

# An explanation of metastability in the viscous Burgers equation with periodic boundary conditions via a spectral analysis

January 15, 2016

## Abstract

A “metastable solution” to a differential equation typically refers to a family of solutions for which nearby initial data converges to the family much faster than evolution along the family. Metastable families have been observed both experimentally and numerically in various contexts; they are believed to be particularly relevant for organizing the dynamics of fluid flows. In this work we propose a candidate metastable family for the Burgers equation with periodic boundary conditions. Our choice of family is motivated by our numerical experiments. We furthermore explain the metastable behavior of the family without reference to the Cole–Hopf transformation, but rather by linearizing the Burgers equation about the family and analyzing the spectrum of the resulting operator. We hope this may make the analysis more readily transferable to more realistic systems like the Navier–Stokes equations. Our analysis is motivated by ideas from singular perturbation theory and Melnikov theory.

## 1 Introduction

In the study of differential equations one often is interested in understanding the long-term asymptotic behavior of solutions; the long term behavior could include, for example, convergence to a periodic orbit or a steady-state. One typical approach is to prove the existence of a particular solution and then to argue that nearby initial data converge to that solution; in the case of a steady-state or periodic orbit, such arguments often involve computations of the linear spectrum.

In this work we address a slightly different question, which arises when the asymptotic state only emerges after a “long” time; in this case, it may be that the intermediate transient behavior of the system is physically relevant. In other words, we are not interested in what the asymptotic state is, but how a wide class of initial data approach it. To address this question we analyze what are known as “metastable” solutions. The term metastable solution often refers to a family of profiles with the following properties: (1) the asymptotic behavior of the system is not contained within the family; (2) a profile in the family evolves within the family towards an asymptotic state of the system; (3) “nearby” initial data remain near the family for all forward times; and (4) the timescale on which nearby initial data approach the family is much faster than the evolution within the family towards the asymptotic state. Property (4) is what makes metastable solutions of physical interest.

Metastable solution families are of particular interest in fluid dynamics. For example, in the Navier–Stokes equation with periodic boundary conditions

$$\begin{aligned} \partial_t \vec{u} &= \nu \Delta \vec{u} - \vec{u} \cdot \nabla \vec{u} + \nabla p, & \nabla \cdot \vec{u} &= 0, & \vec{u} &\in \mathbb{R}^2, \nu \ll 1 \\ \vec{u}(x, y, t) &= \vec{u}(x + 2\pi, y, t), & \text{and} & & \vec{u}(x, y, t) &= \vec{u}(x, y + 2\pi, t), \end{aligned} \tag{1.1}$$

which describes two-dimensional viscous fluid flows, vortex pairs known as “dipoles” were numerically observed [10, 17]; the dipoles emerge quickly and persist for long times before eventually converging to the trivial state.

The metastable states described in [10, 17] are characterized in terms of their vorticity  $\omega$ , defined as  $\omega := \nabla \times \vec{u}$ . In [17] a second metastable family known as “bar” states—solutions with constant vorticity in one spatial direction and periodic vorticity in the other—were observed; which of the two candidate metastable families dominates the dynamics depends on the initial data.

A related context in which metastability has been observed and studied is Burgers equation. Although the Burgers equation is unphysical, it is nevertheless relevant to fluid dynamics since it is, in some sense, the one-dimensional simplified analog of the Navier–Stokes equation. Thus, one often uses the Burgers equation as a test case for Navier–Stokes: one hopes that by first observing and analyzing some phenomenon in the Burgers equation, that insight can be translated into an understanding of related phenomena in Navier–Stokes. Metastable solutions in Burgers equation were observed numerically in the viscous Burgers equation on an unbounded domain [7] in the so-called “scaling variables”

$$\partial_\tau w = \nu \partial_\xi^2 w + \frac{1}{2} \partial_\xi(\xi w) - w w_\xi \quad w \in \mathbb{R}, \nu \ll 1. \quad (1.2)$$

The scaling variables

$$\xi = \frac{x}{\sqrt{1+t}}, \quad \tau = \ln(t+1), \quad \text{and} \quad u(x, t) = \frac{1}{\sqrt{1+t}} w \left( \frac{x}{\sqrt{1+t}}, \ln(1+t) \right)$$

have been defined so that a diffusion wave—a strictly positive triangular profile which approaches zero for  $|x| \rightarrow \infty$ —is a steady state solution to (1.2) (otherwise, all solutions to Burgers equation in the unscaled variables  $\partial_t u = \nu \partial_x^2 u - uu_x$  approach the zero solution). In [7] the authors observe that “diffusive N-waves”—profiles with a negative triangular region immediately followed by a positive triangular region so that the profile resembles a lopsided backwards “N”—quickly emerge before the solution converges to a diffusion wave.

Burgers equation is much more amenable to analysis than the Navier–Stokes equation and there has been a fair amount of theoretical work to explain the observations of [7]. Already in [7], the authors used the Cole–Hopf transformation to derive an analytical expression for the diffusive N-waves. In [1], the authors provide a more dynamical systems motivated explanation of metastability. First they constructed a center-manifold for (1.2) consisting of the diffusion waves, denoted  $A_M(\xi)$ , which is parametrized by the solution mass. Each of these diffusion waves represents the long-time asymptotic state of all integrable solutions with initial mass  $M$  and they are also fixed points in the scaling variables. Through each of these fixed points there is a one-dimensional manifold, parameterized by  $\tau$ , consisting of exactly the diffusive  $N$ -waves. Then, using the Cole–Hopf transformation, the authors show that solutions converge toward the manifold of  $N$ -waves on a time scale of order  $\tau = \mathcal{O}(|\ln \nu|)$ , that solutions remain near  $w_N(\xi, \tau)$  for all future times, and that that evolution along  $w_N(\xi, \tau)$  towards  $A_M(\xi)$  is on a time scale of the order  $\tau = \mathcal{O}(1/\nu)$ . In particular, convergence to the family is faster than the subsequent evolution along the family. We emphasize that their analysis makes strong use of the Cole–Hopf transformation.

In [2] the authors proposed an explanation of the metastability of the bar-states of (1.1) as follows. They first propose as candidates for the metastable family the exact solutions of the Navier–Stokes equations with vorticity distribution

$$\omega^b(x, y, t) = e^{-\nu t} \cos(x)^1,$$

which is again parametrized by time. Solutions in this family converge to the long-time limit (which is the zero solution in this case) on the viscous time scale  $t \sim \frac{1}{\nu}$ . In order to understand the convergence of nearby initial data to the metastable family, the authors linearize the vorticity formulation of (1.1)

$$\partial_t \omega = \nu \Delta \omega - \vec{u} \cdot \nabla \omega, \quad \vec{u} = (-\partial_y \Delta^{-1} \omega, \partial_x \Delta^{-1} \omega). \quad (1.3)$$

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<sup>1</sup>Alternatively, the bar state could be  $\tilde{w}^b(x, y, t) = e^{-\nu t} \sin(x)$ , or the solution could instead be periodic in the  $y$  direction and constant in the  $x$  direction.

about  $\omega^b(x, y, t)$ . The linearization results in a nonlocal time-dependent linear operator

$$\mathcal{L}(t) = \nu\Delta - ae^{-\nu t} \sin x \partial_y (1 + \Delta^{-1}).$$

Using hypercoercivity techniques motivated by the work of Villani [14] and Gallagher, Gallay, and Nier [6], the authors show that solutions to a modified operator  $\mathcal{L}^a(t) = \nu\Delta - ae^{-\nu t} \sin x \partial_y$ , which differs from  $\mathcal{L}(t)$  by removing the non-local, but relatively compact, term, decay with rate at least  $e^{-\sqrt{\nu}t}$ . Additionally, they provide numerical evidence that the real part of the least negative eigenvalue for the nonlocal operator  $\mathcal{L}(t)$  is proportional to  $\sqrt{\nu}$ . These arguments, in combination with the fact that the rate of decay of solutions to (1.3) to zero is given by the much slower viscous time scale provides a mathematical explanation for the metastable behavior of the family of bar states.

