+s)^{-\frac{2}{3}-\lfloor n\rfloor_1}ds \leq CE_n^2 t^{-\frac{5}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}+\frac{1}{5}-\frac{3\lfloor n\rfloor_1}{5}}$$

This same improved bound can be obtained for $\rho_N(t)$ and $\vec{\omega}_N(t)$ as well. □

We now use the estimates just proven, together with a bootstrapping argument, to obtain more refined estimates of the temporal decay of the nonlinear remainder. For this purpose we define the rate $\tilde{b}_{n,p}$ to measure the excess decay of $u_{NR}(t)$ above the linear rate via

$$\tilde{b}_{n,p} = \begin{cases} \frac{1-\lfloor n\rfloor_1}{2} + \min(2n-\frac{1}{3}, n, \frac{1}{2}) & \text{for } 1 \leq p \leq 2 \\ \frac{1-\lfloor n\rfloor_1}{2} + \min(n-\frac{1}{2}, \frac{1}{2}) & \text{for } p = \infty \\ (\tilde{b}_{n,\infty}-\tilde{b}_{n,2})(1-\frac{2}{p}) + \tilde{b}_{n,2} & \text{for } 2 < p < \infty \end{cases} \tag{58}$$

Theorem 3. *Let $n \in [0,2]$, $k \geq 1$ and let $u_0 = (\rho_0, a_0, \vec{\omega}_0)^T \in \cap_{1\leq p\leq \frac{3}{2}} W^{1,p}(n) \times L^p(n) \times \mathbb{L}^p_\sigma(n)$. If $u(t) = (\rho(t), a(t), \vec{\omega}(t))^T$ is the solution in $Y_{n,k} \times X_{n,k}$ given by Theorem 2 and Corollary 3.1 with initial condition u_0, then the nonlinear remainder $u_{NR}(t)$ in (54) satisfies*

$$\|\partial_x^\alpha \rho_{NR}(t)\|_{\mathring{L}^p(\mu)} \leq CE_n^2(1+E_n^2)t^{-r_{\alpha,p}}(1+t)^{-\ell_{n,p,\mu}+\frac{1}{2}-\tilde{b}_{n,p}}$$
$$\|\partial_x^\alpha a_{NR}(t)\|_{\mathring{L}^p(\mu)} \leq CE_n^2(1+E_n^2)t^{-r_{\alpha,p}}(1+t)^{-\ell_{n,p,\mu}-\tilde{b}_{n,p}}$$
$$\|\partial_x^\alpha \vec{\omega}_{NR}(t)\|_{\mathring{\mathbb{L}}^p(\mu)} \leq CE_n^2(1+E_n^2)t^{-r_{\alpha,p}}(1+t)^{-\tilde{\ell}_{n,p,\mu}-\tilde{b}_{n,p}}$$

for $1 \leq p \leq \infty$, $0 \leq \mu \leq n$ and $|\alpha| \leq \min(1, k-1)$.

Proof. Again the estimates for $t < 1$ are identical to those in the existence proof, so we only consider $t > 1$. By definition we see that the nonlinear remainder u_{NR} must satisfy the following equation:

$$u_{NR}(t) = -\int_0^t e^{\mathcal{L}(t-s)}\Big[\mathcal{Q}\big(u_H, u_{LR}+u_N\big) + \mathcal{Q}\big(u_{LR}+u_N, u_H\big) + \mathcal{Q}\big(u_{LR}+u_N, u_{LR}+u_N\big)\Big]ds$$

We start by looking at the Duhamel term corresponding to a_{NR}. By expanding the nonlinearity, we see that for an arbitrary weight $0 \leq \mu \leq n$ we need to bound the norms of terms of the form

$$\int_0^t \Big\|\partial_t w(t-s) * \partial_x^\alpha K_\nu(t-s) * \Big[\partial_{x_i}\partial_{x_j}\big[m_{I,i}(s)m_{J,j}(s)\big]\Big]\Big\|_{\mathring{L}^p(\mu)}ds$$
$$\leq \int_0^t \Big\|\partial_t w(t-s) * K_\nu(\frac{t-s}{2})\Big\|_{\mathring{L}^q(\mu)}\Big\|\partial_x^\alpha K_\nu(\frac{t-s}{2}) * \Big[\partial_{x_i}\partial_{x_j}\big[m_{I,i}(s)m_{J,j}(s)\big]\Big]\Big\|_{L^{q_1}}ds$$
$$+\int_0^t \Big\|\partial_t w(t-s) * K_\nu(\frac{t-s}{2})\Big\|_{L^q}\Big\|\partial_x^\alpha K_\nu(\frac{t-s}{2}) * \Big[\partial_{x_i}\partial_{x_j}\big[m_{I,i}(s)m_{J,j}(s)\big]\Big]\Big\|_{\mathring{L}^{q_1}(\mu)}ds$$

for pairs of indices $(I,J) = (H,LR), (H,N), (LR,LR), (LR,N)$ and (N,N). We will bound the second term, and the bounds for the first can then be obtained by repeating the same analysis by using the weighted bounds in Prop. 2.5 as described previously. We split the second term into two:

$$\int_0^t \Big\|\partial_t w(t-s) * K_\nu(\frac{t-s}{2})\Big\|_{L^q}\Big\|\partial_x^\alpha K_\nu(\frac{t-s}{2}) * \Big[\partial_{x_i}\partial_{x_j}\big[m_{I,i}(s)m_{J,j}(s)\big]\Big]\Big\|_{\mathring{L}^{q_1}(\mu)}ds$$
$$\leq \left(\int_0^{t/2} + \int_{t/2}^t\right)\Big\|\partial_t w(t-s) * K_\nu(\frac{t-s}{2})\Big\|_{L^q}\Big\|\partial_x^\alpha K_\nu(\frac{t-s}{2}) * \Big[\partial_{x_i}\partial_{x_j}\big[m_{I,i}(s)m_{J,j}(s)\big]\Big]\Big\|_{\mathring{L}^{q_1}(\mu)}ds = I_1^{IJ} + I_2^{IJ}$$

Bounds for I_1^{IJ} and I_2^{IJ} can be obtained for $(I,J)=(H,LR),(H,N),(LR,LR)$, and (LR,N) using very similar arguments. We bound these first, then bound (N,N) later. For I_1^{IJ} we use the heat estimate to take all of the derivatives off of the nonlinear term, and then use Hölder's inequality as follows:

$$I_1^{IJ} \le \int_0^{t/2} (t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})-1-\frac{|\alpha|}{2}}(1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\|m_{I,i}(s)\|_{\mathring{L}^{p_1}(\mu)}\|m_{J,j}(s)\|_{L^{p_2}}ds \tag{59}$$

For $(I,J)=(H,LR),(LR,LR)$ we choose $p_1=p_2=2$ for $1\le p\le\infty$ and use our estimates in (55) and our estimates of the linear remainder in Prop 4.3, and we obtain

$$I_1^{H,LR} \le CE_n^2 t^{-\frac{5}{2}(1-\frac{1}{p})-\frac{1}{2}-\frac{|\alpha|}{2}+\mu}L_{n,0}(t) \quad\text{and}\quad I_1^{LR,LR} \le CE_n^2 t^{-\frac{5}{2}(1-\frac{1}{p})-\frac{1}{2}-\frac{|\alpha|}{2}+\mu+\max(\frac{1}{2}-n,0)}L_{n,1/2}(t)$$

For $(I,J)=(H,N),(LR,N)$ we use Cor. 2.2 **(b)** and pull the first factors out of the integral to obtain

$$I_1^{IJ} \le t^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})-1-\frac{|\alpha|}{2}}(1+t)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\int_0^{t/2}\|m_{I,i}(s)\|_{\mathring{L}^{p_1}(\mu)}\big(\|a_J(s)\|_{L^{p_3}}+\|\vec{\omega}_J(s)\|_{\mathbb{L}^{p_3}}\big)ds$$

and we then set $p_1=p_3=3/2$ for $1\le p\le\infty$ and use our estimates in (55) and in Prop 4.3 together with the estimate of the nonlinear term in Lemma 4.4. We find

$$I_1^{H,N} \le CE_n^3 t^{-\frac{5}{2}(1-\frac{1}{p})-\frac{1}{2}-\frac{|\alpha|}{2}+\mu+\max(\frac{1}{6}-n,0)}L_{n,1/6}(t)$$
$$I_1^{LR,N} \le CE_n^3 t^{-\frac{5}{2}(1-\frac{1}{p})-\frac{1}{2}-\frac{|\alpha|}{2}+\mu+\max(\frac{5}{6}-n-b_{n,3/2},0)}L_{n,16/33}(t)$$

For I_2^{IJ} we leave all the derivatives on the nonlinear term and obtain

$$I_2^{IJ} \le \int_{t/2}^{t}(t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})}(1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\big\|\partial_x^\alpha\partial_{x_i}\partial_{x_j}\big[m_{I,i}(s)m_{J,j}(s)\big]\big\|_{\mathring{L}^{q_1}(\mu)}ds \tag{60}$$

Using Liebniz's rule and Hölder's inequality we have

$$\big\|\partial_x^{\alpha+e_i+e_j}\big[m_{I,i}(s)m_{J,j}(s)\big]\big\|_{\mathring{L}^{q_1}(\mu)} \le \sum\nolimits_{\gamma_1+\gamma_2=\alpha+e_i+e_j}\|\partial_x^{\gamma_1}m_{I,i}\|_{\mathring{L}^{p_1}(\mu)}\|\partial_x^{\gamma_2}m_{J,j}\|_{L^{p_2}} \tag{61}$$

For $(I,J)=(H,LR),(LR,LR)$ we choose $p_1=p_2=2$ for $1\le p\le 2$ and make use of our estimates in Prop 4.3 and 2.9. Here we obtain

$$I_2^{H,LR} \le CE_n^2 t^{-\frac{5}{2}(1-\frac{1}{p})-\frac{1+n}{2}-\frac{|\alpha|}{2}+\mu} \quad\text{and}\quad I_2^{LR,LR} \le CE_n^2 t^{-\frac{5}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}+\mu-n}$$

whereas for $p=\infty$ we choose $p_1=p_2=4$ and use Prop 4.3 and 2.9 to obtain

$$I_2^{H,LR} \le CE_n^2 t^{-\frac{5}{2}-\frac{n}{2}-\frac{|\alpha|}{2}+\mu} \quad\text{and}\quad I_2^{LR,LR} \le CE_n^2 t^{-\frac{5}{2}-\frac{|\alpha|}{2}+\mu+\frac{1}{2}-n}$$

