

Orbital stability of localized structures via Bäcklund transformations

A. Hoffman^{*†} C.E. Wayne[‡]

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Abstract

The Bäcklund Transform, first developed in the context of differential geometry, has been classically used to obtain multi-soliton states in completely integrable infinite dimensional dynamical systems. It has recently been used to study the stability of these special solutions. We offer here a dynamical perspective on the Bäcklund Transform, prove an abstract orbital stability theorem, and demonstrate its utility by applying it to the sine-Gordon equation and the Toda lattice.

1 Introduction

In this paper we survey some recent work on the use of Bäcklund transformations to study the stability of localized structures in infinite dimensional Hamiltonian systems. For finite dimensional Hamiltonian systems the constraints imposed by the Hamiltonian structure mean that the stability of stationary solutions can be reduced either to showing that all the eigenvalues of the linearized system at the fixed point lie on the imaginary axis (for spectral stability) or that the full, nonlinear system exhibits Lyapunov stability. The stability of periodic orbits can be studied by similar methods by reducing the problem to the consideration of a fixed point of a Poincaré map.

For infinite dimensional systems the situation can be more subtle. There, the possible presence of dispersive phenomena means that one may have asymptotic stability of such systems, in appropriate norms, a phenomenon that is impossible in finite dimensional systems.

Consequently, the study of stability in such systems has followed two rather different tracks. On one hand, methods to prove Lyapunov (or orbital) stability of localized solutions like traveling waves, solitons, or multi-solitons have been developed which rely on regarding the solution as a minimizer, or critical point, of some energy functions,

^{*}Franklin W. Olin College of Engineering; Needham, MA

[†]corresponding author: aaron.hoffman@olin.edu

[‡]Boston University Department of Mathematics and Statistics; Boston, MA

often subject to appropriate constraints. Examples of this type of approach are [2], [5], [3], [9].

The second approach typically begins by analyzing the linearization of the system about the solitary wave. The spectrum of the linearization is then considered and one shows that on the complement of the point spectrum the linearized evolution generates a dispersive evolution. If the dispersive decay is sufficiently rapid, this can then (sometimes) be used in conjunction with duHamel’s formula to derive nonlinear, asymptotic stability of the underlying solitary wave. Examples of this approach include [17], [11]. Closely related to this approach are stability or instability results based on invariant manifold theorems [7], [19]. Here, one typically shows that the nonlinear equations possesses an invariant manifold associated with the family of localized solutions and examines the behavior of solutions near this manifold to understand stability properties of the underlying family.

Recently, an **old tool has been adapted** to study both of these types of stability, namely Bäcklund transformations. Bäcklund transformations define a relationship between two functions (often, through a differential equation, or some more complicated equation) such that if one of the functions satisfies a given partial differential equation, so does the second. The partial differential equation satisfied by the second function may be the same PDE satisfied by the first function (in which case one speaks of an auto-Bäcklund transform) or it may be a different PDE. In the study of infinite-dimensional Hamiltonian systems Bäcklund transform have mostly been used in the context of completely integrable infinite dimensional systems to obtain explicit formulas for soliton, multi-soliton, or other special solutions of the equations. Thus, for instance, the auto-Bäcklund transform for the Korteweg-de Vries (KdV) equation relates the zero solution to the one-soliton solution, the one-soliton solution to the two-soliton solution and so on and so forth.

However, the Bäcklund transformation is also turning out to be a useful tool to investigate the stability of such special solutions as well. It may not be clear at first glance why this is of interest. Since in principle, one knows “everything” about solutions of a completely integrable system, the stability or instability of such solutions might seem an obvious by-product of their integrability. In practice, however, it may be difficult to see from the formulas defining the solutions in these integrable systems what the asymptotic behavior of solutions with initial conditions close to a soliton are. Furthermore, stability results based on Bäcklund transformations have yielded at least two new insights not available from the complete integrability machinery:

- First, in some circumstances, they allow one to establish stability in much less regular spaces than can be treated either with completely integrable structure, or with energy methods. A first example of this approach is the work of Merle and Vega, [12]. They used the Gardner transformation which maps solutions of the KdV equation close to the soliton into solutions of the modified KdV equation near a kink solution. (The Gardner transformation is an example of a Bäcklund transformation which links two different equations.) They then use the stability of modified KdV kinks in the energy space, plus the fact that the Gardner transformation also maps L^2 solutions in KdV into H_{loc}^1 solutions of modified KdV

to conclude that KdV solitons are actually stable in L^2 . This approach has since been extended to conclude that multi-soliton solutions of KdV are also stable in L^2 , [15], and also that the soliton solution of the nonlinear Schrödinger equation is stable in L^2 , [18].

- A second advantage of Bäcklund transformation methods is that they can sometimes be used as the starting point for a perturbative argument which yields insight into the behavior of other non-integrable systems. Thus, the Bäcklund transformation-based study of the stability of soliton solutions of the (integrable) Toda-lattice in [14] served as the basis for a simple proof of stability of solitary waves in a general class of non-integrable Fermi-Pasta-Ulam models [6].

While this paper will focus on rigorous applications of the Bäcklund transformation method it is worth noting that similar ideas have been used in non-rigorous settings (sometimes in advance of the rigorous applications) to compute explicit approximate solutions with initial conditions close to solitary waves. Thus, in [10], Mann used a linearized Bäcklund transformation to compute the Green's function for the KdV equation linearized about the soliton solution and then in turn used this to study the evolution of initial conditions close to the soliton. Likewise, Tsigaridas, et al [20] make a more general study of this same question and apply these ideas to compute approximate solutions of both the nonlinear Schrödinger equation and KdV equations with the aid of linearized Bäcklund transformations.