What is notable is that the mechanism for metastability as well as the relevant time scales are different in each case [1] versus [2]. Thus, the goal of this work is to re-visit the Burgers equation, albeit with periodic boundary conditions so that the boundary conditions are more similar to those of (1.1), in order to devise a mathematical explanation for metastability which is more easily transferable to Navier–Stokes. To that end, we intentionally avoid the Cole–Hopf transformation and instead use spectral analysis from the linearization about the candidate metastable family. We find that the convergence to the metastable family does not depend, to leading order, on  $\nu$ , even though our analysis depends on the presence of the viscosity term in the equation and thus the calculations below do not apply to the inviscid equation. This is in contrast to the results from [2] for the Navier–Stokes equation in which the rate of approach toward the metastable solutions occurs at a  $\nu$  dependent rate, albeit a much faster rate than the  $\nu$  dependent time of approach toward the final asymptotic state.

From a technical perspective, the linearization about the metastable states leads to a singularly perturbed eigenvalue problem, in which the perturbation parameter is the viscosity  $\nu$ . Our strategy is to construct the eigenfunction-eigenvalue pairs in each of two spatial scaling regimes (denoted the “slow” and “fast” scales) and then to glue the eigenfunction pieces together in an appropriate “overlap” region (see Figure 3 for a schematic representation). We show, in fact, that the eigenvalues are given, to leading order, by the slow-scale eigenvalues; the rigorous “gluing” of the fast and slow solutions is done with the aid of a Melnikov-like computation which gives the first order correction of the eigenvalues. The use of such Melnikov-like computations for piecing together solutions has a long tradition, generally called Lin’s method [8], which has been applied to the construction of eigenfunctions in, for example, [12]. The idea of piecing together slow and fast eigenfunctions in a singularly perturbed eigenvalue problem follows, for example, from [5].

It is worth noting another context in which singularly perturbed eigenvalue problems have arisen in connection with a slightly different type of metastability, including in variants of Burger’s equation. In [13, 15] metastability refers to the very slow motion of internal layers in nearly steady states of reaction diffusion equations and diffusively perturbed conservation laws. While different in details and physical context, the notion of metastability in these papers is similar in spirit to our discussion in that it also describes the slow motion along a family of solutions (in this cases, solutions in which the internal layer occurs at different positions) before the system reaches its final state. The motion of those internal layers is explained by an exponentially small shift in the zero eigenvalue of the operator describing the equation linearized about a stationary state. In contrast, in our problem, the zero eigenvalue is unchanged, regardless of which member of the family of metastable solutions we linearize around, but the remaining eigenvalues (or at least the four additional eigenvalues that we compute here) undergo exponentially small shifts.

Another recent study of metastability in the Navier–Stokes equation, which is similar to our work in context, but very different in methods is the study of the inviscid limit of the Navier–Stokes equations in the neighborhood of the Couette flow, by Bedrossian, Masmoudi and Vicol [4] (see also [3]). In this paper the authors prove an enhanced stability of the Couette flow by using carefully chosen energy functionals. They prove that for times less than  $\mathcal{O}(Re^{1/3})$ , the system approaches the Couette flow in a way governed by the inviscid limit (i.e. the Euler equations) while for time scales longer than this viscosity effects dominate; here  $Re$  is the Reynold’s

number of the flow. Since our results show that our metastable family attracts nearby solutions at a rate which is, to leading order, independent of the viscosity, we believe that they are analogous to the initial phase of the evolution analyzed in [4] in which inviscid effects dominate. It would be interesting to see if the transition to viscosity dominated evolution could be observed in this Burgers equation context as well.

## 2 Set-up and statement of main results

In this section we discuss our candidate family of metastable solutions, denoted  $W(x, t; \nu, x_0, c)$ , to the viscous Burgers equation with periodic boundary conditions

$$\begin{aligned} \partial_t u &= \nu \partial_x^2 u - uu_x & \nu \ll 1, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+ \\ u(x, 0) &= u_0(x) & u_0 \in H_{per}^1([0, 2\pi)) \\ u(x + 2\pi, t) &= u(x, t). \end{aligned} \tag{2.1}$$

We also present numerical and analytical justification for our choice. The analytical justification given in Section 2.2 relies, again, heavily on the Cole-Hopf transformation. Thus, although it provides powerful evidence for the behavior of solutions near  $W(x, t; \nu, x_0, c)$ , the result provides no insight into techniques one might use to analyze Navier–Stokes. Thus we provide an alternative explanation which relies on information about the spectrum of the linear operator obtained from linearizing (2.1) about the metastable family  $W(x, t; \nu, x_0, c)$ ; the statement and discussion of these results can be found in Sections 2.4 and 2.5. In what follows we make the technical assumption that the primitive of  $u_0(x)$  attains a unique global maximum on  $[0, 2\pi)$ . We remark that this assumption is generic since if the primitive of  $u_0(x)$  does not attain a global maximum on  $[0, 2\pi)$  then for all  $\epsilon > 0$  there exists a function  $v(x)$  with  $\|v\|_{H_{per}^1} \leq \epsilon$  such that the primitive of  $u_0(x) + v(x)$  does attain a global maximum, where

$$\|v\|_{H_{per}^1}^2 = \int_0^{2\pi} [v(x)^2 + v'(x)^2] dx$$

is the usual periodic  $H^1$  norm.

### 2.1 Family of metastable solutions

It is well known, using the Cole-Hopf transformation, that

$$u(x, t) = -2\nu \frac{\psi_x(x, t)}{\psi(x, t)} \tag{2.2}$$

is a solution to Burgers on the real line if  $\psi(x, t)$  satisfies the heat equation

$$\psi_t = \nu \psi_{xx} \quad \nu \ll 1, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+. \tag{2.3}$$

A family of periodic solutions to (2.3) can be constructed by placing heat sources on the real line spaced  $2\pi$  apart centered at  $x = \pi(2n - 1)$

$$\psi^W(x, t; \nu) := \frac{1}{\sqrt{4\pi\nu t}} \sum_{n \in \mathbb{Z}} \exp \left[ \frac{-(x + \pi - 2n\pi)^2}{4\nu t} \right]. \tag{2.4}$$

Then every function in the family

$$W_0(x, t; \nu) := -2\nu \frac{\psi_x^W}{\psi^W} = \frac{1}{t} \frac{\sum_{n \in \mathbb{Z}} (x + \pi - 2n\pi) \exp \left[ \frac{-(x + \pi - 2n\pi)^2}{4\nu t} \right]}{\sum_{n \in \mathbb{Z}} \exp \left[ \frac{-(x + \pi - 2n\pi)^2}{4\nu t} \right]} \tag{2.5}$$

is  $2\pi$ -periodic and hence a solution to (2.1). We have denoted solutions (2.5) by  $W_0$  to indicate the fact that one can find them in, for example, the classic text by G.B. Whitham [16, §4.6]. Using formula (2.5) one can check that  $W_0(n\pi, t; \nu) = 0$  and that  $W_0$  is an odd function about  $n\pi$ , for  $n \in \mathbb{Z}$ .