On the other hand, for $(I,J)=(H,N),(LR,N)$ we use Cor. 2.2 part **(b)** on $m_{N,j}$ when $\gamma_2=0$ and choose $p_1=p_3=3/2$ for $1\le p\le 2$ to obtain

$$I_2^{H,N} \le CE_n^3 t^{-\frac{5}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}+\mu-\frac{1}{6}-\frac{\lfloor n\rfloor_1}{2}-b_{n,3/2}} \quad\text{and}\quad I_2^{LR,N} \le CE_n^3 t^{-\frac{5}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}+\mu+\frac{1}{3}-\frac{n+\lfloor n\rfloor_1}{2}-b_{n,3/2}}$$

whereas we choose $p_1=3,p_3=2$ for $p=\infty$ to obtain

$$I_2^{H,N} \le CE_n^3 t^{-\frac{5}{2}-\frac{|\alpha|}{2}+\mu-\frac{\lfloor n\rfloor_1}{2}-b_{n,2}} \quad\text{and}\quad I_2^{LR,N} \le CE_n^3 t^{-\frac{5}{2}-\frac{|\alpha|}{2}+\mu+\frac{1}{2}-\frac{n+\lfloor n\rfloor_1}{2}-b_{n,2}}$$

If $\gamma_2\neq 0$ then we use Cor. 2.2 part **(a)** on $m_{N,j}$ and choose $p_1=p_2=2$ for $1\le p\le 2$ to obtain

$$I_2^{H,N} \le CE_n^3 t^{-\frac{5}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}+\mu-\frac{1}{2}-\frac{\lfloor n\rfloor_1}{2}-b_{n,2}} \quad\text{and}\quad I_2^{LR,N} \le CE_n^3 t^{-\frac{5}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}+\mu-\frac{n+\lfloor n\rfloor_1}{2}-b_{n,2}}$$

whereas we choose $p_1=\infty,p_2=2$ for $p=\infty$ to obtain

$$I_2^{H,N} \le CE_n^3 t^{-\frac{5}{2}-\frac{|\alpha|}{2}+\mu-\frac{\lfloor n\rfloor_1}{2}-b_{n,2}} \quad\text{and}\quad I_2^{LR,N} \le CE_n^3 t^{-\frac{5}{2}-\frac{|\alpha|}{2}+\mu+\frac{1}{2}-\frac{n+\lfloor n\rfloor_1}{2}-b_{n,2}}$$

We also need to bound the norms of the terms for which $(I,J)=(N,N)$. For this we will need to bound $\mu=0$ and $\mu=n$ separately, and the remaining bounds follow from interpolation. Starting with $\mu=0$ we first bound I_1^{NN} by removing all derivatives from the nonlinearity using the heat estimate and use Hölder's inequality as in (59), but we then use Cor. 2.2 **(b)** on both terms to obtain

$$I_1^{NN} \le \int_0^{t/2}(t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})-1-\frac{|\alpha|}{2}}(1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\big(\|a_N\|_{L^{p_3}}+\|\vec{\omega}_N\|_{\mathbb{L}^{p_3}}\big)\big(\|a_N\|_{L^{p_4}}+\|\vec{\omega}_N\|_{\mathbb{L}^{p_4}}\big)ds$$

We can then choose $p_3 = p_4 = 6/5$ for $1 \le p \le \infty$ to obtain

$$I_1^{NN} \le CE_n^4 t^{-\frac{5}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}+\mu-\frac{1}{2}+\max(\frac{7}{6}-\lfloor n\rfloor_1-2b_{n,6/5},0)} L_{n,\frac{17}{36}}(t)$$

On the other hand for I_2^{NN} we leave all of the derivatives on the nonlinearity and use Liebniz and Hölder as in (60), (61). Without loss of generality, we assume $|\gamma_1| \ge |\gamma_2|$, and that for some $\tilde{k}$, $\gamma_1 = \tilde{\gamma}_1 + e_k$. We then use Cor. 2.2 **(a)** on the first term and Cor. 2.2 **(b)** on the second to obtain

$$I_2^{NN} \le \int_{t/2}^{t} (t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})}(1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\big(\|\partial_x^{\tilde{\gamma}_1} a_N\|_{L^{p_1}} + \|\partial_x^{\tilde{\gamma}_1}\vec{\omega}_N\|_{\mathbb{L}^{p_1}}\big)\big(\|\partial_x^{\gamma_2} a_N\|_{L^{p_3}} + \|\partial_x^{\gamma_2}\vec{\omega}_N\|_{\mathbb{L}^{p_3}}\big)ds$$

For $1 \le p \le 2$ we choose $p_1 = p_3 = 3/2$ to obtain

$$I_2^{NN} \le CE_n^4 t^{-\frac{5}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}+\mu+\frac{1}{3}-\lfloor n\rfloor_1-2b_{n,3/2}}$$

whereas for $p = \infty$ we choose $p_1 = p_3 = 12/5$ to obtain

$$I_2^{NN} \le CE_n^4 t^{-\frac{5}{2}-\frac{|\alpha|}{2}+\mu+\frac{1}{2}-\lfloor n\rfloor_1-2b_{n,12/5}}$$

Finally we need to consider the weighted norms when $\mu = n$. For I_1^{NN} we remove all derivatives from the nonlinearity using the heat estimate, but we need to split the weight between the two terms. For $0 < n < 1$ we split the weight evenly between the two terms and we can then apply Cor. 2.2 **(b)** to both terms and pull out the first factors from the integral via

$$\begin{aligned}I_1^{NN} &\le \int_0^{t/2} (t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})-1-\frac{|\alpha|}{2}}(1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\|m_{N,i}(s)\|_{\mathring{L}^{p_1}(\frac{n}{2})}\|m_{N,j}(s)\|_{\mathring{L}^{p_2}(\frac{n}{2})}ds \\ &\le t^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})-1-\frac{|\alpha|}{2}}(1+t)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\int_0^{t/2}\big(\|a_N\|_{L^{p_3}(\frac{n}{2})} + \|\vec{\omega}_N\|_{\mathbb{L}^{p_3}(\frac{n}{2})}\big)\big(\|a_N\|_{L^{p_4}(\frac{n}{2})} + \|\vec{\omega}_N\|_{\mathbb{L}^{p_4}(\frac{n}{2})}\big)ds\end{aligned}$$

whereas for $1 \le n < 2$ we split the weight unevenly between the two terms and apply Cor. 2.2 **(b)** to the term with less weight and Cor. 2.2 **(c)** to the term with more weight to obtain

$$\begin{aligned}I_1^{NN} &\le \int_0^{t/2} (t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})-1-\frac{|\alpha|}{2}}(1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\|m_{N,i}(s)\|_{\mathring{L}^{p_1}(1+\frac{n-1}{2})}\|m_{N,j}(s)\|_{\mathring{L}^{p_2}(\frac{n-1}{2})}ds \\ &\le t^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})-1-\frac{|\alpha|}{2}}(1+t)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\int_0^{t/2}\big(\|a_N\|_{L^{p_3}(1+\frac{n-1}{2})} + \|\vec{\omega}_N\|_{\mathbb{L}^{p_3}(1+\frac{n-1}{2})}\big)\big(\|a_N\|_{L^{p_4}(\frac{n-1}{2})} + \|\vec{\omega}_N\|_{\mathbb{L}^{p_4}(\frac{n-1}{2})}\big)ds\end{aligned}$$

In both cases the choice of $p_3 = p_4 = 6/5$ satisfies the constraints imposed by the use of Cor. 2.2 **(b)**, **(c)**, so for $1 \le p \le \infty$ we obtain

$$I_1^{NN} \le CE_n^4 t^{-\frac{5}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}+n-\frac{1}{2}+\max(\frac{7}{6}-\lfloor n\rfloor_1-2b_{n,6/5},0)} L_{n,\frac{17}{36}}(t)$$

For $1 < n \le 2$ we can obtain a different bound, and note that in the overlapping region $1 < n < 2$ we can use the better of the two estimates. We split the weight unevenly in a different way and apply Cor. 2.2 **(b)**, **(c)** to the terms with respectively less and more weight to obtain

$$\begin{aligned}I_1^{NN} &\le \int_0^{t/2} (t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})-1-\frac{|\alpha|}{2}}(1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\|m_{N,i}(s)\|_{\mathring{L}^{p_1}(1+\frac{n-1}{3})}\|m_{N,j}(s)\|_{\mathring{L}^{p_2}(\frac{2(n-1)}{3})}ds \\ &\le t^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})-1-\frac{|\alpha|}{2}}(1+t)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\int_0^{t/2}\big(\|a_N\|_{L^{p_3}(1+\frac{n-1}{3})} + \|\vec{\omega}_N\|_{\mathbb{L}^{p_3}(1+\frac{n-1}{3})}\big)\big(\|a_N\|_{L^{p_4}(\frac{2(n-1)}{3})} + \|\vec{\omega}_N\|_{\mathbb{L}^{p_4}(\frac{2(n-1)}{3})}\big)ds\end{aligned}$$

In this case the choice of $p_3 = 15/13$, $p_4 = 5/4$ satisfies the constraints imposed by the use of Cor. 2.2 **(b)**, **(c)**, so for $1 \le p \le \infty$ we have

$$I_1^{NN} \le CE_n^4 t^{-\frac{5}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}+n-\frac{1}{2}}$$

For I_2^{NN} we leave all derivatives on the nonlinearity as in (60), use Liebniz and Hölder as in (61) and put the weight on the term having fewer derivatives to obtain

$$I_2^{NN} \le \int_{t/2}^{t} (t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})}(1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\|\partial_x^{\gamma_1} m_{N,i}\|_{L^{p_1}}\|\partial_x^{\gamma_2} m_{N,j}\|_{\mathring{L}^{p_2}(n)}ds$$

where without loss of generality we assume $|\gamma_1| \geq |\gamma_2|$. We can use Cor. 2.2 **(a)** on the first term, and either Cor. 2.2 **(b)** or **(c)** on the second term, depending on n. In either case one obtains

$$I_2^{NN} \leq \int_{t/2}^{t} (t-s)^{-\frac{3}{2}(\frac{1}{q_1}-\frac{1}{p})}(1+t-s)^{\frac{1}{2}-(\frac{1}{q_1}-\frac{1}{p})}\left(\|\partial_x^{\gamma_1-e_{\tilde{k}}}a_N\|_{L^{p_1}}+\|\partial_x^{\gamma_1-e_{\tilde{k}}}\vec{\omega}_N\|_{\mathbb{L}^{p_1}}\right)\left(\|\partial_x^{\gamma_2}a_N\|_{L^{p_3}(n)}+\|\partial_x^{\gamma_2}\vec{\omega}_N\|_{\mathbb{L}^{p_3}(n)}\right)ds$$

for some index $\tilde{k}$. For $0 < n < 1$ we use Cor. 2.2 **(b)** and choose $p_1 = p_3 = 3/2$ for $1 \leq p \leq 2$ to obtain

$$I_2^{NN} \leq CE_n^4 t^{-\frac{5}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}+n+\frac{1}{3}-\lfloor n \rfloor_1-2b_{n,3/2}}$$

whereas we can obtain the exact same bound for $1 < n < 2$ using Cor. 2.2 **(c)** with $p_1 = p_3 = 3/2$. We can also obtain the same bound for $1 \leq n < 3/2$ using Cor. 2.2 **(c)** with $p_1 = 2$, $p_3 = 6/5$ for $1 \leq p \leq 2$, and also for $3/2 < n \leq 2$ using Cor. 2.2 **(c)** with $p_1 = 6/5$, $p_3 = 2$. Finally, for $p = \infty$ and $0 < n < 7/4$ we can use Cor. 2.2 **(b)** by choosing $p_1 = p_3 = 12/5$ and we obtain

$$I_2^{NN} \leq CE_n^4 t^{-\frac{5}{2}-\frac{|\alpha|}{2}+n+\frac{1}{2}-\lfloor n \rfloor_1-2b_{n,12/5}}$$

and for $7/4 < n \leq 2$ we can obtain the same bound by using Cor. 2.2 **(c)**. For $3/2 < n \leq 2$ we can use Cor. 2.2 **(c)** with $p_1 = 3$, $p_3 = 2$ and we obtain

$$I_2^{NN} \leq CE_n^4 t^{-\frac{5}{2}-\frac{|\alpha|}{2}+n+\frac{1}{2}-\lfloor n \rfloor_1-b_{n,3}-b_{n,2}}$$