The classical view of the Bäcklund transform for the sine-Gordon equation is geometric in nature. It relates angles between curves of zero curvature on patches of pseudo-spherical surfaces. As is common in differential geometry, partial differential equations arise. Here the partial differential equation relates the aforementioned angle as a state variable to the coordinates on the manifold as independent variables. Thus the geometric relationship between these angles on a pair of pseudo-spherical surfaces manifests in the PDE world as a relationship between a pair of solutions. Moreover, the zero solution is related to the one-kink in this way, the one-kink to the two-kink and so forth. Thus such a perspective is very useful for constructing multi-soliton solutions. In studying the stability of these kink and multi-kink solutions, however, the theory of dynamical systems is necessarily brought in and from this perspective the classical view of the Bäcklund transform is not entirely natural. The reason for this is as follows. The typical strategy of proof in the nascent literature of stability via Bäcklund transform is effectively to conjugate the flow about a soliton (or multi-soliton) with the flow about the zero solution, leveraging the stability of the zero solution to obtain the stability of the soliton or multi-soliton. The problem from the perspective of dynamical systems is that when conjugating a flow one makes use of a map that acts on the phase space and not on the much larger space of solution trajectories in the phase space. One of the key ideas in this paper is to redefine the Bäcklund transform as a map that acts on the phase space. An orbital stability result for solitons and multi-solitons then follows very quickly from this definition with the aid of well-developed and classical ideas in dynamical systems.

2 Abstract orbital stability

Definition Let X and Y be open subsets of affine subspaces of Banach spaces and let $\Phi(t) : X \rightarrow X$ and $\Psi(t) : Y \rightarrow Y$ be semiflows. Let Λ be a finite dimensional manifold and let Z be a Banach space. Let $F : X \times Y \times \Lambda \rightarrow Z$ such that for each $\lambda \in \Lambda$, $\mathcal{M}_\lambda := F(\cdot, \cdot, \lambda)^{-1}(0)$ is an invariant set for the product flow $\Phi \times \Psi : X \times Y \rightarrow X \times Y$. Assume further

(H0) there is some $(\bar{x}, \bar{y}, \bar{\lambda}) \in X \times Y \times \Lambda$ such that $F(\bar{x}, \bar{y}, \bar{\lambda}) = 0$.

(H1) $D_x F(x, y, \lambda) : T_x X \rightarrow T_{F(x, y, \lambda)} Z$ is Fredholm and injective whenever $F(x, y, \lambda) = 0$.

(H2) $D_{y, \lambda} F(x, y, \lambda) : T_y Y \rightarrow T_{F(x, y, \lambda)} Z$ is an isomorphism whenever $F(x, y, \lambda) = 0$.

Then we say that F **Bäcklund-conjugates** the flows Φ and Ψ . In the case that $\Phi = \Psi$ we say that F **auto-Bäcklund-conjugates** Φ with itself.

Theorem 2.1. *Assume (H0)-(H2). Let $(\bar{x}, \bar{y}, \bar{\lambda})$ be given as in (H0) and let $H \subset Y$ be an invariant manifold for Ψ that contains \bar{y} and is stable in sense of Lyapunov: There is an $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0]$ there is a $\delta = \delta(\varepsilon) > 0$ such that for any $t > 0$, we have $d_Y(\Psi(t)y, H) \leq \varepsilon$ whenever $d_Y(y, H) \leq \delta$. Assume further that*

C1 Given any compact subset $\Lambda_0 \subset \Lambda$ there is a constant C such that for each $(x, y, \lambda) \in F^{-1}(0)$, there is a subspace \tilde{X} complementary to the kernel of $D_x F(x, y, \lambda)$ such that

$$\left\| (D_x F(x, y, \lambda)|_{\tilde{X}})^{-1} \right\| \leq C$$

with the estimate uniform among $y \in Y$ with $d_Y(y, H) < \delta(\varepsilon_0)$, $x \in X$ and $\lambda \in \Lambda_0$ with $F(x, y, \lambda) = 0$.

C2 the norm of $(D_{(y, \lambda)} F(x, y, \lambda))^{-1}$ is bounded above uniformly among (x, y, λ) such that $d_Y(y, H) < \delta(\varepsilon_0)$ and $\lambda \in \Lambda_0$ compact and $F(x, y, \lambda) = 0$.

C3 the norm of $D_x F(x, y, \lambda)$ is bounded above uniformly among (x, y, λ) with $d_Y(y, H) < \delta(\varepsilon_0)$, with $\lambda \in \Lambda_0$ compact and $F(x, y, \lambda) = 0$.

Then there is an invariant manifold M for Φ containing \bar{x} , a function $\lambda^ : M \rightarrow \Lambda$, a decomposition of M into invariant manifolds $M^\lambda = (\lambda^*)^{-1}(\lambda)$, as well as a constant C such that*

$$d_X(\Phi(t)x, M^{\lambda^*(x)}) \leq C\varepsilon$$

whenever $d_X(x, M) \leq \frac{1}{C}\delta$. Moreover, M^λ is precisely the set of $x \in X$ such that $F(x, y, \lambda) = 0$ for some $y \in H$.

Proof. Let $(\bar{x}, \bar{y}, \bar{\lambda})$ be given as in (H0). It follows from (H2) and the implicit function theorem that there are smooth functions y^* and λ^* mapping a neighborhood of \bar{x} to neighborhoods of \bar{y} and $\bar{\lambda}$ respectively such that $F(x, y^*(x), \lambda^*(x)) = 0$. Moreover, these functions are unique in that if (x, y, λ) is close to $(\bar{x}, \bar{y}, \bar{\lambda})$ and $F(x, y, \lambda) = 0$, then $y = y^*(x)$ and $\lambda = \lambda^*(x)$. Furthermore, since condition (H2) holds on all of $F^{-1}(0)$, it follows that y^* and λ^* can be extended to all of X .