The family of solutions (2.5) is parametrized by  $t$ . We can extend the family to include two additional parameters as follows. Firstly, we can replace  $x$  by  $x - x_0$ , effectively shifting the origin of the  $x$ -axis. Next, suppose  $u(x, t)$  is a solution to (2.1). Then  $u_c(x, t) := c + u(x - ct, t)$  solves (2.1) as well since

$$\partial_t u_c = \partial_t u - c \partial_x u = \nu \partial_x^2 u_c - (u_c - c) \partial_x u_c - c \partial_x u = \nu \partial_x^2 u_c - u_c (u_c)_x.$$

Thus we define an extension of (2.5) by  $W(x, t; \nu, x_0, c) := c + W_0(x - x_0 - ct, t; \nu)$ . We remark that if  $\psi(x, t)$  is periodic,

$$\int_{-\pi}^{\pi} -2\nu \partial_x \psi(x, t) dx = 0$$

and thus, since

$$\int_{-\pi}^{\pi} W(x, t; \nu, x_0, c) dx = 2\pi c,$$

$W(x, t; \nu, x_0, c)$  can not be obtained via the Cole-Hopf transformation of a periodic function unless  $c = 0$ .

We will need the following estimates of  $W_0$  and its derivatives.

**Proposition 2.1** *Fix  $\nu > 0$ ,  $0 < \epsilon_0 \ll 1$ . Then there exists  $0 < C(\epsilon_0) < \infty$  such that*

$$\begin{aligned} \sup_{|x| \leq \pi} \left| W_0(x, t; \nu) - \frac{1}{t} \left[ x - \pi \tanh \left( \frac{\pi x}{2\nu t} \right) \right] \right| &\leq \frac{C(\epsilon_0)}{t} e^{-1/\nu t} \\ \sup_{|x| \leq \pi} \left| \partial_x W_0(x, t; \nu) - \frac{1}{t} \left[ 1 - \frac{\pi^2}{2\nu t} \operatorname{sech}^2 \left( \frac{\pi x}{2\nu t} \right) \right] \right| &\leq \frac{C(\epsilon_0)}{t^2} e^{-1/\nu t} \\ \sup_{|x| \leq \pi} \left| \partial_t W_0(x, t; \nu) - \frac{1}{t^2} \left[ -x + \pi \tanh \left( \frac{\pi x}{2\nu t} \right) + \frac{\pi^2 x}{2\nu t} \operatorname{sech}^2 \left( \frac{\pi x}{2\nu t} \right) \right] \right| &\leq \frac{C(\epsilon_0)}{t^3} e^{-1/\nu t} \end{aligned} \quad (2.6)$$

for all  $0 < \nu t < \epsilon_0$ .

We remark that since  $W_0(x, t; \nu)$  is periodic, these  $L^\infty$  estimates can be converted into  $L^p_{per}$  estimates for any  $1 \leq p < \infty$ .

**Proof.** Due to the fact that  $W_0(x, t; \nu)$  is an odd function centered about  $x = 0$ , we prove the estimates for  $x \in [0, \pi]$ . Define

$$S(x, t; \nu) := -1 + \frac{2 \sum_{n \in \mathbb{Z}} n \exp \left[ \frac{-(x + \pi - 2n\pi)^2}{4\nu t} \right]}{\sum_{n \in \mathbb{Z}} \exp \left[ \frac{-(x + \pi - 2n\pi)^2}{4\nu t} \right]}$$

so that

$$\begin{aligned} W_0(x, t; \nu, -\pi) &= \frac{x}{t} - \frac{\pi}{t} S(x, t; \nu) \\ \partial_x W_0(x, t; \nu, -\pi) &= \frac{1}{t} - \frac{\pi}{t} S_x(x, t; \nu) \\ \partial_t W_0(x, t; \nu, -\pi) &= -\frac{x}{t^2} + \frac{\pi}{t^2} S(x, t; \nu) - \frac{\pi}{t} S_t(x, t; \nu). \end{aligned}$$

Thus it remains to estimate  $S(x, t; \nu)$  and its derivatives. We factor  $\exp[-(x + \pi)^2/4\nu t]$  out of both the numerator and denominator, define

$$\exp_n(x, t; \nu) := \exp \left[ -\pi[-nx + n^2\pi - n\pi]/\nu t \right] = \left\{ \begin{array}{ll} \exp \left[ \frac{-\pi n[(n-1)\pi - x]}{\nu t} \right] & : n \geq 0 \\ \exp \left[ \frac{\pi n[(-n+1)\pi + x]}{\nu t} \right] & : n \leq 0 \end{array} \right\}, \quad (2.7)$$

and rearrange to get

$$\begin{aligned} &= \frac{-\exp\left[\frac{-\pi x}{2\nu t}\right] + \exp\left[\frac{\pi x}{2\nu t}\right] + \exp\left[\frac{-\pi x}{2\nu t}\right] \sum_{n \neq 0,1} (2n-1) \exp_n(x; t, \nu)}{\exp\left[\frac{-\pi x}{2\nu t}\right] + \exp\left[\frac{\pi x}{2\nu t}\right] + \exp\left[\frac{-\pi x}{2\nu t}\right] \sum_{n \neq 0,1} \exp_n(x; t, \nu)} \\ &= \tanh\left(\frac{\pi x}{2\nu t}\right) + \mathcal{R}(x; \nu, t) \end{aligned}$$

where

$$\mathcal{R}(x; \nu, t) := \frac{\exp\left[\frac{-\pi x}{2\nu t}\right] \sum_{n \neq 0,1} [2n-1 - \tanh\left(\frac{\pi x}{2\nu t}\right)] \exp_n(x; t, \nu)}{\exp\left[\frac{-\pi x}{2\nu t}\right] \sum_{n \in \mathbb{Z}} \exp_n(x; t, \nu)}$$

Define  $r := \exp[-\pi^2/\nu t]$ ; we have that  $0 \leq r < 1$  for all  $0 \leq \nu t \leq \epsilon_0$ . Then, using (2.7), we see that for all  $x \in [0, \pi]$

$$\exp_n(x; t, \nu) \leq r^{|n|} \quad \forall n \neq 0, 1, 2$$

and

$$\exp\left[\frac{-\pi x}{2\nu t}\right] \exp_2(x; t, \nu) = \exp\left[\frac{-\pi(4\pi - 3x)}{2\nu t}\right] \leq \exp\left[\frac{-\pi^2}{2\nu t}\right] = r^{1/2}.$$

Using the fact that the denominator of  $\mathcal{R}$  greater than or equal to one since it is a sum of positive terms and the leading term

$$\exp\left[\frac{-\pi x}{2\nu t}\right] \exp_1(x; \nu, t) = \exp\left[\frac{\pi x}{2\nu t}\right] \geq 1 \quad \forall x \in [0, \pi],$$

we find

$$\begin{aligned} |\mathcal{R}(x; \nu, t)| &\leq 4r^{1/2} + \exp\left[\frac{-\pi x}{2\nu t}\right] \sum_{n \neq 0,1,2} 2(|n| + 1)r^{|n|} \\ &\leq 4r^{1/2} + 4\frac{r(2-r)}{(1-r)^2}. \end{aligned}$$

Thus, there exists  $0 < C(\epsilon_0) < \infty$  such that  $|\mathcal{R}(x; \nu, t)| \leq C(\epsilon_0)e^{-\pi^2/2\nu t}$  for all  $0 \leq \nu t \leq \epsilon_0$  and  $x \in [0, \pi]$ . The same transformations and estimates give

$$\left|S_x(x, t; \nu) - \frac{\pi}{2\nu t} \operatorname{sech}^2\left(\frac{\pi x}{2\nu t}\right)\right| \leq \frac{C(\epsilon_0)}{t} e^{-1/\nu t} \quad \text{and} \quad \left|S_t(x, t; \nu) + \frac{\pi x}{2\nu t^2} \operatorname{sech}^2\left(\frac{\pi x}{2\nu t}\right)\right| \leq \frac{C(\epsilon_0)}{t^2} e^{-1/\nu t}$$

after potentially making  $C(\epsilon_0)$  larger. ■

## 2.2 Solutions via the Cole-Hopf transformation

Based on our numerical simulations (see Section 2.3), we anticipate that solutions to (2.1) rapidly approach a profile in the family  $W(x, t; \nu, x_0, c)$ , and that the specific member in the family that the solution approaches depends on the initial data  $u_0(x)$ . In Section 2.1 we discussed the Cole-Hopf transformation but did not take the initial data into account; we address the initial value problem now and show how the initial data can be used to determine which specific profile  $W(x, t; \nu, x_0, c)$  the solution is expected to approach.