The excess decay rate $\tilde{b}_{n,p}$ can therefore be found by collecting these results and finding the slowest decay, and the bounds for the terms ρ_{NR} and $\vec{\omega}_{NR}$ can be obtained similarly. These calculations are carried out in full form in [8]. □

5. Conclusions

We can now discuss the implications of the results from the previous section for the asymptotic approximation theory of the modified compressible Navier-Stokes system. It is desirable to make approximations which are efficient, in the sense that they are easily evaluated, and it is also desirable that the approximations are accurate, in the sense that the error is small relative to the size of the approximation. For concreteness, let us describe the results for $\rho(t)$. We have decomposed $\rho(t)$ into $\rho_L(t)$ and $\rho_N(t)$ in (51), used the Hermite expansion to decompose $\rho_L(t)$ in $\rho_H(t)$ and $\rho_{LR}(t)$ in (52), and we have decomposed $\rho_N(t)$ into $\rho_{HP}(t)$ and $\rho_{NR}(t)$ in (54). We list these terms in order of efficiency, which we will define qualitatively as the computational complexity required to evaluate each term at time $t > 0$, as follows:

- The term $\rho_H(t)$ can be evaluated directly from the formulas in Appendix D, once the moments of the initial conditions ρ_0, a_0 are calculated.
- The term $\rho_{LR}(t)$, as well as $\rho_L(t)$, must be evaluated by computing a convolution of the initial conditions ρ_0, a_0 with the heat-wave kernels.
- The term $\rho_{HP}(t)$, can also be evaluated by computing a convolution with the heat-wave kernels, and then integrating this convolution up to time $t > 0$. In principle, an explicit formula for the function $\rho_{HP}(t)$ could be obtained, but it would take further analysis to determine its form.
- The term $\rho_{NR}(t)$, as well as $\rho_N(t)$, require knowledge of the true solution at all times $0 < s < t$. With this on hand, these terms can then be evaluated by computing a convolution with the heat-wave kernels, and then integrating this convolution up to time $t > 0$.

On the other hand by collecting the results from Prop. 2.8, Prop. 2.9, equation (55), Lemma 4.4 and Theorem 3, we have the following for all $t > 1$, $1 \leq p \leq \infty$, $n \in [0, 2]$, $0 \leq \mu \leq n$:

$$\|\rho_H(t)\|_{\mathring{L}^p(\mu)} \leq t^{-\frac{5}{2}(1-\frac{1}{p})+\mu+\frac{1}{2}}$$

$$\|\rho_{LR}(t)\|_{\mathring{L}^p(\mu)} \leq Ct^{-\frac{5}{2}(1-\frac{1}{p})+\mu+1-\frac{n}{2}}$$

$$\|\rho_{HP}(t)\|_{\mathring{L}^p(\mu)} \leq Ct^{-\frac{5}{2}(1-\frac{1}{p})+\mu}$$

Recall that the bounds for $\rho_{NR}(t)$ were obtained separately for $1 \leq p \leq 2$ and for $p = \infty$, and for $2 < p < \infty$ the bounds were obtained by interpolation. For all $t > 1$, $n \in [0, 2]$, $0 \leq \mu \leq n$, we have:

$$\|\rho_{NR}(t)\|_{\mathring{L}^p(\mu)} \leq C \begin{cases} t^{-\frac{5}{2}(1-\frac{1}{p})+\mu+\frac{1}{2}-\min(2n-\frac{1}{3},n,\frac{1}{2})} & \text{for } 1 \leq p \leq 2 \\ t^{-\frac{5}{2}+\mu+\frac{1}{2}-\min(n-\frac{1}{2},\frac{1}{2})} & \text{for } p = \infty \end{cases}$$

Note that our explicit bounds show that the bounds on ρ_H are sharp. While we have not obtained lower bounds on ρ_{HP}, our analysis suggests that these estimates are sharp as well. The bounds on ρ_{LR} depend on the properties of ρ_0 and a_0, but in general our example in Remark 2.4 indicates that these bounds are saturated as well. Finally, it is unknown to us whether the bound for ρ_{NR} is saturated.

For $0 \leq n \leq 2$, there are two relevant choices of approximations for $\rho(t)$ that one could make: $\rho_{app}(t) = \rho_L(t)$ and $\rho_{app}(t) = \rho_H(t)$. Comparing the estimates above for the various values of n, we can summarize how the localization affects the relative asymptotic behavior of the various terms ρ_I, and consequently determine the optimal choice of asymptotic approximation as follows:

- First, we see that for all $n > 0$ and all $1 \leq p \leq \infty$, $\rho_N(t)$ decays more quickly than $\rho_L(t)$, although our findings indicate that we need to take $n \geq 2/9$ to achieve the $t^{-1/2}$ extra decay of $\rho_N(t)$ above the rate of $\rho_L(t)$ for $1 \leq p \leq 2$, and we need to take $n \geq 1$ to achieve the $t^{-1/2}$ extra decay for $p = \infty$.
- For $0 \leq n < 1$ $\rho_{LR}(t)$ in general can decay more slowly than $\rho_H(t)$, hence we need to take $\rho_{app}(t) = \rho_L(t)$ to capture the leading order behavior for $\rho(t)$.
- For $n > 1$ we need only evaluate the explicit functions $\rho_H(t)$ to obtain the leading order behavior.
- For $1 < n < 2$ the next order of behavior is given by $\rho_{LR}(t)$, while $\rho_{HP}(t)$ and $\rho_{NR}(t)$ decay faster still. Hence we could either use $\rho_{app}(t) = \rho_L(t)$ or $\rho_{app}(t) = \rho_H(t)$.
  - In the first case, the error decays $t^{-1/2}$ faster than $\rho_H(t)$, hence this is a more accurate, but less efficient, approximation.
  - In the second case the error decays $t^{-(n-1)/2}$ faster than $\rho_H(t)$, hence this is a more efficient, but less accurate, approximation.
- Finally, for $n = 2$ there is no loss in accuracy by taking $\rho_{app}(t) = \rho_H(t)$.
- The Hermite-Picard term $\rho_{HP}(t)$ decays more quickly than $\rho_{LR}(t)$ for $n > 2$, hence in this regime we would either take $\rho_{app}(t) = \rho_L(t) + \rho_{HP}(t)$ or $\rho_{app}(t) = \rho_H(t) + \rho_{HP}(t)$. However, we do not consider $n > 2$ in the present paper for reasons discussed below.

Precisely the same statements can be made regarding the asymptotic approximation of $a(t)$ and $\vec{\omega}(t)$.

In order to contextualize these findings, let us compare the results obtained here for the modified compressible Navier-Stokes system to those obtained by Hoff and Zumbrun for the compressible Navier-Stokes system. First, note that our results are specified in terms of ρ, a, and $\vec{\omega}$, whereas Hoff and Zumbrun's results are specified in terms of ρ and $\vec{m}$. As remarked, we make use of a and $\vec{\omega}$ in order to avoid the delocalization effect of Brandolese, but in addition this has the benefit of allowing us to consider initial conditions with less restrictive smoothness and localization requirements, since for instance an initial divergence a_0 in L^1 can correspond to a momentum field $\vec{m}_0$ which is not in L^1. Furthermore, as noted prior to Prop. 4.3, if we naively use the estimates in Cor. 2.2 to obtain the decay rate of $\Pi a(t)$, we obtain a bound which is in all likelihood not sharp. While we work around this for a_H in Prop. 4.1 and a_{LR} in Prop. 4.3, it is more complicated to work around for a_N since this involves nonlinear estimates. We defer this obstacle to future work. Therefore it is more natural to compare our results on a and $\vec{\omega}$ to Hoff and Zumbrun's results on the derivatives of $\vec{m}$. We see that in the parameter regime $0 \leq n < 1$ the more lenient localization requirements allow our solutions to decay more slowly, and that our nonlinear term can in general decay less quickly relative to our linear term. Furthermore, while Hoff and Zumbrun show that their linear approximation $u_{app}(t) = u_L(t)$ is sufficient to obtain $t^{-1/2}$ extra decay relative to the linear rate, our analysis shows that it is in fact necessary in this regime. In the regime $n \geq 1$ we obtain the same decay rates as Hoff and Zumbrun.

Comparing our results with those of Kagei and Okita also presents several points of interest. We see from (6) that the next order term in the Hermite expansion appears in the expansion of Kagei and Okita. Also, for all values of n the error made by our best asymptotic approximation achieves at most $t^{-1/2}$ extra decay relative to the linear rate, while the error of Kagei and Okita's approximation achieves $t^{-3/4}$ extra decay. The key difference between their approximation and ours is given by the last term in (6). This term contains an integral which requires knowledge of the solution for all time $0 \leq s < \infty$, hence this approximation cannot be made a priori.