Note that

$$\begin{pmatrix} D_x y^*(x) \\ D_x \lambda^*(x) \end{pmatrix} = -D_{y,\lambda} F(x, y^*(x), \lambda^*(x))^{-1} F_x(x, y^*(x), \lambda^*(x)),$$

and hence the implicitly defined functions y^* and λ^* are Lipschitz on any set for which $x \mapsto \|D_{y,\lambda} F(x, y^*(x), \lambda^*(x))^{-1}\|$ and $x \mapsto \|F_x(x, y^*(x), \lambda^*(x))\|$ enjoy a uniform bound. Define $H_\delta = \{y \in Y \mid d_Y(y, H) < \delta\}$ and define $M_\delta = (y^*)^{-1}(H_\delta \cap y^*(X))$ with M denoting M_0 . It follows from conditions (C2) and (C3) that y^* and λ^* are Lipschitz on $M_\delta \subset X$ for $\delta \leq \delta(\varepsilon_0)$.

Let $T_{\bar{x}}X = \bar{K} \times \bar{X}_1$ be a Lyapunov-Schmidt decomposition of $T_{\bar{x}}X$ subordinate to $D_x F(\bar{x}, \bar{y}, \bar{\lambda})$. Here \bar{K} denotes the kernel and \bar{X}_1 is chosen as in (C1). It follows from (H1) that there is a smooth implicitly defined function \bar{x}_1^* taking a neighborhood of $(\bar{y}, \bar{\lambda}, 0) \in Y \times \Lambda \times K$ to \bar{X}_1 such that $F(\bar{x}_1^*(y, \lambda, k) + k, y, \lambda) = 0$ for any k in the given neighborhood of $\{0\} \subset K$. Since the domain of y^* is all of X , it follows that $y = y^*(\bar{x}_1^*(y, \lambda, k) + k)$ and more specifically that for $y \in H$ we have $\bar{x}_1^*(y, \lambda, k) + k \in M$ whenever this quantity is defined.

Because of (H1) the point $(\bar{x}, \bar{y}, \bar{\lambda})$ is not distinguished among points in $F^{-1}(0)$. Thus given any $(x, y, \lambda) \in F^{-1}(0)$ there is a similar Lyapunov-Schmidt decomposition $T_x X = K \times X_1$ and a similar implicitly defined function x_1^* . It follows from (C1) that x_1^* has a Lipschitz constant which is uniform in the choice of base point (x, y, λ) for the implicit function theorem.

Let C denote the Lipschitz constant of y^* . Since we have assumed in the statement of the theorem that x is δ/C -close to M it follows that $y^*(x)$ is δ -close to H . Recall that this is the neighborhood of Lyapunov stability for H corresponding to the given small number ε : $d_Y(\Psi(t)y^*(x), H) < \varepsilon$. Since the pair $(x, y^*(x))$ lies on the invariant manifold $\mathcal{M}_{\lambda^*(x)} = F(\cdot, \cdot, \lambda^*(x))^{-1}(0)$ it follows that

$$F(\Phi(t)x, \Psi(t)y^*(x), \lambda^*(x)) = 0,$$

hence that $\Phi(t)x = x_1^*(\Psi(t)y^*(x), \lambda^*(x), k) + k$ for one of the local functions x_1^* . This point is well approximated by $x_1^*(y^*(x), \lambda^*(x), k) + k$, a point on M . The error is bounded by $C(\Psi(t) - I)y^*(x)$ where C has been made larger if necessary to dominate the Lipschitz constant of x_1^* . Since $d_Y(\Psi(t)y^*(x), H) < \varepsilon$ we have that $d_X(\Phi(t)x, M) < C\varepsilon$ as desired. \square

3 Examples of orbital stability

3.1 Sine-Gordon equation

As a first application of Theorem 2.1, we consider the orbital stability of the kink solutions of the Sine-Gordon equation. While the stability of these solutions is not surprising and could probably be proved using the energy methods discussed above, it gives a simple illustration of our approach. Furthermore, as we indicate at the end of this section, we suspect that with some additional work this method will also yield the stability of multi-kink solutions for this equation.

The classical Bäcklund transform for the Sine-Gordon equation relates two solutions \bar{u} and \bar{u}' by the pair of equations

$$\begin{aligned}\bar{u}_x - \bar{u}_t &= \bar{u}'_x - \bar{u}'_t + 2a \sin\left(\frac{\bar{u} + \bar{u}'}{2}\right) \\ \bar{u}_x + \bar{u}_t &= -\bar{u}'_x - \bar{u}'_t + \frac{2}{a} \sin\left(\frac{\bar{u} - \bar{u}'}{2}\right)\end{aligned}\tag{1}$$