A solution  $u(x, t)$  given by the Cole-Hopf transformation (2.2) will satisfy the Burgers equation on the real line with initial data  $u_0(x)$  provided  $\psi(x, t)$  satisfies the initial value problem

$$\begin{aligned} \psi_t &= \nu \psi_{xx} & \nu &\ll 1, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}^+ \\ \psi(x, 0) &= \psi_0(x) = e^{\frac{1}{2\nu} F(x; u_0)}, & F(x; u_0) &:= - \int_0^x u_0(s) ds. \end{aligned} \tag{2.8}$$

Solutions to (2.8) can be expressed as a convolution with the heat kernel  $G_t : \mathbb{R} \rightarrow \mathbb{R}^+$

$$\psi(x, t) = \int_{-\infty}^{\infty} \psi_0(y) G_t(x - y) dy = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} e^{\frac{1}{2\nu} [F(y; u_0) - \frac{1}{2t}(x-y)^2]} dy.$$

As was argued in [11], if one additionally assumes that  $\int_0^{2\pi} u_0(s) ds = 0$  then  $\psi_0(x)$  is  $2\pi$ -periodic, and hence so are  $\psi(x, t)$  and

$$u_0^{CH}(x, t; \nu, u_0) := -2\nu \frac{\psi_x(x, t)}{\psi(x, t)} = \frac{1}{t} \frac{\int_{-\infty}^{\infty} (x - y) \exp \left[ \frac{1}{2\nu} \left( -\frac{(x-y)^2}{2t} + F(y; u_0) \right) \right] dy}{\int_{-\infty}^{\infty} \exp \left[ \frac{1}{2\nu} \left( -\frac{(x-y)^2}{2t} + F(y; u_0) \right) \right] dy}.$$

Thus  $u_0^{CH}(x, t; \nu, u_0)$  is a solution to the periodic problem (2.1) with initial data  $u_0^{CH}(x, t; \nu, u_0) = u_0(x)$ . We assume that  $F(y; u_0)$  has a single global maximum in the interval  $y \in [-\pi, \pi)$  located at  $y = y_0$

$$y_0 = \operatorname{argmax}_{y \in [-\pi, \pi]} \left( - \int_0^y u_0(s) ds \right).$$

Then the solution  $u_0^{CH}$  can be estimated as

$$u_0^{CH}(x, t; \nu, u_0) = \frac{1}{t} \left[ x - y_0 - \pi - \pi \tanh \left( \frac{\pi(x - y_0 - \pi)}{2\nu t} \right) + \mathcal{O} \left( \sqrt{\nu} + \frac{1}{t} \right) \right], \quad (2.9)$$

which can be seen by using, for example, Laplace's method; since the goal of this work is to get away from the Cole-Hopf transformation, we leave the details to the reader. Comparison of (2.9) with (2.6) indicates that solutions to (2.1) will asymptotically approach  $W_0(x, t; \nu, x_0)$ , and that  $x_0$  is close to  $y_0 + \pi$ , where  $y_0$  depends on the initial data. If  $\int_0^{2\pi} u_0(s) ds = c \neq 0$  then

$$u^{CH}(x, t; \nu, u_0, c) = c + u_0^{CH}(x - ct, t; \nu, u_0).$$

### 2.3 Numerical results

The discussion in Sections 2.1 and 2.2 indicates that  $W(x, t; \nu, x_0, c)$  should be our candidate metastable solution. Numerical simulations indicate the same result. We numerically computed solutions to (2.1) in Python using Gudonov's scheme for conservative PDEs. Letting  $h = dx$  and  $k = dt$ , the CFL condition is

$$k = \min \left\{ \frac{\lambda h}{\max[u(x, 0)]}, \lambda h^2 \right\}$$

for  $\lambda < 1$ . We used  $\lambda = 0.5$ . The initial condition  $u(x, 0)$  was given by

$$u(x, 0) = a_0 + \sum_{j=1}^m [a_j \sin(jx) + b_j \cos(jx)],$$

where  $m$  is the number of modes and the coefficients  $a_j$  were randomly generated. Due to the symmetry of the modes for  $j \geq 1$ , the mean of  $u(x, 0)$ , denoted  $\overline{u(x, 0)}$ , is given by  $a_0$ ; furthermore, due to the periodic boundary conditions the mean of any solution is preserved since

$$\frac{d}{dt} \bar{u} = \int_{-\pi}^{\pi} u_t dx = \int_{-\pi}^{\pi} [\nu u_{xx} - u u_x] dx = \left[ \nu u_x - \frac{1}{2} u^2 \right] \Big|_{-\pi}^{\pi} = 0.$$

The time series for a solution with  $a_0 = 0$  is shown in Figure 1. We find that  $u(x, t)$  rapidly approaches a solution  $W_0(x, t; \nu, x_0)$ , defined in (2.5); for all future times, the solution converges to zero in a manner resembling the behavior of  $W_0(x, t; \nu, x_0)$ . When  $a_0 \neq 0$  we find that the solution centers around  $a_0$  moves to the left for  $a_0 < 0$  and to the right for  $a_0 > 0$ . Although we show only one sample time series here, we ran multiple experiments with different initial conditions; our results indicate that the evolution of a wide class of initial data evolve in a qualitatively similar fashion to that shown in Figure 1.