Our analysis suggests that it is necessary to include terms which cannot be computed a priori in order to achieve additional accuracy beyond the $t^{-1/2}$ extra decay achieved by Hoff and Zumbrun. The reason turns out to be visible from the analysis in Lemma 4.4. Specifically, note that for $t > 1$ the term J_1 in (57) contains the term

$$\tilde{J}_1 = \int_0^1 (t-s)^{-\frac{3}{2}(1-\frac{1}{p})}(1+t-s)^{\frac{1}{2}-(1-\frac{1}{p})}\|K_\nu(t-s) * \big[\nabla \cdot N\big(a(s), \vec{\omega}(s)\big)\big]\|_{L^1} ds$$

Here, the only option available is to pull both derivatives off of the nonlinear term using the heat estimate, and one obtains

$$\tilde{J}_1 \leq t^{-\frac{3}{2}(1-\frac{1}{p})-1}(1+t)^{\frac{1}{2}-(1-\frac{1}{p})} \max_{i,j} \int_0^1 \|m_i(s) m_j(s)\|_{L^1} ds$$

However, now the integral no longer depends on t, so since we know that these estimates are sharp, it seems that this decay rate cannot be improved upon. If we include this term in our approximation, the same reasoning would then apply to the integral over $s \in [1, 2]$. Thus we must find a way to include some of the nonlinear term in our approximation. For instance, one could include all of the nonlinear terms present in J_1. However, from the form of J_1 in (57) this would mean that one would have to compute the true solution up to time $t^{3/5}$ in order to obtain an approximate solution at time t. In other words, the approximation could not be made a priori. While this would mean improved accuracy, it would come at an increased computational cost. However, our analysis strongly suggests that one could obtain approximations which are less computationally expensive to evaluate at time $t > 0$ than the true solution.

Finally, we discuss how our results will aid in obtaining higher order approximations for the compressible Navier Stokes equations. Our Hermite expansion allows us to extend the approximation made in (5) to arbitrary order. However, since the connection to the original compressible Navier-Stokes system is via the series of approximations (2), (4), (5), we stop our analysis of the modified compressible Navier-Stokes system at $n = 2$. In order to obtain higher order approximations for the original compressible Navier-Stokes system, it is necessary to improve both the approximation in (2) and in (4). We leave this to future work.

In the present paper, we study the effects of localization by working with solutions of the curl-divergence representation of the modified compressible Navier-Stokes system in weighted spaces, which has not previously been considered, and obtain several insights into how these improvements might be made. The weighted estimates obtained for the Π, B, heat and heat-wave operators can be used in the analysis of the original compressible Navier-Stokes system directly. The weighted estimates on the Π and B operators especially help to prepare for the investigation of the delocalization effect of Brandolese for solutions of the compressible Navier-Stokes. Furthermore, the analysis of the quadratic nonlinear term of the modified system sets up a framework to handle those of the original system, since one of the nonlinear terms is identical, several others are quadratic as well, and higher order nonlinear terms should decay more quickly. Finally, the nonlinear analysis suggests how one can achieve additional accuracy with approximation terms which are not computable a priori, while preserving a standard of efficiency.

Appendix A. Proof of the estimates on Π and B in Proposition 2.1

We begin the proof with the following Lemmas:

Lemma A.1. *For p_2, p_3 and n chosen as in Proposition 2.1 part **(b)** above, and given f, g such that*

$$f(x) = \int_{\mathbb{R}^3} \frac{g(y)}{|x-y|^2} dy$$

we have

$$\|f\|_{L^{p_2}(n)} \leq C \|g\|_{L^{p_3}(n)}$$

Proof. The proof is based on a dyadic decomposition

$$\mathbb{R}^3 = \cup_{j=0}^{\infty} A_j$$

where $A_0 = \{x \in \mathbb{R}^3 : |x| \leq 1\}$ and $A_j = \{x \in \mathbb{R}^3 : 2^{j-1} < |x| < 2^j\}$ for $j \in \mathbb{N}$. Let $f_i = f\chi_{A_i}$ and $g_j = g\chi_{A_j}$. Clearly $f_i = \sum_{j \in \mathbb{N}} \Delta_{ij}$, where

$$\Delta_{ij}(x) = \chi_{A_i}(x) \int_{A_j} \frac{g_j(y)}{|x-y|^2} dy$$

For the case $|i-j|\le 1$ note that if

$$h_j(x)=\int_{A_j}\frac{g_j(y)}{|x-y|^2}dy$$

then by the Hardy-Littlewood-Sobolev inequality ([10] Theorem V.1), we have

$$\|\Delta_{ij}\|_{L^{p_2}}\le \|h_j(x)\|_{L^{p_2}}\le C\|g_j\|_{L^{p_3}}\le \tilde{C}2^{-\alpha|i-j|}\|g_j\|_{L^{p_3}}$$

for an $\alpha\in(0,1)$ of our choosing. Next, we consider the case $i\ge j+2$. By the triangle inequality

$$\|\Delta_{ij}\|_{L^{p_2}}\le\Big(\int\chi_{A_i}(x)\Big(\int\chi_{A_j}(y)\frac{|g_j(y)|}{|x-y|^2}dy\Big)^{p_2}dx\Big)^{1/p_2}$$

Since $i\ge j+2$ we have $|x-y|\ge 2^{i-2}$ and hence

$$\begin{aligned}\|\Delta_{ij}\|_{L^{p_2}} &\le \Big(\int\chi_{A_i}(x)\Big(\int\chi_{A_j}(y)\frac{|g_j(y)|}{|x-y|^2}dy\Big)^{p_2}dx\Big)^{1/p_2}\le\frac{16}{2^{2i}}\Big(\int\chi_{A_i}(x)\Big(\int\chi_{A_j}(y)|g_j(y)|dy\Big)^{p_2}dx\Big)^{1/p_2}\\ &= \frac{16}{2^{2i}}\Big(\int\chi_{A_i}(x)dx\Big)^{1/p_2}\int\chi_{A_j}(y)|g_j(y)|dy\le\frac{16}{2^{2i}}\Big(\int\chi_{A_i}(x)dx\Big)^{1/p_2}\Big(\int\chi_{A_j}(y)dy\Big)^{1-1/p_3}\|g_j\|_{L^{p_3}}\\ &= \frac{C}{2^{2i}}2^{3i/p_2}2^{3j(1-\frac{1}{p_3})}\|g_j\|_{L^{p_3}}=C2^{-3(1-\frac{1}{p_3})(i-j)}\|g_j\|_{L^{p_3}}\ ,\end{aligned}\tag{62}$$

where in the last step we used (16). By a very similar argument, if $j\ge i+2$ we have

$$\|\Delta_{ij}\|_{L^{p_2}}\le C2^{-3(\frac{1}{p_3}-\frac{1}{3})(j-i)}\|g_j\|_{L^{p_3}}\ .$$

Recalling the limits on the support of f_i and its decomposition in terms of $\Delta_{i,j}$ we have the inequality:

$$\begin{aligned}\|f_i\|_{L^{p_2}(n)} &\le C2^{ni}\|f_i\|_{L^{p_2}}\le C2^{ni}\sum_{j\in\mathbb{N}}2^{-|i-j|-3(\frac{2}{3}-\frac{1}{p_3})(i-j)}\|g_j\|_{L^{p_3}}\\ &\le C\sum_{j\in\mathbb{N}}2^{-|i-j|-3(\frac{2}{3}-\frac{1}{p_3}-\frac{n}{3})(i-j)}\|g_j\|_{L^{p_3}(n)}\le\sum_{j\in\mathbb{N}}C2^{-\alpha|i-j|}\|g_j\|_{L^{p_3}(n)}\ ,\end{aligned}\tag{63}$$

for some $\alpha>0$, since $-1<3(\frac{2-n}{3}-\frac{1}{p_3})<1$. Considering now f itself, we have

$$\|f\|^{p_2}_{L^{p_2}(n)}=\sum_i\|f_i\|^{p_2}_{L^{p_2}(n)}$$

and since

$$\sum_i\|f_i\|^{p_2}_{L^{p_2}(n)}\le\sum_i\Big(\sum_j C2^{-\alpha|i-j|}\|g_j\|_{L^{p_3}(n)}\Big)^{p_2}=\sum_i\Big(\sum_j C2^{-\alpha|i-j|(1-\frac{1}{p_3})}2^{-\alpha|i-j|\frac{1}{p_3}}\|g_j\|_{L^{p_3}(n)}\Big)^{p_2}\tag{64}$$

we can then apply Hölder's inequality and interchange the order of summation to obtain

$$\|f\|^{p_2}_{L^{p_2}(n)}\le C\sum_i\Big[\sum_j 2^{-\alpha|i-j|}\|g_j\|^{p_3}_{L^{p_3}(n)}\Big]^{\frac{p_2}{p_3}}\le C\Big[\sum_i\sum_j 2^{-\alpha|i-j|}\|g_j\|^{p_3}_{L^{p_3}(n)}\Big]^{\frac{p_2}{p_3}}\le C\|g\|^{p_2}_{L^{p_3}(n)}\tag{65}$$

where in the last step we compute the geometric sum and use convexity since $\frac{p_2}{p_3}=\frac{3}{3-p_3}>1$. □

Lemma A.2. *For* $1<p_3<p_2<\infty$ *and* $n\in[0,2)$ *chosen such that*

$$\frac{2-n}{3}<\frac{1}{p_3}<\frac{3-n}{3}$$

and given f,g *such that*

$$f(x)=\int_{\mathbb{R}^3}\frac{g(y)}{|x-y|}dy$$

we have

$$\|f\|_{L^{p_2}(n-1)}\le C\|g\|_{L^{p_3}(n)}$$

Proof. Defining f_i, g_j, Δ_{ij} and h_j analogously to the above much of the proof follows in almost identical fashion. The key difference arises from the fact that p_3 lies in a different range in this case. In the step analogous to (63), we have

$$\begin{aligned} \|f_i\|_{L^{p_2}(n-1)} &\leq C2^{(n-1)i}\|f_i\|_{L^{p_2}} \leq C2^{(n-1)i}\sum_{j\in\mathbb{N}} 2^{-\frac{1}{2}|i-j|-3(\frac{1}{2}-\frac{1}{p_3})(i-j)}\|g_j\|_{L^{p_3}(1)} \\ &\leq C\sum_{j\in\mathbb{N}} 2^{-\frac{1}{2}|i-j|-3(\frac{5}{6}-\frac{1}{p_3}-\frac{n}{3})(i-j)}\|g_j\|_{L^{p_3}(n)} \leq \sum_{j\in\mathbb{N}} C2^{-\alpha|i-j|}\|g_j\|_{L^{p_3}(n)}\ , \end{aligned}$$

for some $\alpha > 0$, since $-\frac{1}{2} < 3(\frac{5}{6}-\frac{1}{p_3}-\frac{n}{3}) < \frac{1}{2}$. The estimate in the lemma now follows by a summation similar to that in (64) and (65). $\square$

Proof of Proposition 2.1. The operators $\partial_{x_i}\Pi$ and $\partial_{x_i}B$ are singular integral operators formed by kernels of Calderon-Zygmund type, so part **(a)** follows from Theorem II.3 in [10]. Examining the form of the Π and B operators, we see that part **(b)** follows directly the result of the Lemma *A*.1.