If we introduce phase space variables $u = \bar{u}$, $v = \bar{u}_t$ and $u' = \bar{u}'$, $v' = \bar{u}'_t$, we see that the Bäcklund transform can be written as

$$F(u, v, u', v', a) = \begin{pmatrix} u_x + v' - a \sin\left(\frac{u+u'}{2}\right) - \frac{1}{a} \sin\left(\frac{u-u'}{2}\right) \\ v + u'_x - \frac{1}{a} \sin\left(\frac{u-u'}{2}\right) + a \sin\left(\frac{u+u'}{2}\right) \end{pmatrix} = 0\tag{2}$$

and it is to this function that we apply Theorem 2.1

Recall that the Bäcklund transform for the Sine-Gordon equation maps the zero solution to the 1-kink solution and then successively maps the k -kink to the $k + 1$ -kink, for any positive integer k [8]. With this in mind, let $\bar{X} = \{(u, v) \mid \sin(\frac{u}{2}) \in H^1 \text{ and } v \in L^2 \text{ and } u(-\infty) = 0\}$. Consider the decomposition $\bar{X} = \cup_{k=-\infty}^{\infty} X_k$ where $X_k = \{(u, v) \in \bar{X} \mid u(\infty) = 2\pi k\}$.¹ Let $\Phi(t) = \Psi(t)$ denote the time t map for the sine-Gordon equation $u_{tt} = u_{xx} - \sin u$ and let $Z = L^2 \times L^2$. Given the properties of the k -kink solution, it is natural to study its evolution in the space X_k . For the time being, since we want to concentrate on the 1-kink solution we will focus on the spaces X_0 and X_1 , and consider the function:

$$F : X_1 \times X_0 \times (0, 1) \rightarrow Z\tag{3}$$

We now have:

Theorem 3.1. *Let \bar{u} be a 1-kink solution for the sine-Gordon equation, let $\varepsilon > 0$ be given and let u^0 be ε -close to $\bar{u}(t)$ in $H^1 \times L^2$ for time $t = 0$. (i.e. $(u_0 - \bar{u}(0))$ is small in $H^1 \times L^2$). Let u^t denote the time-evolution of u^0 . Then for all time, u^t remains $\sqrt{\varepsilon}$ -close to some 1-kink solution with to the speed of \bar{u}*

Proof. We first check that hypotheses (H0), (H1), and (H2) are satisfied. We take $y = (u', v') = (0, 0)$, and then we can solve explicitly for those points $x = (u, v)$ for which F is zero and we find $u = 4 \arctan(\exp(ax + \delta))$, $v = 4av \exp(ax + \delta)/(1 + \exp(2(ax + \delta)))$, where v and a are related by $a\sqrt{1-v^2} = 1$. (Note that this calculation insures that (H0) is satisfied.) However, we will use this explicit form of the kink solution very little, in order to set the stage for our discussion of the stability of the k -kink solution later in this section.

Now differentiate F with respect to (u, v) to obtain:

$$D_{(u,v)}F(u, v, u', v', , a) = \begin{pmatrix} \partial_x - \frac{a}{2} \cos\left(\frac{u+u'}{2}\right) - \frac{1}{2a} \cos\left(\frac{u-u'}{2}\right) & 0 \\ \partial_x - \frac{1}{2a} \cos\left(\frac{u+u'}{2}\right) + \frac{a}{2} \cos\left(\frac{u-u'}{2}\right) & 1 \end{pmatrix}\tag{4}$$

¹Note that the fact that $\sin(u/2) \in H^1$ implies that the jump in u from $-\infty$ to ∞ is an integer multiple of 2π .

To invert this operator we need to solve the system of ODE's:

$$\begin{pmatrix} \partial_x - \frac{a}{2} \cos(\frac{u+u'}{2}) - \frac{1}{2a} \cos(\frac{u-u'}{2}) & 0 \\ \partial_x - \frac{1}{2a} \cos(\frac{u+u'}{2}) + \frac{a}{2} \cos(\frac{u-u'}{2}) & 1 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \quad (5)$$

Note that the operator is lower triangular, and the second row gives ψ in terms ϕ_x , a bounded (invertible) multiplication operator acting on ϕ , and g .

Thus, we focus on the first row which gives ϕ as the solution to a first order non-autonomous ODE (in x) with inhomogeneous term given by f . More precisely, we must solve

$$\partial_x \phi - \left(\frac{a}{2} \cos(\frac{u+u'}{2}) + \frac{1}{2a} \cos(\frac{u-u'}{2}) \right) \phi = f. \quad (6)$$

We analyze this equation, and similar equations below with the aid of the following lemma.

Lemma 3.2. *Consider the ODE*

$$u_x - \alpha(x)u = f(x). \quad (7)$$

1. *Assume that $\alpha_{\pm} = \lim_{x \rightarrow \pm\infty} \alpha(x)$ are defined with $\alpha_- > 0 > \alpha_+$. Assume further that $\int_0^{\infty} |\alpha(t) - \alpha_+| dt < \infty$, and $\int_{-\infty}^0 |\alpha(t) - \alpha_-| dt < \infty$. Then there exists a constant C_{α} , such that the unique solution of (7) with $u(x_0) = 0$ satisfies $\|u\|_{H^1} \leq C_{\alpha} \|f\|_{L^2}$.*
2. *Assume that $\alpha_- < 0 < \alpha_+$, with $|\alpha(x)| \leq \kappa < \infty$ and that $\int_{\mathbb{R}} f(t)\phi(t)dt = 0$ for the unique (up to constant multiple), non-zero, bounded ϕ solving the adjoint ODE $\dot{\phi} = -\alpha(t)\phi$. Then there is a unique choice of u_0 for which $u \in L^2$ and for this choice of u_0 we have $\|u\|_{H^1} \leq C_{\alpha} \|f\|_{L^2}$.*

Remark This lemma is a simple and explicit example of the relationship between Fredholm properties of operators and exponential dichotomies, which has been very useful in the theory of dynamical systems [16]. In particular, the fact that we require $\alpha(x)$ to converge to its limiting values in L^1 is a very natural assumption in this context.