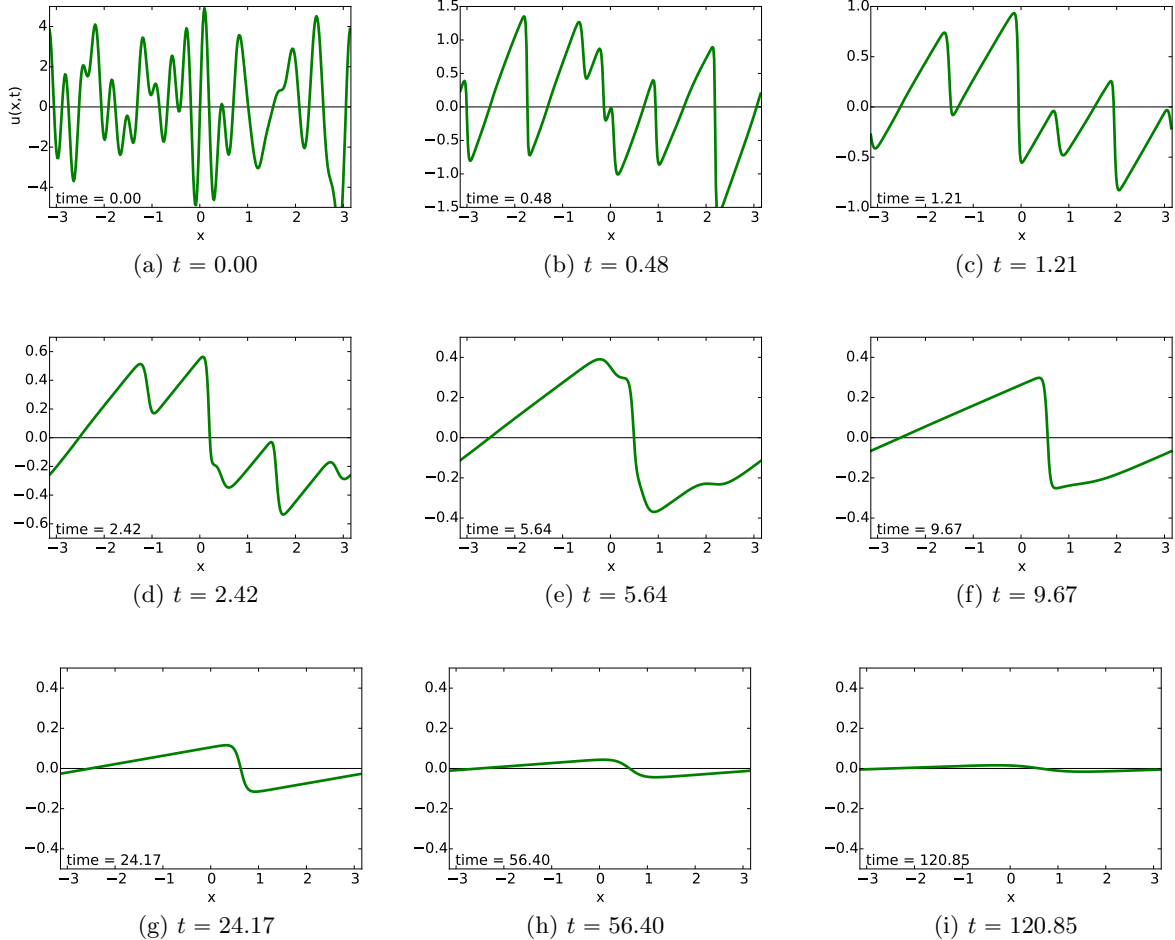


Figure 1: A numerically computed solution to (2.1) with  $\nu = 0.008$  and random initial data. Solution computed in Python using Gudonov's method with  $h = 2\pi/350$ , CFL constant  $\lambda = 0.5$ ,  $m = 20$  modes for the random initial data,  $\bar{u}(x, t) = a_0 = 0$ , and  $y_0 := \operatorname{argmax}_{x \in [-\pi, \pi]} \int^x u(y, 0) dy \approx -2.53$ . We find that  $u(x, t)$  rapidly approaches a solution  $W_0(x, t; \nu, x_0)$  and then converges to 0 in a manner consistent with the time evolution of  $W_0$ . Our computations are consistent with the discussion in Sections 2.1 and 2.2, which indicates that  $x_0$  should be near  $y_0 + \pi = 0.611$ . The scale for (a-d) is not the same as for all other figures. Numerical experiments with different initial data evolved in a qualitatively similar fashion to that shown here.

## 2.4 Statement of the main results

Our main result concerns the spectrum of the linearization of the viscous Burgers equation about one of the solutions  $W(x, t_0; \nu, x_0, c_0) = c_0 + W_0(x - x_0 - c_0 t_0, t_0; \nu)$ . We show that if we fix the time, we can compute the spectrum of the resulting linearized operator and we find that nearby solutions approach one of the solutions  $W(x, t_*; \nu, x_*, c_*)$  (with  $|t_0 - t_*|, |c_0 - c_*|, |x_0 - x_*| \ll 1$ ) at a much faster rate than the solutions  $W(x, t; \nu, x_0, c)$  themselves evolve, justifying our identification of  $W(x, t; \nu, x_0, c)$  as the metastable states of the system. The linearization about  $W(x, t; \nu, x_0, c)$  in the moving frame  $x - x_0 - ct \mapsto x$  takes the form

$$v_t = \nu v_{xx} - (W_0(x, t; \nu) v)_x, \quad (2.10)$$

and the resulting eigenvalue problem is

$$\mathcal{L}(\nu, t)\phi = \lambda\phi, \quad \mathcal{L}(\nu, t)\phi := \nu\phi_{xx} - (W_0(x, t; \nu)\phi)_x, \quad (2.11)$$



where  $\mathcal{L}(\nu, t)$  is considered as an operator  $\mathcal{L}(\nu, t) : H_{per}^2([-\pi, \pi]) \rightarrow L_{per}^2([-\pi, \pi])$  for every fixed  $\nu$  and  $t$ . We use the standard inner product on  $L_{per}^2([-\pi, \pi])$

$$\langle u, v \rangle := \int_{-\pi}^{\pi} u(x)v(x)dx$$

and norm  $\|u\|_{L_{per}^2} = \langle u, u \rangle$ . Motivated by the discussion of the solutions  $W(x, t; \nu, x_0, c)$  and  $u^{CH}(x, t; \nu, u_0, c)$  above we define the small parameter  $\epsilon^2 := 2\nu t$ . Then our main result is as follows.

**Theorem 1** *There exists  $\epsilon_0 > 0$  such that for all  $\nu, t$  such that  $0 < \epsilon \leq \epsilon_0$  with  $\epsilon = \sqrt{2\nu t}$ , the spectrum for (2.11) consists entirely of ordered eigenvalues with  $\lambda_0 = 0$  and the remaining eigenvalues contained on the negative real-axis. In particular,*

$$\begin{aligned} \lambda_1 &= -1/t + \mathcal{O}(\epsilon^{1/2}e^{-1/\epsilon^2}), & \lambda_2 &= -2/t + \mathcal{O}(\epsilon^{-2}e^{-1/\epsilon^2}), \\ \lambda_3 &= -3/t + \mathcal{O}(\epsilon^{-7/2}e^{-1/\epsilon^2}), & \lambda_4 &= -4/t + \mathcal{O}(\epsilon^{-6}e^{-1/\epsilon^2}). \end{aligned} \quad (2.12)$$

and  $\lambda_j \leq \lambda_4$  for all  $j > 4$ .

Denoting the eigenfunction associated with  $\lambda_n$  by  $\phi_n(x - x_0 - ct; t, \nu)$  we also show

**Theorem 2** *Fix  $\gamma_0 \ll 1$  and let  $u(x, t_0; \nu) = W(x, t_0; \nu, x_0, c_0) + v_0(x; t_0, x_0, c_0; \nu)$  be a solution to (2.1), with  $\|v_0\|_{H_{per}^1} = \gamma \leq \gamma_0$ . Then there exists  $x_*, t_*, c_*$  such that  $v_*(x; t_*, x_*, c_*; \nu) := u_0(x, t_0; \nu) - W(x, t_*; \nu, x_*, c_*)$  is orthogonal to the first three eigenfunctions for (2.11)*

$$\langle v_*(x; t_*, x_*, c_*; \nu), \phi_j(x - x_* - c_*t_*; t_*, \nu) \rangle = 0 \quad \text{for } j = 0, 1, 2.$$

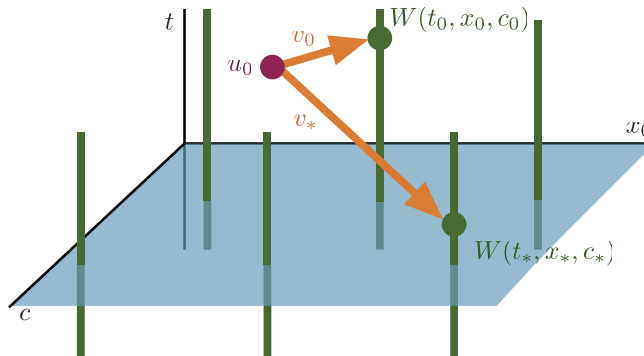


Figure 2:  $u_0$  is a solution to (2.1) which at a fixed time  $t$  is known to be close to a solution  $W(x, t_0; \nu, x_0, c_0)$ .

We show that by adjusting the parameters  $(c_0, t_0, x_0)$  slightly we can also write  $u_0 = W(x, t_*; \nu, x_*, c_*) + v_*(x; t_*, x_*, c_*; \nu)$  where  $v_*$  is orthogonal to the first three eigenfunctions for (2.11).