For part **(c)**, we use our modified versions of $B.2$, $B.3$ to complete the analogous proof in [3]. Write

$$(\Pi a)_i = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\left(\frac{x_i-y_i}{|x-y|^3}-\frac{x_i}{|x|^3}\right)a(y)dy$$

using the moment zero condition. Using the identity

$$|x|^3(x_i-y_i)-|x-y|^3x_i = (x_i-y_i)|x|^2(|x|-|x-y|)+|x-y|(2x_i(x\cdot y)-y_i|x|^2-x_i|y|^2)$$

it follows that

$$\big||x|^3(x_i-y_i)-|x-y|^3x_i\big| \leq C|x-y||x||y|(|x|+|y|) \leq C(|x-y||x|^2|y|+|x-y|^2|x||y|)$$

and hence $|(\Pi a)_i| \leq C(u_1+u_2)$ where

$$u_1(x) = \frac{1}{|x|}\int\frac{|y||a(y)|}{|x-y|^2}dy \qquad , \qquad u_2(x) = \frac{1}{|x|^2}\int\frac{|y||a(y)|}{|x-y|}dy$$

Therefore, using Lemma *A*.1, *A*.2 with $f_1 = |x|u_1$, $f_2 = |x|^2u_2$ and $g_1 = g_2 = |y|\ |a(y)|$ we have

$$\begin{aligned} \|\Pi a\|_{L^{p_2}(n)} &\leq C\big\|\chi_{|\cdot|\leq 1}\Pi a\big\|_{L^{p_2}} + C\big\|\chi_{|\cdot|>1}|\cdot|^n\Pi a\big\|_{L^{p_2}} \\ &\leq C\big\|\Pi a\big\|_{L^{p_2}} + C\Big\|\chi_{|\cdot|>1}|\cdot|^n u_1\Big\|_{L^{p_2}} + C\Big\|\chi_{|\cdot|>1}|\cdot|^n u_2\Big\|_{L^{p_2}} \\ &\leq C\big\|\Pi a\big\|_{L^{p_2}(n-1)} + C\big\|f_1\big\|_{L^{p_2}(n-1)} + C\big\|f_2\big\|_{L^{p_2}(n-2)} \\ &\leq C\big\|a\big\|_{L^{p_3}(n-1)} + C\big\|g_1\big\|_{L^{p_3}(n-1)} + C\big\|g_2\big\|_{L^{p_3}(n-1)} \leq C\big\|a\big\|_{L^{p_3}(n)} \end{aligned}$$

The proof for $B\vec{\omega}$ is analogous.

$\square$

Appendix B. Proof of heat estimate in Proposition 2.3

Proof. We prove that

$$\|\partial_x^\alpha K_\nu(t)*f\|_{\mathring{L}^p(\mu)} \leq C(\nu t)^{-\frac{|\alpha|}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-\frac{n-\mu}{2}}\|f\|_{L^q(n)}$$

and the result then holds by estimating the $\mathring{L}^p(\mu)$ norms separately for $\nu t < 1$ and $\nu t \geq 1$ using different values for q. Write

$$\|\partial_x^\alpha K_\nu(x,t)*f\|_{\mathring{L}_x^p(\mu)} = \|\Big(\int_{|y|\geq\sqrt{\nu t}} + \int_{|y|<\sqrt{\nu t}}\Big)\partial_x^\alpha K_\nu(x-y,t)f(y)dy\|_{\mathring{L}_x^p(\mu)} \leq S_1+S_2+S_3$$

where

$$S_1 = \| \int_{\mathbb{R}^3} \partial_x^\alpha K_\nu(x-y,t) f(y) \chi_{|y|\geq\sqrt{\nu t}} dy \|_{\mathring{L}^p_x(\mu)}$$

$$S_2 = \sum_{|\beta|\leq\tilde{n}} \frac{1}{\beta!} \| \int_{\mathbb{R}^3} \partial_x^{\alpha+\beta} K_\nu(x,t) y^\beta f(y) \chi_{|y|<\sqrt{\nu t}} dy \|_{\mathring{L}^p_x(\mu)}$$

$$S_3 = \sum_{|\beta|=\tilde{n}+1} \frac{\tilde{n}+1}{\beta!} \Big\| \int_{|y|<\sqrt{\nu t}} y^\beta f(y) \int_0^1 (1-s)^{\tilde{n}} \partial_x^{\alpha+\beta} K_\nu(x-sy,t) ds dy \Big\|_{\mathring{L}^p_x(\mu)}$$

and where we used Taylor's theorem

$$\partial_x^\alpha K_\nu(x-y,t) = \sum_{|\beta|\leq\tilde{n}} (-1)^{|\beta|} \frac{\partial_x^{\alpha+\beta} K_\nu(x,t)}{\beta!} y^\beta + \sum_{|\beta|=\tilde{n}+1} (-1)^{\tilde{n}+1} \frac{\tilde{n}+1}{\beta!} y^\beta \int_0^1 (1-s)^{\tilde{n}} \partial_x^{\alpha+\beta} K_\nu(x-sy,t) ds$$

For S_1, we change variables and use $|\tilde{x}| \leq |\tilde{x}-\tilde{y}| + |\tilde{y}|$:

$$\| \int_{\mathbb{R}^3} \partial_x^\alpha K_\nu(x-y,t) f(y) \chi_{|y|\geq\sqrt{\nu t}} dy \|_{\mathring{L}^p_x(\mu)} \leq (\nu t)^{\frac{\mu}{2}-\frac{|\alpha|}{2}+\frac{3}{2p}} \Big[\| \int_{\mathbb{R}^3} |\tilde{x}-\tilde{y}|^\mu \partial_{\tilde{x}}^\alpha K_\nu(\tilde{x}-\tilde{y}) f(\sqrt{\nu t}\tilde{y}) \chi_{|\tilde{y}|\geq 1} d\tilde{y} \|_{\mathring{L}^p_{\tilde{x}}} + \| \int_{\mathbb{R}^3} |\tilde{y}|^\mu \partial_{\tilde{x}}^\alpha K_\nu(\tilde{x}-\tilde{y}) f(\sqrt{\nu t}\tilde{y}) \chi_{|\tilde{y}|\geq 1} d\tilde{y} \|_{\mathring{L}^p_{\tilde{x}}} \Big]$$

We can then use Young's inequality and change back to our original variables:

$$\begin{aligned} \| \int_{\mathbb{R}^3} \partial_x^\alpha K_\nu(x-y,t) f(y) \chi_{|y|\geq\sqrt{\nu t}} dy \|_{\mathring{L}^p_x(\mu)} &\leq (\nu t)^{\frac{\mu}{2}-\frac{|\alpha|}{2}+\frac{3}{2p}} \Big[\| \partial_{\tilde{x}}^\alpha K_\nu(\tilde{x}) \|_{\mathring{L}_{\tilde{x}}^{\frac{pq}{pq+q-p}}(\mu)} \| f(\sqrt{\nu t}\tilde{y}) \chi_{|\tilde{y}|\geq 1} \|_{L^q_{\tilde{y}}} \\ &\quad + \| \partial_{\tilde{x}}^\alpha K_\nu(\tilde{x}) \|_{L_{\tilde{x}}^{\frac{pq}{pq+q-p}}} \| |\tilde{y}|^\mu f(\sqrt{\nu t}\tilde{y}) \chi_{|\tilde{y}|\geq 1} \|_{L^q_{\tilde{y}}} \Big] \\ &\leq C(\nu t)^{\frac{\mu-n}{2}-\frac{|\alpha|}{2}+\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \| f(y) \|_{\mathring{L}^q_y(n)} \end{aligned}$$

For S_2, we can factor out the y dependent terms from the L^p_x norm

$$\| \int_{\mathbb{R}^3} \partial_x^{\alpha+\beta} K_\nu(x,t) y^\beta f(y) \chi_{|y|<\sqrt{\nu t}} dy \|_{\mathring{L}^p_x(\mu)} \leq (\nu t)^{-\frac{|\alpha|+|\beta|}{2}+\frac{\mu}{2}+\frac{3}{2}(\frac{1}{p}-1)} \| \partial_{\tilde{x}}^{\alpha+\beta} K_\nu(\tilde{x}) \|_{L^p_{\tilde{x}}(\mu)} \Big| \int_{|y|<\sqrt{\nu t}} y^\beta f(y) dy \Big|$$

Since $|\beta| \leq \tilde{n} < n - 3(1-\frac{1}{q})$ we use the zero moment property to obtain

$$\begin{aligned} \Big| \int_{|y|<\sqrt{\nu t}} y^\beta f(y) dy \Big| &= \Big| \int_{|y|\geq\sqrt{\nu t}} \frac{|y|^{n-|\beta|}}{|y|^{n-|\beta|}} |y|^{|\beta|} |f(y)| dy \Big| \\ &\leq \| |y|^{-n+|\beta|} \chi_{|y|\geq\sqrt{\nu t}} \|_{L_y^{\frac{q}{q-1}}} \| f \|_{\mathring{L}^q(n)} \leq C(\nu t)^{-\frac{n}{2}+\frac{|\beta|}{2}-\frac{3}{2}(\frac{1}{q}-1)} \| f \|_{\mathring{L}^q(n)} \end{aligned}$$

For S_3, write

$$\begin{aligned} \| \int_{|y|<\sqrt{\nu t}} \Phi_\beta y^\beta f(y) dy \|_{\mathring{L}^p_x(\mu)} &= \Big\| |x|^\mu \int_{|y|<\sqrt{\nu t}} y^\beta f(y) \Big[\int_0^1 (1-s)^{\tilde{n}} \partial_x^{\alpha+\beta} K_\nu(x-sy,t) ds \Big] dy \Big\|_{L^p_{\tilde{x}}} \\ &= (\nu t)^{\frac{\mu}{2}-\frac{|\alpha|}{2}+\frac{3}{2p}} \Big\| |\tilde{x}|^\mu \int_{|\tilde{y}|<1} \tilde{y}^\beta f(\sqrt{\nu t}\tilde{y}) \Big[\int_0^1 (1-s)^{\tilde{n}} \partial_{\tilde{x}}^{\alpha+\beta} K_1(\tilde{x}-s\tilde{y}) ds \Big] d\tilde{y} \Big\|_{L^p_{\tilde{x}}} \\ &\leq (\nu t)^{-\frac{|\alpha|}{2}+\frac{\mu}{2}+\frac{3}{2p}} \Big\| |\tilde{x}|^\mu \int_{|\tilde{y}|<1} |\tilde{y}|^{\tilde{n}+1} |f(\sqrt{\nu t}\tilde{y})| \Big[\int_0^1 \Big| \partial_{\tilde{x}}^{\alpha+\beta} K_1(\tilde{x}-s\tilde{y}) \Big| ds \Big] d\tilde{y} \Big\|_{L^p_{\tilde{x}}} \end{aligned}$$