Proof. (of Lemma) Define $\mu(x) = \exp(-\int_{x_0}^x \alpha(t)dt)$. Then the unique solution of (7) with $u(x_0) = u_0$ is

$$u(x) = u_0/\mu(x) + \int_{x_0}^x \frac{\mu(y)}{\mu(x)} f(y)dy. \quad (8)$$

Write $u = u^> + u^<$, where $u^>(x) = u(x)$ for $x \geq x_0$, and zero otherwise, and $u^<(x) = u(x)$ for $x < x_0$, and zero otherwise. Then $\|u\|_{L^2}^2 = \|u^>\|_{L^2}^2 + \|u^<\|_{L^2}^2$. We will bound $\|u^>\|_{L^2}^2$ and leave the estimate on $\|u^<\|_{L^2}^2$ as an exercise.

We now consider specifically the situation in Case 1, where $\alpha_- > 0 > \alpha_+$. Define $h^>(x) = |h(x)|$ if $x > x_0$ and $h^>(x) = 0$ for $x \leq x_0$. Likewise define $\mathcal{E}^>(x) = \exp(\alpha_+x)$

if $x > 0$ and zero otherwise. Then we can estimate

$$\begin{aligned}
|u^>(x)| &= \left| \int_{x_0}^x e^{\int_y^x \alpha(t) dt} h(y) dy \right| \\
&= \left| \int_{x_0}^x e^{\alpha_+(x-y) + \int_y^x (\alpha(t) - \alpha_+) dt} h(y) dy \right| \\
&\leq C_\alpha^+ \int_{x_0}^x e^{\alpha_+(x-y)} |h(y)| dy = C_\alpha^+ \int_{-\infty}^\infty \mathcal{E}_+(x-y) h^>(y) dy
\end{aligned}$$

From this last expression we immediately obtain $\|u^>\|_{L^2} \leq \tilde{C}_\alpha^+ \|\mathcal{E}_+\|_{L^1} \|h^>\|_{L^2}$, from Young's inequality. The L^2 norm of the derivative of $u^>$ can be estimated in a similar fashion which completes the estimate of the H^1 norm, and the proof of Case 1.

Now turn to Case 2. Rewrite (8) as

$$\mu(x)u(x) = u_0 + \int_{x_0}^x \mu(y)f(y)dy \quad (9)$$

The assumptions on α imply that $\mu(x) \rightarrow 0$ as $|x| \rightarrow \infty$ and thus, in order for the solution $u(x)$ to be bounded we must have

$$u_0 + \int_{x_0}^\infty \mu(y)f(y)dy = u_0 + \int_{x_0}^{-\infty} \mu(y)f(y)dy = 0, \quad (10)$$

which uniquely defines u_0 provided

$$0 = \int_{x_0}^\infty \mu(y)f(y)dy - \int_{x_0}^{-\infty} \mu(y)f(y)dy = \int_{-\infty}^\infty \mu(y)f(y)dy \quad (11)$$

Note that $\mu(x)$ is the solution of the adjoint ODE, and hence the hypothesis of Case 2 is satisfied. The resulting bound on the norm of u then follows as in Case 1. \square

We now apply the Lemma to (6) where we see that

$$\alpha(x) = \frac{a}{2} \cos\left(\frac{u+u'}{2}\right) + \frac{1}{2a} \cos\left(\frac{u-u'}{2}\right) \quad (12)$$

Recalling that $u' \in H^1$ and that u is an H^1 perturbation of the kink, we see that $u \pm u'$ will approach their limits as $x \rightarrow \pm\infty$ in L^2 and hence that $\cos(\frac{u \pm u'}{2})$ will approach their limits in L^1 as required. Furthermore, $\alpha_\pm = \mp(\frac{a}{2} + \frac{1}{2a})$, so we are in Case 1 of the Lemma. Thus, (6) can be solved for any $f \in L^2$, which gives the invertibility of $D_{u,v}F$ and verifies (H1).

Next consider hypothesis (C1) of Theorem 2.1. Note that from the calculation above, we see that $D_{(u,v)}F$ does have a one-dimensional kernel spanned by $\phi(x) = \mu(x)$ and $\psi(x) = \left(\left(\frac{1}{2a}\right) \cos\left(\frac{u+u'}{2}\right) - \frac{a}{2} \cos\left(\frac{u-u'}{2}\right)\right) \mu(x) - \mu'(x)$. Consider the subspace orthogonal to this kernel. Note that we need a uniform bound on the inverse of $D_{(u,v)}F$ only for $(u', v') \in H = \{(0, 0)\}$ and for a in some compact subset of $(0, 1)$. A bound on the inverse is easily derived from the proof of the lemma and is proportional to the constant C_α which we again see from the proof of the lemma is determined by

$\int_{x_0}^{\pm\infty} |\alpha(t) - \alpha_{\pm}| dt$. Since $u' = 0$, we know that $u(x)$ is given by a 1-kink solution of the Sine-Gordon equation and hence $\alpha(x) = \frac{a}{2} \cos(\frac{u}{2}) + \frac{1}{2a} \cos(\frac{u}{2})$. If we choose the point x_0 to be the midpoint of the kink, then it is easy to show that the quantities $\int_{x_0}^{\pm\infty} |\alpha(t) - \alpha_{\pm}| dt$, and hence the constants C_{α} , can be bounded uniformly for all kinks with parameter a in some compact subinterval of $(0, 1)$. This verifies hypothesis (C1).