See Figure 2. We remark that the discussion in Section 2.2 indicates that the condition  $u(x, t_0; \nu) = W(x, t_0; \nu, x_0, c_0) + v_0(x; t_0, x_0, c_0; \nu)$  with  $\|v\|_{H_{per}^1} \ll 1$  holds for most initial data provided that  $\nu, 1/t \ll 1$ .

## 2.5 Justification of $W$ as a family of metastable solutions

Finally, we discuss why the combination of Theorems 1 and 2 justifies our identification of the states  $W$  as a metastable family. If we attempt to analyze the dynamics of solutions near the Whitham solutions the resulting linear equation is non-autonomous. However, for parabolic non-autonomous partial differential equations, the method of “freezing” the coefficients provides good approximations to the evolution, at least over short time intervals. Since we know that over long-times, all solutions will tend toward this zero solution, this “frozen” time evolution should give us a reasonable picture of what happens near the Whitham solutions over times of  $\mathcal{O}(1)$ .

If we think of the spectral picture of the linearized equation (2.10)

$$\partial_t u = \mathcal{L}(\nu, t^*)u = \nu u_{xx} - (W_0 u)_x ,$$

(where  $W_0$  is now evaluated at a fixed time  $t^*$ ), then at first glance it looks as if the solutions don't tend toward the Whitham family at all, since due to the zero eigenvalue of  $\mathcal{L}(\nu, t^*)$  the linear evolution is not contractive. However, the point of Theorem 2 is that by choosing the parameters,  $c$ ,  $x_0$  and  $t_0$  of  $W_0$  appropriately, the perturbation of the Whitham solution will actually be orthogonal to the eigenfunctions  $\phi_0$ ,  $\phi_1$  and  $\phi_2$ . Thus, we expect that the linear evolution will result in the perturbation decaying toward the family of Whitham solutions with a rate governed by fourth eigenvalue, which according to Theorem 1 satisfies

$$\lambda_3 \approx -\frac{3}{t^*} .$$

Thus, if we write  $t = t^* + \tau$ , and denote the perturbation of the Whitham solution as  $p(\tau)$ , then the size of the perturbation will decay like

$$\|p(\tau)\|_{L^2} \sim e^{-\frac{3}{t^*}\tau} .$$

If not rewrite

$$\frac{1}{(t^* + \tau)^3} = \frac{1}{(t^*)^3(1 + \tau/t^*)^3} = \frac{e^{-3\ln(1 + \frac{\tau}{t^*})}}{(t^*)^3} = \frac{e^{-\frac{3}{t^*}\tau}}{(t^*)^3}$$

so we have

$$\|p(\tau)\|_{L^2} \sim \frac{1}{t^3} .$$

Since the evolution along the family behaves like  $1/t$ , as can be seen from equation (2.6)

$$W_0(x, t; \nu, ) = \frac{1}{t} \left[ x - \pi \tanh\left(\frac{\pi x}{2\nu t}\right) + \mathcal{O}\left(e^{-1/\nu t}\right) \right],$$

solutions approach the family at a rate that is much faster than the evolution along the family justifying our characterization of these states as metastable.

### 3 Eigenvalue problem

In this section we prove Theorems 1 and 2. Without loss of generality we let  $c = 0$  and  $x_0 = 0$  (otherwise make the substitution  $y = x - x_0 - ct$ ). If we consider the eigenvalue equation for the linear operator (2.11) with  $\lambda = 0$  we have

$$\partial_x^2 \phi_0 - \frac{1}{\nu} (W_0(x, t; \nu) \phi_0)_x = 0.$$

Integrating this equation twice we find

$$\phi_0(x; t, \nu) := \exp\left[\frac{1}{\nu} \int^x W_0(s, t; \nu) ds\right] = \frac{C}{[\psi^W(x, t; \nu)]^2} \quad (3.1)$$

is an exact eigenfunction for (2.11) with  $\lambda = 0$ , where the function  $\psi^W(x, t; \nu)$  was defined in (2.4). To find the rest of the spectrum we define the transformation

$$\phi(x; t, \nu) = \exp\left[\frac{1}{2\nu} \int^x W(s, t; \nu) ds\right] \tilde{\phi}(x; t, \nu) = \frac{\tilde{C}}{\psi^W(x, t; \nu)} \tilde{\phi}(x; t, \nu). \quad (3.2)$$

Without loss of generality we choose  $\tilde{C} = 1$ . A straightforward computation shows that  $\lambda$  is an eigenvalue for (2.11) with associated eigenvector  $\phi(x; t, \nu)$  if, and only if,  $\lambda$  is an eigenvalue for the self-adjoint problem (3.3)

$$\tilde{\mathcal{L}}(\nu, t)\tilde{\phi} = \lambda\tilde{\phi}, \quad \tilde{\mathcal{L}}(\nu, t)\tilde{\phi} := \nu\tilde{\phi}_{xx} - \frac{1}{2} \left[ \partial_x W_0(x, t; \nu) + \frac{1}{2\nu} W_0^2(x, t; \nu) \right] \tilde{\phi} \quad (3.3)$$

with associated eigenfunction  $\tilde{\phi}$  given by (3.2), where we again consider  $\tilde{\mathcal{L}}(\nu, t)$  as an operator

$$\tilde{\mathcal{L}}(\nu, t) : H_{per}^2([-\pi, \pi]) \rightarrow L_{per}^2([-\pi, \pi])$$

for every fixed  $\nu$  and  $t$ . In particular, since the transformation  $\phi \mapsto \tilde{\phi}$  is bounded with bounded inverse, the spectra of  $\mathcal{L}$  and  $\tilde{\mathcal{L}}$  are identical. Owing to Sturm-Liouville theory for periodic self-adjoint scalar eigenvalue problems (c.f. [9, Thm 2.1, 2.14]), the eigenvalues for (3.3) are ordered  $\lambda_0 > \lambda_1 \geq \lambda_2 > \lambda_3 \geq \lambda_4 > \dots$ . Furthermore, the eigenfunctions  $\tilde{\phi}_{2n-1}$  and  $\tilde{\phi}_{2n}$  have exactly  $2n$  zeros in  $x \in [x_0 - \pi, x_0 + \pi]$ ; since the transformation (3.2) is strictly positive, the eigenfunctions  $\phi_{2n-1}$  and  $\phi_{2n}$  for (2.11) have exactly  $2n$  zeros in  $x \in [x_0 - \pi, x_0 + \pi]$  as well. From (3.1) we see that  $\phi_0(x; t, \nu) > 0$  has no zeros in  $x \in [-\pi, \pi]$  since  $W$  is continuous; hence, all other eigenvalues  $\lambda_j$  are contained on the negative real axis. The following Proposition completes the proof of Theorem 1.