Now using the fact that $s \leq 1, |\tilde{y}| \leq 1$ we have

$$\Big| \partial_{\tilde{x}}^{\alpha+\beta} K_1(\tilde{x}-s\tilde{y}) \Big| = \Big| \sum_{j=0}^{\tilde{n}+1+|\alpha|} c_j (\tilde{x}_j - s\tilde{y}_j)^j \exp[-\frac{|\tilde{x}-s\tilde{y}|^2}{4}] \Big| \leq C(1+|\tilde{x}|)^{\tilde{n}+1+|\alpha|} \exp[-\frac{|\tilde{x}-s\tilde{y}|^2}{4}]$$

and

$$\exp\Big[-\frac{|\tilde{x}-s\tilde{y}|^2}{4}\Big] = \exp\Big[-\frac{|\tilde{x}|^2}{8}\Big] \exp\Big[-\frac{|\tilde{x}|^2}{8} + \frac{s\tilde{x}\cdot\tilde{y}}{2} - \frac{s^2|\tilde{y}|^2}{4}\Big] \leq C \exp\Big[-\frac{|\tilde{x}|^2}{8}\Big]$$

If we let $\delta > 0$ be such that $n - 3(1-\frac{1}{q}) + \delta < \tilde{n}+1$, then we have

$$\|\int_{|y|<\sqrt{\nu t}} \Phi_\beta y^\beta f(y)dy\|_{\mathring{L}^p_x(\mu)} \le (\nu t)^{-\frac{|\alpha|}{2}+\frac{\mu}{2}+\frac{3}{2p}} \Big\|(1+|\tilde{x}|)^{\mu+\tilde{n}+1+|\alpha|} \int_{|\tilde{y}|<1} |\tilde{y}|^{\tilde{n}+1}|f(\sqrt{\nu t}\tilde{y})| \exp[-\frac{|\tilde{x}|^2}{8}]d\tilde{y}\Big\|_{L^p_x}$$

and since the integral in $\tilde{y}$ no longer depends on $\tilde{x}$ we have

$$\begin{aligned}\|\int_{|y|<\sqrt{\nu t}} \Phi_\beta y^\beta f(y)dy\|_{\mathring{L}^p_x(\mu)} &= (\nu t)^{-\frac{|\alpha|}{2}+\frac{\mu}{2}+\frac{3}{2p}} \Big\|(1+|\tilde{x}|)^{\mu+\tilde{n}+1+|\alpha|} \exp[-\frac{|\tilde{x}|^2}{8}]\Big\|_{L^p_x} \int_{|\tilde{y}|<1} |\tilde{y}|^{\tilde{n}+1}|f(\sqrt{\nu t}\tilde{y})|d\tilde{y} \\ &\le C(\nu t)^{-\frac{|\alpha|}{2}+\frac{\mu}{2}+\frac{3}{2p}} \int_{|\tilde{y}|<1} |\tilde{y}|^{n-d(1-\frac{1}{q})+\delta}|f(\sqrt{\nu t}\tilde{y})|d\tilde{y} \\ &\le C(\nu t)^{-\frac{|\alpha|}{2}-\frac{n-\mu}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\|f\|_{\mathring{L}^q(n)}\end{aligned}$$

□

Appendix C. Proof of the heat-wave estimates

C.1. **Proof of Proposition 2.5.** We first obtain point-wise estimates. Recalling the form of the Kirchhoff formula, we need a bound on the spherical integral of the Gaussian, so we begin with the following estimate:

Lemma C.1. *There exists a constant $C > 0$ depending only on c and ν such that*

$$\int_{|z|=1} \mathrm{e}^{\frac{-|x+ctz|^2}{\nu t}} dS(z) \le C\,(1+t)^{-1}\,\mathrm{e}^{-\frac{(|x|-ct)^2}{3\nu t}}.$$

Proof. We recall the proof given by [5]. First note that the integral above is rotationally invariant so that we may, without loss of generality, set $x = |x|e_1$. It then suffices to integrate over the set $\{z : |z| = 1,\ z_1 \le 0\}$, since the other part is smaller, and we will relabel z with $-z$ for convenience. Note that for such x and z,

$$\begin{aligned}3|x-ctz|^2 &\ge (|x|-ct)^2 + 2|x-ctz|^2 \\ &= (|x|-ct)^2 + 2(|x|^2 - 2|x|z_1ct + c^2t^2|z|^2) \\ &\ge (|x|-ct)^2 + c^2t^2 + 2|x|^2 - 2|x|ct + c^2t^2 \\ &= (|x|-ct)^2 + c^2t^2 + (\sqrt{2}|x|-ct)^2 + 2(\sqrt{2}-1)|x|ct \\ &\ge (|x|-ct)^2 + c^2t^2(1-z_1^2)\end{aligned}$$

This can then be used to obtain the estimate

$$\int_{|z|=1,z_1\ge 0} \mathrm{e}^{\frac{-||x|e_1-ctz|^2}{\nu t}} dS(z) \le \mathrm{e}^{\frac{-(|x|-ct)^2}{3\nu t}} \int_{|z|=1,z_1\ge 0} \mathrm{e}^{-\frac{c^2t(1-z_1^2)}{3\nu}} dS(z) = C\Big(\frac{ct}{\nu}\Big)^{-1}\mathrm{e}^{\frac{-(|x|-ct)^2}{3\nu t}}$$

by a simple calculation using the parameterization $z_1 = \sqrt{1-(z_2^2+z_3^2)}$ of the hemispherical integral.

We can remove the blow up as $t \to 0$ as follows. Note that for $|z| = 1$, we have

$$|x+ctz|^2 = |x|^2 + c^2t^2 - 2ctz_1|x| \ge \frac{|x|^2}{3} - c^2t^2\ ,$$

so

$$\int_{|z|=1} e^{-\frac{|x+ctz|^2}{\nu t}} dS(z) \le \int_{|z|=1} e^{-\frac{|x|^2}{3\nu t}}\, e^{\frac{c^2t}{\nu}} dS(z) \le Ce^{-\frac{|x|^2}{3\nu t}}\ .$$

□

Proof of Proposition 2.5. We first derive pointwise bounds for the Green's functions $w * K_{\nu t}, \partial_t w * K_{\nu t}$, and $\partial_t^2 w * K_{\nu t}$. Using (14) and the above lemmas, we find

$$\begin{aligned}|w * K_{\nu t}(x)| &\le \Big|b_0ct\int_{|z|=1} K_{\nu t}(x+ctz)dS(z)\Big| \\ &\le C(ct)^{1-\frac{3}{2}} \int_{|z|=1} e^{-\frac{|x+ctz|^2}{5\nu t}} dS(z) \\ &\le Ct^{-\frac{1}{2}}(1+t)^{-1}\mathrm{e}^{-\frac{(|x|-ct)^2}{15\nu t}}\end{aligned}$$

for some constant C. Using the analogous bounds we then find

$$|\partial_t w * K_{\nu t}(x)| \le t^{-\frac{3}{2}}(1+t)^{-\frac{1}{2}} \mathrm{e}^{-\frac{(|x|-ct)^2}{15\nu t}} \qquad \text{and} \qquad |\partial_t^2 w * K_{\nu t}(x)| \le Ct^{-2}(1+t)^{-\frac{1}{2}} \mathrm{e}^{-\frac{(|x|-ct)^2}{15\nu t}}$$

The desired $\mathring{L}^q(n)$ bounds then follow from an estimate of the $\mathring{L}^q(n)$ norm of the translating exponential:

$$\begin{aligned}
\|\mathrm{e}^{-\frac{(|\cdot|-ct)^2}{15\nu t}}\|^q_{\mathring{L}^q(n)} &= \int_{\mathbb{R}^d} (|x|^n)^q \mathrm{e}^{-p\frac{(|x|-ct)^2}{15\nu t}} dx = C\int_0^\infty (r^n)^q \mathrm{e}^{-q\frac{(r-ct)^2}{15\nu t}} r^2 dr \\
&= \int_0^\infty (t^{1/2}\tilde{r})^{nq+2} \mathrm{e}^{-\frac{q(\tilde{r}-ct^{1/2})^2}{15\nu}} t^{1/2} d\tilde{r}, \qquad\qquad r = \tilde{r}t^{1/2} \\
&= t^{(nq+3)/2}\int_0^\infty \tilde{r}^{nq+2}\mathrm{e}^{-\frac{q(\tilde{r}-ct^{1/2})^2}{15\nu}} d\tilde{r} = t^{(nq+3)/2}\int_{-t^{1/2}}^\infty (\rho+ct^{1/2})^{nq+2}\mathrm{e}^{-\frac{q\rho^2}{15\nu}}d\rho, \qquad \tilde{r} = \rho + ct^{1/2} \\
&\lesssim t^{(nq+3)/2}\int_{\mathbb{R}} (\rho^{nq+2} + t^{(nq+2)/2})\mathrm{e}^{-\frac{q\rho^2}{15\nu}} d\rho \lesssim Ct^{(nq+3)/2}(1+t^{(nq+2)/2})
\end{aligned}$$

and hence

$$\|\mathrm{e}^{-\frac{(|\cdot|-ct)^2}{15\nu t}}\|_{\mathring{L}^q(n)} \lesssim t^{\frac{n}{2}+\frac{3}{2q}}(1+t)^{\frac{n}{2}+\frac{1}{q}}$$

□

C.2. **Proof of Proposition 2.6.**

Proof. The proof follows by putting one of the derivatives in (14) on ρ_0. Specifically, we have

$$\begin{aligned}
\Big|\partial_t^2 w * K_{\nu t} * \rho_0\Big| &\le \sum_{1\le|\tilde{\alpha}|\le 2} c_{\tilde{\alpha}} t^{|\tilde{\alpha}|-1}\Big|\int_{|z|=1} D_x^{\tilde{\alpha}}\big[K_{\nu t} * \rho_0(x+ctz)\big] z^{\tilde{\alpha}} dS(z)\Big| \\
&\le \sum_{0\le|\alpha|\le 1}\sum_{j=1}^3 c_{\alpha+e_j} t^{|\alpha|}\int_{|z|=1} D_x^\alpha D_{x_j}\big[K_{\nu t} * \rho_0(x+ctz)\big] z_j z^\alpha dS(z) \\
&\le \sum_{0\le|\alpha|\le 1}\sum_{j=1}^3 c_{\alpha+e_j} t^{\frac{|\alpha|}{2}-\frac{3}{2}}\int_{\mathbb{R}^3}\Big[\int_{|z|=1} \exp\big[-\frac{|x-y+ctz|^2}{5\nu t}\big] dS(z)\Big]\big|D_{x_j}\rho_0(y)\big| dy
\end{aligned}$$

and we can then use Lemma C.1

$$\begin{aligned}
\Big|\partial_t^2 w * K_{\nu t} * \rho_0\Big| &\le C\sum_{j=1}^3 t^{-\frac{3}{2}}(1+t)^{-\frac{1}{2}}\int_{\mathbb{R}^3}\exp\big[-\frac{(|x-y|-ct)^2}{15\nu t}\big]\big|D_{x_j}\rho_0(y)\big| dy \\
&= C\sum_{j=1}^3 t^{-\frac{3}{2}}(1+t)^{-\frac{1}{2}}\exp\big[-\frac{(|\cdot|-ct)^2}{15\nu t}\big] * \big|D_{x_j}\rho_0\big|(x)
\end{aligned}$$

hence for small times we can make the estimate

$$\begin{aligned}
\|\partial_t^2 w * K_{\nu t} * \rho_0\|_{\mathring{L}^p(\mu)} &\le C\sum_{j=1}^3 t^{-\frac{3}{2}}(1+t)^{-\frac{1}{2}}\Big\|\exp\big[-\frac{(|\cdot|-ct)^2}{20\nu t}\big] * \big|D_{x_j}\rho_0\big|\Big\|_{\mathring{L}^p(\mu)} \\
&\le C\sum_{j=1}^3 t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}(1+t)^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})}\Big[\|D_{x_j}\rho_0\|_{\mathring{L}^q(\mu)} + t^{\frac{\mu}{2}}(1+t)^{\frac{\mu}{2}}\|D_{x_j}\rho_0\|_{L^q}\Big]
\end{aligned}$$

whereas for large times we use the Young's inequality together with the estimate in Prop. 2.5.