We now turn to (H2) and (C2). In this case, we must solve the equations

$$D_{(u',v',a)}F(u, v, u', v', a) \begin{pmatrix} \phi \\ \psi \\ \delta a \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \quad (13)$$

where

$$D_{(u',v',a)}F(u, v, u', v', a) = \begin{pmatrix} -\frac{a}{2} \cos(\frac{u+u'}{2}) + \frac{1}{2a} \cos(\frac{u-u'}{2}) & 1 & -\sin(\frac{u+u'}{2}) + a^{-2} \sin(\frac{u-u'}{2}) \\ \partial_x + \frac{1}{2a} \cos(\frac{u-u'}{2}) + \frac{a}{2} \cos(\frac{u+u'}{2}) & 0 & a^{-2} \sin(\frac{u-u'}{2}) + \sin(\frac{u+u'}{2}) \end{pmatrix} \quad (14)$$

We can solve the first equation to find ψ in terms of ϕ , f , g and a . Thus, we focus on the second equation

$$\partial_x \phi + \left(\frac{a}{2} \cos(\frac{u+u'}{2}) + \frac{1}{2a} \cos(\frac{u-u'}{2}) \right) \phi = g - b(x) \delta a \quad (15)$$

where $b(x) = a^{-2} \sin(\frac{u-u'}{2}) + \sin(\frac{u+u'}{2})$. This is remarkably similar to equation (6) except that the sign in front of the non-autonomous term $\left(\frac{a}{2} \cos(\frac{u+u'}{2}) + \frac{1}{2a} \cos(\frac{u-u'}{2}) \right)$ has changed. This means that we are in Case 2 of the Lemma, rather than Case 1, and in order to solve the equation we must check that the right hand side of (15) is orthogonal to the solution of the adjoint ODE. In this case, one can check that the solution of the adjoint ODE is $\mu(x) = \exp(\int_{x_0}^x \alpha(t) dt)$, where in this case $\alpha(x) = \left(\frac{a}{2} \cos(\frac{u+u'}{2}) + \frac{1}{2a} \cos(\frac{u-u'}{2}) \right)$. We can insure that the RHS of (15) is orthogonal to μ by picking δa appropriately, provided $\int_{-\infty}^{\infty} b(x) \mu(x) dx \neq 0$.

So far, we have not found any way of demonstrating that this integral is non-zero for an arbitrary choice of $u \in X_1$ and $u' \in X_0$. However, if we take $u' = 0$ and u equal to a 1-kink, we have $b(x) = 2(1 + a^{-2}) \exp(ax) / (1 + \exp(2ax)) > 0$. Likewise, $\mu(x) > 0$ for all x , so $\int_{-\infty}^{\infty} b(x) \mu(x) dx \neq 0$. Since both b and μ depend smoothly on u and u' , this condition will also hold for all u near the 1-kink and all u' near the zero-solution. Thus, Theorem 3.1, holds on such a neighborhood. This verifies hypothesis (H2). To check (C1) we need only derive a uniform estimate the solution ϕ of (15) for $u' = 0$ and a in some compact subinterval of $(0, 1)$. This follows in a very similar fashion to estimates on solutions of (8), and we leave the details as an easy exercise. \square

Remark We note that there is a natural path to attempt to build on the preceding result to establish the stability of an arbitrary k -kink solution. It is known that the

Bäcklund transformation (1) links the k -kink to the $k + 1$ -kink. Thus, we can repeat the above proof, this time considering

$$F : X_2 \times X_1 \times (0, 1) \rightarrow Z \quad (16)$$

and considering the the base point of our theorem (u', v') to be a 1-kink. Then the manifold H is the family of 1-kinks. If we then consider the linearizations $D_{(u,v)}F$ and $D_{(u',v',a)}F$, the verification of (H1), (H2), (C1) and (C2) proceed much as above. The only points that need to be checked are the uniform estimates on the inverses. These require uniform estimates on the analogues of (6) and (15). In our estimates of the stability of the 1-kink, we had the freedom to choose the point x_0 to be the center of the kink. This made it simple to establish uniform estimates. For multi-kink solutions, there is no such distinguished point and we need an analysis of the form of the 2-kink solution to show that for large time, the solution of these equations can be treated essentially by regarding the 2-kink as a sum of two 1-kink solutions which were estimated above. This program is carried out in detail to establish the stability of the multi-soliton solutions of the Toda lattice in [1]. Once the stability of the 2-kink solution is established, it can be used in conjunction with the Bäcklund transformation to establish the stability of the 3-kink, and so-on and so-forth.

3.2 The Toda Lattice

As a second application of Theorem 2.1 we study the orbital stability of the multi-soliton solutions in the Toda Lattice

$$\dot{q}_j = p_j; \quad \dot{p}_j = e^{q_{j-1}-q_j} - e^{q_j-q_{j+1}} \quad (17)$$

posed in the energy space $\ell^2 \times \ell^2$. Here q and p can be regarded as the position and momentum, respectively, of the j th particle in an infinite chain where neighboring particles resist compression quite strongly but resist extension only weakly.

Because of the lattice discreteness, some of the quantities that are most easily used to obtain multi-soliton solutions as constrained minimizers of a Lyapunov function in the PDE case are no longer conserved and so the Lyapunov function approach is not easily extended. For single solitons more detailed stability results have been established, specifically orbital stability with asymptotic phase, and moreover asymptotic stability in a weighted space [13]. The techniques used here have been a combination of the Bäcklund approach we take here with the dispersive approach mentioned in the introduction . Using a linearized version of the Bäcklund transform, asymptotic stability of multi-solitons has also been obtained, albeit in an exponentially weighted space [1].