**Proposition 3.1** *Let  $\epsilon := \sqrt{2\nu t}$ . There exists  $0 < \epsilon_0 \ll 1$  such that for all  $\epsilon \leq \epsilon_0$  the next four eigenvalues for (3.3) after  $\lambda_0 = 0$  are*

$$\begin{aligned} \lambda_1 &= -1/t + \mathcal{O}\left(\epsilon^{1/2}e^{-1/\epsilon^2}\right), & \lambda_2 &= -2/t + \mathcal{O}\left(\epsilon^{-2}e^{-1/\epsilon^2}\right), \\ \lambda_3 &= -3/t + \mathcal{O}\left(\epsilon^{-7/2}e^{-1/\epsilon^2}\right), & \lambda_4 &= -4/t + \mathcal{O}\left(\epsilon^{-6}e^{-1/\epsilon^2}\right). \end{aligned} \quad (3.4)$$

Furthermore, defining  $I_s(\epsilon) := [\epsilon^{3/2}, 2\pi - \epsilon^{3/2}]$ ,  $I_f(\epsilon) := [-\epsilon^{3/2}, \epsilon^{3/2}]$ , there exists a  $0 < C(\epsilon_0) < \infty$  such that the following estimates of the first two associated eigenfunctions hold for all  $\epsilon \leq \epsilon_0$

$$\tilde{\phi}_1 : \left\{ \begin{array}{ll} \sup_x \left| e^{(x-\pi)^2/2\epsilon^2} \tilde{\phi}_1(x; t, \nu) + 1 \right| \leq C(\epsilon_0)\epsilon^{3/2} & : x \in I_s(\epsilon) \\ \sup_x \left| \frac{\epsilon^2}{2\pi^2} e^{\pi^2/2\epsilon^2} \operatorname{sech}\left(\frac{\pi x}{\epsilon^2}\right) \tilde{\phi}_1(x; t, \nu) - \left[ \operatorname{sech}^2\left(\frac{\pi x}{\epsilon^2}\right) \left(1 + \frac{x^2}{2\epsilon^2} + \frac{\epsilon^2}{2\pi^2}\right) - \frac{\epsilon^2}{2\pi^2} \right] \right| \leq C(\epsilon_0)\epsilon^{5/2} & : x \in I_f(\epsilon) \end{array} \right\} \quad (3.5a)$$

$$\tilde{\phi}_2 : \left\{ \begin{array}{ll} \sup_x \left| \frac{\epsilon}{x-\pi} e^{(x-\pi)^2/2\epsilon^2} \tilde{\phi}_2(x; t, \nu) + 1 \right| \leq C(\epsilon_0)\epsilon & : x \in I_s(\epsilon) \\ \sup_x \left| \frac{\epsilon}{2\pi} e^{\pi^2/2\epsilon^2} \tilde{\phi}_2(x; t, \nu) - \left[ \sinh\left(\frac{\pi x}{\epsilon^2}\right) + \frac{\pi x}{\epsilon^2} \operatorname{sech}\left(\frac{\pi x}{\epsilon^2}\right) \right] \right| \leq C(\epsilon_0)\epsilon & : x \in I_f(\epsilon) \end{array} \right\} \quad (3.5b)$$

See Figure 3 for a representation of  $I_s(\epsilon)$  and  $I_f(\epsilon)$ . We will prove Proposition 3.1 in Sections 3.2-3.4 by computing the associated eigenfunctions  $\phi_j(x; t, \nu)$  and showing that  $\phi_{1,2}(x; t, \nu)$  have two zeros in  $x \in [-\pi, \pi]$  and  $\phi_{3,4}(x; t, \nu)$  have four zeros in  $x \in [-\pi, \pi]$ . The intervals  $I_{s,f}$  come up naturally in the proof of Proposition 3.1 and we will discuss them in more detail in Section 3.1. Estimates (3.5) can then be transformed into estimates on the eigenfunctions for (2.11) via (3.2) since, using the same types of computations as were used to derive (2.6) we can derive analogous estimates on the transformation function  $(\psi^W)^{-1}$ , namely

$$\left\{ \begin{array}{ll} \sup_x \left| \exp\left[-\frac{(x-\pi)^2}{2\epsilon^2}\right] (\psi^W(x; t, \nu))^{-1} - 1 \right| \leq C(\epsilon_0)e^{-1/\sqrt{\epsilon}} & : x \in I_s(\epsilon) \\ \sup_x \left| 2 \exp\left[-\frac{x^2}{2\epsilon^2}\right] \exp\left[-\frac{\pi^2}{2\epsilon^2}\right] (\psi^W(x; t, \nu))^{-1} - \operatorname{sech}\left(\frac{\pi x}{\epsilon^2}\right) \right| \leq C(\epsilon_0)e^{-1/\epsilon^2} & : x \in I_f(\epsilon) \end{array} \right\}.$$

Thus, the following Proposition is an immediate corollary to Proposition 3.1 and the fact that  $\phi_0(x; t, \nu) = 1/(\psi^W(x; t, \nu))^2$ .

**Proposition 3.2** *Let  $y = x - x_0 - ct$  and  $\epsilon := \sqrt{2\nu t}$ . There exists  $0 < \epsilon_0 \ll 1$  and  $0 < C(\epsilon_0) < \infty$  such that for*

---

$\int_0^{\epsilon^{3/2}} \operatorname{sech}^2\left(\frac{\pi x}{\epsilon^2}\right) dx$	$\frac{\epsilon^2}{\pi} + \mathcal{O}(\epsilon^2 e^{-1/\sqrt{\epsilon}})$
$\int_0^{\epsilon^{3/2}} x^2 \operatorname{sech}^2\left(\frac{\pi x}{\epsilon^2}\right) dx$	$\frac{\epsilon^6}{12\pi} + \mathcal{O}(\epsilon^5 e^{-1/\sqrt{\epsilon}})$
$\int_0^{\epsilon^{3/2}} x^4 \operatorname{sech}^2\left(\frac{\pi x}{\epsilon^2}\right) dx$	$\frac{7\epsilon^{10}}{240\pi} + \mathcal{O}(\epsilon^8 e^{-1/\sqrt{\epsilon}})$
$\int_0^{\epsilon^{3/2}} \operatorname{sech}^4\left(\frac{\pi x}{\epsilon^2}\right) dx$	$\frac{2\epsilon^2}{3\pi} + \mathcal{O}(\epsilon^2 e^{-1/\sqrt{\epsilon}})$
$\int_0^{\epsilon^{3/2}} x^2 \operatorname{sech}^4\left(\frac{\pi x}{\epsilon^2}\right) dx$	$\frac{\epsilon^6(\pi^2-6)}{18\pi^3} + \mathcal{O}(\epsilon^5 e^{-1/\sqrt{\epsilon}})$
$\int_0^{\epsilon^{3/2}} x^4 \operatorname{sech}^4\left(\frac{\pi x}{\epsilon^2}\right) dx$	$\frac{\epsilon^{10}(7\pi^2-60)}{360\pi^3} + \mathcal{O}(\epsilon^8 e^{-1/\sqrt{\epsilon}})$
$\int_0^{\epsilon^{3/2}} \tanh^2\left(\frac{\pi x}{\epsilon^2}\right) dx$	$\mathcal{O}(\epsilon^{3/2})$
$\int_0^{\epsilon^{3/2}} x \tanh\left(\frac{\pi x}{\epsilon^2}\right) dx$	$\mathcal{O}(\epsilon^3)$
$\int_0^{\epsilon^{3/2}} x \operatorname{sech}^2\left(\frac{\pi x}{\epsilon^2}\right) \tanh\left(\frac{\pi x}{\epsilon^2}\right) dx$	$\mathcal{O}(\epsilon^4)$