□

Appendix D. Explicit calculations of the Hermite profiles

D.1. **Explicit functional form for the hyperbolic-parabolic Hermite profiles.** The functions ρ_1, a_1, ρ_2, a_2, Πa_1 and Πa_2 are given by the following explicit formulas:

$$\rho_1(x,t) = \frac{(|x|-ct)e^{-\frac{(|x|-ct)^2}{4(1+\nu t)}} + (|x|+ct)e^{-\frac{(|x|+ct)^2}{4(1+\nu t)}}}{2|x|(4\pi(1+\nu t))^{3/2}}$$

$$a_1(x,t) = \frac{c}{2|x|(4\pi(1+\nu t))^{3/2}}\Big[\big[\frac{(|x|+ct)^2}{2(1+\nu t)} - 1\big]e^{-\frac{(|x|+ct)^2}{4(1+\nu t)}} - \big[\frac{(|x|-ct)^2}{2(1+\nu t)} - 1\big]e^{-\frac{(|x|-ct)^2}{4(1+\nu t)}}\Big]$$

$$\rho_2(x,t) = \frac{1}{(4\pi)^{3/2}(1+\nu t)^{1/2}}\frac{e^{-\frac{(|x|+ct)^2}{4(1+\nu t)}} - e^{-\frac{(|x|-ct)^2}{4(1+\nu t)}}}{c|x|}$$

$$a_2(x,t) = \frac{1}{(4\pi(1+\nu t))^{3/2}}\frac{(|x|-ct)e^{-\frac{(|x|-ct)^2}{4(1+\nu t)}} + (|x|+ct)e^{-\frac{(|x|+ct)^2}{4(1+\nu t)}}}{2|x|}$$

$$\Pi a_1 = \frac{cx}{(4\pi)^{3/2}|x|^3(1+\nu t)^{1/2}}\Big[e^{-\frac{(|x|-ct)^2}{4(1+\nu t)}}\Big(\frac{|x|(|x|-ct)}{2(1+\nu t)} + 1\Big) - e^{-\frac{(|x|+ct)^2}{4(1+\nu t)}}\Big(\frac{|x|(|x|+ct)}{2(1+\nu t)} + 1\Big)\Big]$$

$$\Pi a_2 = \frac{1}{(4\pi)^{3/2}}\frac{x}{|x|^3}\Big(- |x|\frac{e^{-\frac{(|x|+ct)^2}{4(1+\nu t)}} + e^{-\frac{(|x|-ct)^2}{4(1+\nu t)}}}{(1+\nu t)^{1/2}} + \mathsf{Erf}(\frac{|x|-ct}{2(1+\nu t)^{1/2}}) + \mathsf{Erf}(\frac{|x|+ct}{2(1+\nu t)^{1/2}})\Big)$$

where

$$\mathsf{Erf}(r) = 2\int_0^r e^{-z^2}dz$$

Given a spherically symmetric initial condition $(u_0, 0)^T$, the solution to the wave equation is given by

$$u(x,t) = \frac{(|x|-ct)u_0(||x|-ct|) + (|x|+ct)u_0(|x|+ct)}{2|x|} \tag{66}$$

Taking u_0 to be $K_\nu(t) * \phi_0$, we obtain the equation for ρ_1. We compute a_1 by plugging $u_0 = K_\nu(s) * \phi_0$ into (66), taking the derivative of $u(x,t)$ with respect to t, multiplying by -1 and then setting $s = t$.

To compute Πa_1, note that

$$\Pi a_1 = \nabla(\Delta^{-1}a_1)$$

and that since a_1 is spherically symmetric it suffices to compute ∇u, where

$$\frac{1}{r^2}\frac{\partial}{\partial r}\big[r^2\frac{\partial u}{\partial r}\big] = a_1$$

The result follows by computing an indefinite radial integral, ensuring the integal is zero at the origin, and making use of

$$\nabla u = \frac{x}{r}\frac{\partial u}{\partial r} \tag{67}$$

To calculate the explicit forms of ρ_2 and a_2 we use the fact that the solution of the wave equation with a spherically symmetric initial condition of the form $(0, u_0(r))^T$ is given by

$$u(x,t) = -\int_0^t \frac{(|x|-cs)u_0(||x|-cs|) + (|x|+cs)u_0(|x|+cs)}{2|x|}ds \tag{68}$$

hence we have the result above for ρ_2, and a_2 is found by using (66). Πa_2 is computed using the same method used for Πa_1.

D.2. **Explicit functional form for the divergence-free vector field Hermite profiles.** We compute $B\vec{g}_i$ where

$$\vec{g}_i := \frac{1}{(4\pi)^{3/2}(1+\epsilon t)^{3/2}} \nabla \times \left(e^{-\frac{|x|^2}{4(1+\epsilon t)}}\ \vec{e}_i\right)$$

and note that in view of the definitions in Table 1 the terms $B\vec{f}_{\tilde{\alpha},j}$ can be computed by taking appropriate derivatives. One can check that the function

$$\frac{1}{(4\pi)^{3/2}(1+\epsilon t)^{3/2}} \Big[e^{-\frac{|x|^2}{4(1+\epsilon t)}} \vec{e}_i - \partial_{x_i} \nabla (\Delta^{-1} e^{-\frac{|x|^2}{4(1+\epsilon t)}}) \Big]$$

has curl equal to $\vec{g}_i$ since the second term is a gradient, hence has zero curl. Furthermore the divergence of the above expression is zero, since the divergence and gradient cancel the inverse Laplacian in the second term. As before we can compute the inverse Laplacian of the Gaussian term by exploiting the spherical symmetry and we get

$$\frac{\partial u}{\partial r} = -\frac{2(1+\epsilon t)}{r} e^{-\frac{r^2}{4(1+\epsilon t)}} + \frac{2(1+\epsilon t)}{r^2} \int_0^r e^{-\frac{z^2}{4(1+\epsilon t)}} dz$$

so using (67) we have

$$B\vec{g}_i = \frac{1}{(4\pi)^{3/2}} \Big[\frac{e^{-\frac{|x|^2}{4(1+\epsilon t)}}}{(1+\epsilon t)^{3/2}} \vec{e}_i - \partial_{x_i} \Big[\frac{x}{|x|^3} \big[-\frac{2|x| e^{-\frac{|x|^2}{4(1+\epsilon t)}}}{(1+\epsilon t)^{1/2}} + 2\mathsf{Erf}\big(\frac{|x|}{2(1+\epsilon t)^{1/2}}\big) \big] \Big] \Big]$$

Appendix E. Analysis of the linear evolution

We show that $\rho_L(t), a_L(t)$ and $\vec{\omega}_L(t)$ defined in defined by (23) and (26) for $t > 0$ and $(\rho_L(t), a_L(t), \vec{\omega}_L(t))^T = (\rho_0, a_0, \vec{\omega}_0)^T$ for $t = 0$ map $[0, \infty)$ continuously into $L^p(n)$ for initial conditions in $L^p(n)$, and that these define differentiable functions of space and time for $t > 0$. We also determine bounds on the temporal evolution of the norms of these terms.

E.1. Smoothness properties.

Proposition E.1. ***(a)*** *Let $n \in \mathbb{R}_{\geq 0}$, $p \geq 1$ and $(\rho_0, a_0, \vec{\omega})^T \in W^{1,p}(n) \times L^p(n) \times \mathbb{L}^p_\sigma(n)$. Then*

$$(\rho_L(t), a_L(t), \vec{\omega}_L(t))^T \in C^0\big[[0,\infty), L^p(n) \times L^p(n) \times \mathbb{L}^p_\sigma(n)\big]$$

(b) *Let $n \in \mathbb{R}_{\geq 0}$ and $(\rho_0, a_0, \vec{\omega})^T \in W^{1,1}(n) \times L^1(n) \times \mathbb{L}^1_\sigma(n)$. Then*

$$(\partial_x^\alpha \rho_L(t), \partial_x^\alpha a_L(t), \partial_x^\alpha \vec{\omega}_L(t))^T \in C^0\big[(0,\infty), L^p(n) \times L^p(n) \times \mathbb{L}^p_\sigma(n)\big]$$

for every $1 \leq p \leq \infty$ and $\alpha \in \mathbb{N}^3$.

Proof. We prove continuity at $t = 0$ for part **(a)**, then prove part **(b)**, and the continuity for $t > 0$ follows from the fact that solutions are differentiable in time, and that these time derivatives can be written in terms of the spatial derivatives by virtue of the differential equation that the solutions satisfy. Starting with $\vec{\omega}_L$ we show continuity at $t = 0$ by first noting that it suffices to consider $\vec{\omega}_0$ which is smooth and has compact support by a density argument, together with the linearity of the heat operator, Young's inequality and the heat estimates in Proposition 2.3. Standard arguments show that for such $\vec{\omega}_0$ we have $\mathbb{K}_\epsilon(t) * \vec{\omega}_0 \to \vec{\omega}_0$ uniformly as $t \to 0$, and the result follows. For $t > 0$ one obtains $\partial_x^\alpha \mathbb{K}_\epsilon(t) * \vec{\omega}_0 \in \mathbb{L}^p(n)$ via Young's inequality and the differentiability as a map into $\mathbb{L}^p_\sigma$ follows from the fact that

$$\lim_{h \to 0} \big\| \frac{\partial_x^\alpha K_\epsilon(t+h) - \partial_x^\alpha K_\epsilon(t)}{h} - \partial_t \partial_x^\alpha K_\epsilon(t) \big\|_{L^1(\mu)} = 0$$

for all μ, together with Young's inequality.