The Bäcklund transform for the Toda lattice, written in our framework, is

$$F_j(q, p, q', p', \kappa) = \begin{pmatrix} p_j + e^{-(q'_j - q_j - \kappa)} + e^{-(q_j - q'_{j-1} + \kappa)} - 2 \cosh \kappa \\ p'_j + e^{-(q'_j - q_j - \kappa)} + e^{-(q_{j+1} - q'_j + \kappa)} - 2 \cosh \kappa \end{pmatrix} \quad (18)$$

where κ is a real parameter. As with the sine-Gordon equation considered above, the zero and one-kink solutions (and more generally the m - and $m + 1$ -kink solutions) are

related via (18) with the parameter κ controlling the amplitude (2κ) and speed $\frac{\sinh \kappa}{\kappa}$ of the additional kink.

Let $\Phi = \Psi$ be the propagator for the Toda lattice. Denote the m -kink solution with amplitude parameter $\kappa_1, \dots, \kappa_m$ and phase parameters $\delta_1, \dots, \delta_m$ by $(q^{\kappa_1, \dots, \kappa_m, \delta_1, \dots, \delta_m}, p^{\kappa_1, \dots, \kappa_m, \delta_1, \dots, \delta_m})$ or more concisely (q^m, p^m) when the parameters are understood. Given an m -kink solution (q^m, p^m) and an $m+1$ -kink solution (q^{m+1}, p^{m+1}) related via Bäcklund transform with parameter κ : $F(q^{m+1}, p^{m+1}, q^m, p^m, \kappa) = 0$, define the affine spaces X_m and X_{m+1} to be the space of ℓ^2 perturbations of (q^m, p^m) and (q^{m+1}, p^{m+1}) respectively.

Theorem 3.3. *The Toda- m -soliton is orbitally stable in ℓ^2 in the sense of Lyapunov: Let M denote the $2m$ -dimensional manifold of m -soliton states with phases free to vary in \mathbb{R} and with amplitudes constrained to any compact set. Let (Q^m, P^m) denote a point on M . For each $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $(q, p) \in \ell^2 \times \ell^2$ satisfies $\|(Q^m, P^m) - (q, p)\|_{\ell^2 \times \ell^2} < \delta$, then its forward evolution $\Phi(t)(q, p)$ under (17) satisfies $d(\Phi(t)(q, p), M) < \varepsilon$.*

The proof proceeds by induction. At the k^{th} stage of the induction, the invariant manifold H is a particular 1-manifold corresponding to the temporal evolution of a particular $k-1$ -soliton while M is a particular 2-manifold corresponding to the temporal evolution of a one-parameter family of k -solitons with the parameter corresponding to the initial phase of the additional soliton. At each state of induction we verify the hypotheses (H0)-(H2) and (C1)-(C2) without reference to the induction. On the other hand, in verifying the hypothesis that the invariant manifold H (M in the previous step) is stable in the sense of Lyapunov, we rely very much on the induction.

We first verify (H0). In the base case we set $(q', p') = (0, 0)$ and solve the equation $F(q, p, 0, 0, \kappa) = 0$ for $q_j = \log \frac{\cosh(\kappa j + \gamma)}{\cosh(\kappa(j+1) + \gamma)}$ and $p_j = e^\kappa \left(\frac{\cosh(\kappa j + \gamma)}{\cosh(\kappa j + \gamma + \kappa)} - 1 \right) + e^{-\kappa} \left(\frac{\cosh(\kappa j + \gamma + \kappa)}{\cosh(\kappa j + \gamma)} - 1 \right)$. More generally, we set $(q', p') = (q^m, p^m)$ and solve $F(q, p, q^m, p^m, \kappa) = 0$ for $(q, p) = (q^{m+1}, p^{m+1})$. To do this computation from scratch is a significant undertaking; we rely on the early literature in the history of the Toda lattice [21].

Note that in this formulation the differences in the asymptotic values satisfy $q'_{-\infty} - q_{-\infty} = 0$ and $q'_{\infty} - q_{\infty} = -2\kappa$. We now check the hypotheses (H1)-(H2).

To check (H1) we differentiate F and obtain

$$D_{(p,q)} F(p, q, p', q', \kappa) = \begin{pmatrix} e^{-(q'-q-\kappa)} - e^{-(q-q'_-+\kappa)} & I \\ e^{-(q'-q-\kappa)} - e^{-(q_+-q'+\kappa)} S & 0 \end{pmatrix}.$$

Here S is the shift operator $(Sq)_j = q_{j+1}$ and the symbols q_{\pm} denote the shifted sequences $S^{\pm 1}q$. We must study

$$\begin{pmatrix} e^{-(q'-q-\kappa)} - e^{-(q-q'_-+\kappa)} & I \\ e^{-(q'-q-\kappa)} - e^{-(q_+-q'+\kappa)} S & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

obtaining solvability conditions for ϕ and ψ in terms of f and g .

Note that ϕ is given as a linear combination of f and the action of a multiplication operator (bounded from $\ell^2 \rightarrow \ell^2$) on ψ . Thus we restrict attention to the second row, which is a first order linear difference equation

$$\psi_+ = e^{-(2q' - q - q_+ - 2\kappa)}\psi + e^{(q_+ - q' + \kappa)}g$$

This equation is of the form $\psi_{j+1} = \alpha_j\psi_j + f_j$ which can be solved explicitly via a summing factor. A discrete analog of Lemma 3.2 holds with the relevant numbers now $|\alpha_{\pm\infty}|$ rather than $\text{sgn}(\alpha_{\pm\infty})$:

Lemma 3.4. *Consider the first order linear recursion $u_{n+1} - \alpha_n u_n = f_n$ and suppose that the limits $\alpha_{\pm} = \lim_{n \rightarrow \pm\infty} \alpha_n$ exist.*

1. *Suppose that $|\alpha_{-\infty}| > 1 > |\alpha_{\infty}|$ and that $\sum_{n=1}^{\infty} |\alpha_n - \alpha_+| < \infty$ as well as $\sum_{n=-\infty}^0 |\alpha_n - \alpha_-| < \infty$. Then there is a constant C_{α} depending on $\sum_{n=0}^{\pm\infty} |\alpha_n - \alpha_{\pm}|$ such that the unique solution u with $u_0 = 0$ satisfies $\|u\|_{\ell^2} \leq C_{\alpha}\|f\|_{\ell^2}$.*
2. *Suppose that $|\alpha_{-\infty}| < 1 < |\alpha_{\infty}|$ and that $\sum_{n=1}^{\infty} |\alpha_n - \alpha_+| < \infty$ as well as $\sum_{n=-\infty}^0 |\alpha_n - \alpha_-| < \infty$. Then there is a constant C_{α} depending on $\sum_{n=0}^{\pm\infty} |\alpha_n - \alpha_{\pm}|$ such that if f satisfies $\sum_{n \in \mathbb{Z}} f_n \phi_n = 0$ for ϕ the unique (up to scalar multiple) solution of the adjoint equation $\phi_{n-1} - \alpha_n \phi_n = 0$ then there is a unique choice of u_0 such that $u \in \ell^2$ and for this choice $\|u\|_{\ell^2} \leq C_{\alpha}\|f\|_{\ell^2}$.*

Its proof is similar to the proof of Lemma 3.2 and can be regarded as a consequence of the theory of exponential dichotomies or as an exercise for the reader.

We compute $\alpha_{\pm} = e^{-2(q_{\pm\infty} - q_{\pm\infty} - \kappa)} = e^{\mp 2\kappa}$. To check the hypotheses of the lemma we must show that $\alpha - \alpha_{\pm}$ is summable over $\pm\mathbb{N}$. To that end, we compute $a - a_+ = e^{-2\kappa} \left(e^{-(2q' - q' - q_+ - 4\kappa)} - 1 \right) \approx q' - (q + 2\kappa) + q' - (q_+ + 2\kappa)$. It follows from Lemma 2.1 in [1] together with the fact that $q_+ - q$ is exponentially localized [4], that this quantity is in ℓ^1 and moreover approaches 2κ in ℓ^1 exponentially fast as $t \rightarrow \infty$ when q and q' are m and $m - 1$ soliton solutions respectively. We remark that this is one place where the restriction to a compact set of κ is necessary. These estimates are not uniform in the limit as any of the wave speeds goes to zero.

The coefficients α_{\pm} satisfy $\alpha_- > 1 > \alpha_+$ and thus we are in case (1) of the lemma. This establishes (H1). In the base case this establishes (C1) as well; after all uniform bounded are not hard to obtain on $H \times M \cong \{(0, 0)\} \times \mathbb{R}/\mathbb{Z}$. The constant C_{α} given in the lemma depends only upon $\sum_{n>0} \alpha_n - \alpha_+$ which has a limit in ℓ^1 as $t \rightarrow \infty$ and hence is bounded uniformly as $t \rightarrow \infty$. This establishes that the constant C_{α} is bounded along any trajectory in $H \times M \cap F^{-1}(0)$ that corresponds to the temporal evolution of the k -soliton state under (17), i.e. it establishes (C1) when the manifold H in the theorem is the orbit of a k -soliton state.

We now check (H2)-(C2). We compute

$$D_{q', p', \kappa} F(q, p, q', p', \kappa) = \begin{pmatrix} e^{-(q - q'_+ + \kappa)} S^{-1} - e^{-(q' - q - \kappa)} & 0 & e^{-(q' - q' \kappa)} - e^{-(q - q'_+ + \kappa)} - 2 \sinh \kappa \\ e^{-(q_+ - q + \kappa)} - e^{-(q' - q - \kappa)} & I & e^{-(q' q - \kappa)} - e^{-(q_+ - q' + \kappa)} - 2 \sinh \kappa \end{pmatrix}$$

and solve

$$\begin{pmatrix} e^{-(q-q'_-+\kappa)}S^{-1} - e^{-(q'-q-\kappa)} & 0 & e^{-(q'-q'\kappa)} - e^{-(q-q'_-+\kappa)} - 2\sinh \kappa \\ e^{-(q_+-q+\kappa)} - e^{-(q'-q-\kappa)} & I & e^{-(q'q-\kappa)} - e^{-(q_+-q'+\kappa)} - 2\sinh \kappa \end{pmatrix} \begin{pmatrix} \phi \\ \psi \\ \delta\kappa \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

The second row gives ψ as a linear combination of g , $\delta\kappa$ and a bounded multiplication operator acting on ϕ thus we restrict attention to the first row which is a first order, non-autonomous linear recurrence for ϕ . The coefficient $\alpha = e^{(2q-q'-q'_-+2\kappa)}$ is well behaved just like the similar coefficient we studied when verifying (H1)-(C1). In particular, the hypotheses of Lemma 2 are satisfied and we are in case 2. To verify (H2) we must show that the (2,3) entry of the derivative matrix is not orthogonal to the kernel of the adjoint. At first glance it appears that one must dirty one's hands with the explicit form of the m -soliton solution in order to do this computation. However, it was shown in [1] that the quantity of interest is independent of time and hence it suffices to analyze the quantity in the limit $t \rightarrow \infty$ where it reduces to the computation for the vacuum-kink pairing. This computation is not difficult and has been checked in [1]. To check (C2) we again make use of the fact that a multi-soliton decomposes into the linear superposition of soliton solutions in ℓ^1 .

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