For $\rho_L(t)$ we start with $\partial_t w(t) * K_\nu(t) * \rho_0$. Again we can assume that ρ_0 is smooth and has compact support using Proposition 2.5. For such ρ_0 the uniform convergence of $\partial_t w(t) * K_\nu(t) * \rho_0$ to ρ_0 as $t \to 0$ is immediate from the formula

$$\partial_t w(t) * K_\nu(t) * \rho_0 = \frac{1}{4\pi} \int_{|z|=1} K_\nu(t) * \rho_0(x + ctz) dS(z)$$

and from the result for $K_\nu(t) * \rho_0$. The continuity in $L^p(n)$ then follows. For $t > 0$ the differentiability follows by the same reasoning as above. The proofs for the smoothness properties of the other terms are similar. □

E.2. **Linear evolution decay rates.** Let $r_{\alpha,p}, \ell_{n,p,\mu}$ and $\tilde{\ell}_{n,p,\mu}$ be as defined in (9) and (10).

Proposition E.2. *Let $n \in \mathbb{R}_{\geq 0}$ be given. Suppose $(\rho_0, a_0, \vec{\omega}_0)^T \in \bigcap_{1\leq\tilde{p}\leq 3/2} W^{1,\tilde{p}}(n) \times L^{\tilde{p}}(n) \times \mathbb{L}^{\tilde{p}}_\sigma(n)$. If $n > 0$, suppose also that a_0 and $\vec{\omega}_0$ have zero total mass. Then*

$$\begin{aligned}
&\|\partial_x^\alpha \rho_L(t)\|_{\mathring{L}^p(\mu)} \leq Ct^{-r_{\alpha,p}}(1+t)^{-\ell_{n,p,\mu}+\frac{1}{2}} \sup_{1\leq\tilde{p}\leq 3/2} \left(\|\rho_0\|_{W^{1,\tilde{p}}(n)} + \|a_0\|_{L^{\tilde{p}}(n)}\right)\\
&\|\partial_x^\alpha a_L(t)\|_{\mathring{L}^p(\mu)} \leq Ct^{-r_{\alpha,p}}(1+t)^{-\ell_{n,p,\mu}} \sup_{1\leq\tilde{p}\leq 3/2} \left(\|\rho_0\|_{W^{1,\tilde{p}}(n)} + \|a_0\|_{L^{\tilde{p}}(n)}\right)\\
&\|\partial_x^\alpha \vec{\omega}_L(t)\|_{\mathring{\mathbb{L}}^p(\mu)} \leq Ct^{-r_{\alpha,p}}(1+t)^{-\tilde{\ell}_{n,p,\mu}} \sup_{1\leq\tilde{p}\leq 3/2} \left(\|\vec{\omega}_0\|_{\mathbb{L}^{\tilde{p}}(n)}\right)
\end{aligned} \tag{69}$$

holds for all $t \in (0,\infty)$, $1 \leq p \leq \infty$, $0 \leq \mu \leq n$ and $\alpha \in \mathbb{N}^3$

Proof. In the following computations we ignore constant proportionality factors for simplicity. The proof follows from Young's inequality, together with the the fact that we can split the weight via $(1+|x|)^\mu \leq (1+|y|)^\mu + (1+|x-y|)^\mu$ and estimate in different L^p norms. For the first term in (23), this is as follows. For large times $t > 1$ we have

$$\begin{aligned}
\|\partial_t w * \partial_x^\alpha K_\nu * \rho_0\|_{\mathring{L}^p(\mu)} &\leq \|\partial_t w * \partial_x^\alpha K_\nu(t)\|_{\mathring{L}^p(\mu)}\|\rho_0\|_{L^1} + \|\partial_t w * \partial_x^\alpha K_\nu(t)\|_{L^p}\|\rho_0\|_{\mathring{L}^1(\mu)}\\
&\leq t^{\frac{\mu}{2}-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}(1+t)^{\frac{\mu}{2}+\frac{1}{2}-(1-\frac{1}{p})}\|\rho_0\|_{L^1} + t^{-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}(1+t)^{\frac{1}{2}-(1-\frac{1}{p})}\|\rho_0\|_{\mathring{L}^1(\mu)}
\end{aligned}$$

whereas for small times $t < 1$ we have

$$\begin{aligned}
\|\partial_t w * \partial_x^\alpha K_\nu * \rho_0\|_{\mathring{L}^p(\mu)} &\leq \|\partial_t w * \partial_x^\alpha K_\nu(t)\|_{\mathring{L}^{\tilde{p}}(\mu)}\|\rho_0\|_{L^{3/2}} + \|\partial_t w * \partial_x^\alpha K_\nu(t)\|_{L^{\tilde{p}}}\|\rho_0\|_{\mathring{L}^{3/2}(\mu)}\\
&\leq t^{\frac{\mu}{2}-\frac{3}{2}(\frac{2}{3}-\frac{1}{p})-\frac{|\alpha|}{2}}(1+t)^{\frac{\mu}{2}+\frac{1}{2}-(\frac{2}{3}-\frac{1}{p})}\|\rho_0\|_{L^{3/2}} + t^{-\frac{3}{2}(\frac{2}{3}-\frac{1}{p})-\frac{|\alpha|}{2}}(1+t)^{\frac{1}{2}-(\frac{2}{3}-\frac{1}{p})}\|\rho_0\|_{\mathring{L}^{3/2}(\mu)}
\end{aligned}$$

for $p \geq 3/2$ and

$$\begin{aligned}
\|\partial_t w * \partial_x^\alpha K_\nu * \rho_0\|_{\mathring{L}^p(\mu)} &\leq \|\partial_t w * \partial_x^\alpha K_\nu(t)\|_{\mathring{L}^1(\mu)}\|\rho_0\|_{L^p} + \|\partial_t w * \partial_x^\alpha K_\nu(t)\|_{L^1}\|\rho_0\|_{\mathring{L}^p(\mu)}\\
&\leq t^{\frac{\mu}{2}-\frac{|\alpha|}{2}}(1+t)^{\frac{\mu}{2}+\frac{1}{2}}\|\rho_0\|_{L^p} + t^{-\frac{|\alpha|}{2}}(1+t)^{\frac{1}{2}}\|\rho_0\|_{\mathring{L}^p(\mu)}
\end{aligned}$$

for $1 \leq p \leq 3/2$ hence these norms blow up at the rate $t^{-\frac{3}{2}(\frac{2}{3}-\frac{1}{p})-\frac{|\alpha|}{2}}$ as $t \to 0$ for $p \geq 3/2$, blow up at the rate $t^{-\frac{|\alpha|}{2}}$ as $t \to 0$ for $1 \leq p \leq 3/2$ and decay at the rate $t^{k-\frac{5}{2}(1-\frac{1}{p})+\frac{1}{2}-\frac{|\alpha|}{2}}$ as $t \to \infty$ for all $1 \leq p \leq \infty$. For the next term in ρ_L we find

$$\begin{aligned}
\|w * \partial_x^\alpha K_\nu * a_0\|_{\mathring{L}^p(\mu)} &\leq \|w * \partial_x^\alpha K_\nu(\frac{t}{2})\|_{\mathring{L}^p(\mu)}\|K_\nu(\frac{t}{2}) * a_0\|_{L^1} + \|w * \partial_x^\alpha K_\nu(\frac{t}{2})\|_{L^p}\|K_\nu(\frac{t}{2}) * a_0\|_{\mathring{L}^1(\mu)}\\
&\leq t^{1+\frac{\mu}{2}-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}(1+t)^{\frac{\mu}{2}-(1-\frac{1}{p})-\frac{\lfloor n\rfloor_1}{2}}\|a_0\|_{\mathring{L}^1(\lfloor n\rfloor_1)} + t^{1-\frac{3}{2}(1-\frac{1}{p})-\frac{|\alpha|}{2}}(1+t)^{-(1-\frac{1}{p})}\|K_\nu(\frac{t}{2}) * a_0\|_{\mathring{L}^1(\mu)}
\end{aligned}$$

for large times. For the case $\mu = 0$ note that the second term on the right hand side does not appear since we can use Young's inequality directly, and if $0 < \mu \leq n$ then we can use

$$\|K_\nu(\frac{t}{2}) * a_0\|_{\mathring{L}^1(\mu)} \leq t^{-\frac{\lfloor n\rfloor_1-\lfloor\mu\rfloor_1}{2}}\|a_0\|_{L^1(n)}$$

For small times we have

$$\begin{aligned}
\|w * \partial_x^\alpha K_\nu * a_0\|_{\mathring{L}^p(\mu)} &\leq \|w * \partial_x^\alpha K_\nu(t)\|_{\mathring{L}^{\tilde{p}}(\mu)}\|a_0\|_{L^{3/2}} + \|w * \partial_x^\alpha K_\nu(t)\|_{L^{\tilde{p}}}\|a_0\|_{\mathring{L}^{3/2}(\mu)}\\
&\leq t^{1+\frac{\mu-|\alpha|}{2}-\frac{3}{2}(\frac{2}{3}-\frac{1}{p})}(1+t)^{\frac{\mu}{2}-(\frac{2}{3}-\frac{1}{p})}\|a_0\|_{L^{3/2}} + t^{1-\frac{3}{2}(\frac{2}{3}-\frac{1}{p})-\frac{|\alpha|}{2}}(1+t)^{-(\frac{2}{3}-\frac{1}{p})}\|a_0\|_{\mathring{L}^{3/2}(\mu)}
\end{aligned}$$

for $p \geq 3/2$ and

$$\begin{aligned}
\|w * \partial_x^\alpha K_\nu * a_0\|_{\mathring{L}^p(\mu)} &\leq \|w * \partial_x^\alpha K_\nu(t)\|_{\mathring{L}^1(\mu)}\|a_0\|_{L^p} + \|w * \partial_x^\alpha K_\nu(t)\|_{L^1}\|a_0\|_{\mathring{L}^p(\mu)}\\
&\leq t^{1+\frac{\mu-|\alpha|}{2}}(1+t)^{\frac{\mu}{2}}\|\rho_0\|_{L^p} + t^{1-\frac{|\alpha|}{2}}\|\rho_0\|_{\mathring{L}^p(\mu)}
\end{aligned}$$

for $1 \leq p \leq 3/2$. The time estimates of the other terms in (23), (26) are obtained similarly. Note the weighted estimates in (69) aren't sharp for $\vec{\omega}_L$, but instead match the decay rate of solutions of (12). ☐

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