---

Table 1: *Estimates on the integrals which arise in the computation of  $\|\phi_j\|$  and the inner-products (3.8). Integrals and expansions computed using Mathematica.*

all  $\epsilon \leq \epsilon_0$  the first three eigenfunctions for (2.11) are

$$\phi_0 : \left\{ \begin{array}{ll} \sup_y |\phi_0(y; t, \nu)| \leq \frac{C(\epsilon_0)}{\epsilon} e^{-1/\sqrt{\epsilon}} & : y \in I_s(\epsilon) \\ \sup_y \left| \sqrt{\frac{4\epsilon^2}{3\pi}} \exp\left[-\frac{y^2}{\epsilon^2}\right] \phi_0(y; t, \nu) - \operatorname{sech}^2\left(\frac{\pi y}{\epsilon^2}\right) \right| \leq C(\epsilon_0)\epsilon^2 & : y \in I_f(\epsilon) \end{array} \right\} \quad (3.6a)$$

$$\phi_1 : \left\{ \begin{array}{ll} \sup_y \left| \sqrt{\frac{4\pi^3}{3\epsilon^2}} \phi_1(y; t, \nu) + 1 \right| \leq C(\epsilon_0)\epsilon^{1/2} & : y \in I_s(\epsilon) \\ \sup_y \left| \sqrt{\frac{4\epsilon^2}{3\pi}} \exp\left[-\frac{y^2}{2\epsilon^2}\right] \phi_1(y; t, \nu) - \left[ \operatorname{sech}^2\left(\frac{\pi y}{\epsilon^2}\right) \left(1 + \frac{y^2}{2\epsilon^2} + \frac{\epsilon^2}{2\pi^2}\right) - \frac{\epsilon^2}{2\pi^2} \right] \right| \leq C(\epsilon_0)\epsilon^{3/2} & : y \in I_f(\epsilon) \end{array} \right\} \quad (3.6b)$$

$$\phi_2 : \left\{ \begin{array}{ll} \sup_y \left| \sqrt{\frac{2\pi^2}{3}} \frac{1}{[y-\pi]} \phi_2(y; t, \nu) + 1 \right| \leq C(\epsilon_0)\epsilon & : y \in I_s(\epsilon) \\ \sup_y \left| \sqrt{\frac{2}{3}} \phi_2(y; t, \nu) - \left[ \tanh\left(\frac{\pi y}{\epsilon^2}\right) + \frac{\pi y}{\epsilon^2} \operatorname{sech}^2\left(\frac{\pi y}{\epsilon^2}\right) \right] \right| \leq C(\epsilon_0)\epsilon & : y \in I_f(\epsilon) \end{array} \right\}. \quad (3.6c)$$

We remark that in going from Proposition 3.1 to Proposition 3.2 we have introduced a scaling constant which ensures that all eigenfunctions in (3.6) have been normalized so that  $\|\phi_j\| = 1$ , which one can check by using

$$\exp\left[\frac{ax^2}{\epsilon^2}\right] = 1 + \frac{ax^2}{\epsilon^2} + \frac{a^2x^4}{2\epsilon^4} + \mathcal{O}(\epsilon^3) \quad \forall x \in I_f(\epsilon)$$

and the integrals in Table 1. We observe that even though the eigenfunctions (3.5) for (3.3) are exponentially small for  $x \in I_f(\epsilon)$  relative to  $x \in I_s(\epsilon)$ , undoing transformation (3.2), which is exponentially localized in  $x \in I_f(\epsilon)$ , the behavior of eigenfunctions (3.6) for (2.11) in  $x \in I_f(\epsilon)$  becomes relevant.

Using Proposition 3.2 we prove Theorem 2.

**Proof. (of Theorem 2)**

$v_*$  is given by

$$v_*(x; t_*, x_*, c_*; \nu) := W(x, t_0; \nu, x_0, c_0) + v_0(x; t_0, x_0, c_0; \nu) - W(x, t_*; \nu, x_*, c_*)$$

We will apply the Implicit Function Theorem to

$$\begin{aligned}
\mathcal{F}(v_0; x_*, t_*, c_*; \nu) &:= \begin{pmatrix} \langle v_*(x; t_*, x_*, c_*; \nu), \phi_0(x - x_* - c_* t_*; t_*, \nu) \rangle \\ \langle v_*(x; t_*, x_*, c_*; \nu), \phi_1(x - x_* - c_* t_*; t_*, \nu) \rangle \\ \langle v_*(x; t_*, x_*, c_*; \nu), \phi_2(x - x_* - c_* t_*; t_*, \nu) \rangle \end{pmatrix} \\
&= \begin{pmatrix} \langle W_0(x - x_0 - c_0 t_0, t_0; \nu) - W_0(x - x_* - c_* t_*; t_*, \nu), \phi_0(x - x_* - c_* t_*; t_*, \nu) \rangle \\ \langle W_0(x - x_0 - c_0 t_0, t_0; \nu) - W_0(x - x_* - c_* t_*; t_*, \nu), \phi_1(x - x_* - c_* t_*; t_*, \nu) \rangle \\ \langle W_0(x - x_0 - c_0 t_0, t_0; \nu) - W_0(x - x_* - c_* t_*; t_*, \nu), \phi_2(x - x_* - c_* t_*; t_*, \nu) \rangle \end{pmatrix} \\
&\quad + \begin{pmatrix} \langle c_0 - c_*, \phi_0(x - x_* - c_* t_*; t_*, \nu) \rangle \\ \langle c_0 - c_*, \phi_1(x - x_* - c_* t_*; t_*, \nu) \rangle \\ \langle c_0 - c_*, \phi_2(x - x_* - c_* t_*; t_*, \nu) \rangle \end{pmatrix} + \begin{pmatrix} \langle v_0(x, t_0; x_0, c_0; \nu), \phi_0(x - x_* - c_* t_*; t_*, \nu) \rangle \\ \langle v_0(x, t_0; x_0, c_0; \nu), \phi_1(x - x_* - c_* t_*; t_*, \nu) \rangle \\ \langle v_0(x, t_0; x_0, c_0; \nu), \phi_2(x - x_* - c_* t_*; t_*, \nu) \rangle \end{pmatrix}
\end{aligned}$$

near  $(v_0; x_*, t_*, c_*) = (0; x_0, t_0, c_0)$ , where the inner product  $\langle v, w \rangle$  is the normal periodic  $L^2$  inner product. Due to Cauchy-Schwartz

$$\langle v_0(x; t_0, x_0, c_0; \nu), \phi_j(x - x_* - c_* t_*; t_*, \nu) \rangle \leq \|v_0\|_{L^2_{per}} \|\phi_j\| \leq \|v_0\|_{H^1_{per}}.$$

Thus,  $\mathcal{F}(v_0; x_0, t_0, c_0; \nu) = 0$  for  $v_0 \equiv 0$ . Next we show that the matrix

$$\left( \begin{array}{ccc} | & | & | \\ \frac{d\mathcal{F}}{dx_*} & \frac{d\mathcal{F}}{dc_*} & \frac{d\mathcal{F}}{dt_*} \\ | & | & | \end{array} \right) \Big|_{(x_*, t_*, c_*; v_0) = (x_0, t_0, c_0; 0)}$$

is invertible. We use the facts that

$$\begin{aligned}
&\frac{d}{dx_*} \langle W_0(x - x_0 - c_0 t_0, t_0; \nu) - W_0(x - x_* - c_* t_*; t_*, \nu), \phi_j(x - x_* - c_* t_*; t_*, \nu) \rangle \Big|_{(x_*, t_*, c_*) = (x_0, t_0, c_0)} \\
&\quad = \langle [\partial_x W_0](x - x_0 - c_0 t_0, t_0; \nu), \phi_j(x - x_0 - c_0 t_0; t_0, \nu) \rangle \\
&\frac{d}{dt_*} \langle W_0(x - x_0 - c_0 t_0, t_0; \nu) - W_0(x - x_* - c_* t_*; t_*, \nu), \phi_j(x - x_* - c_* t_*; t_*, \nu) \rangle \Big|_{(x_*, t_*, c_*) = (x_0, t_0, c_0)} \\
&\quad = c_0 \langle [\partial_x W_0](x - x_0 - c_0 t_0, t_0; \nu), \phi_j(x - x_0 - c_0 t_0; t_0, \nu) \rangle \\
&\quad \quad - \langle [\partial_t W_0](x - x_0 - c_0 t_0, t_0; \nu), \phi_j(x - x_0 - c_0 t_0; t_0, \nu) \rangle \\
&\frac{d}{dc_*} \langle W_0(x - x_0 - c_0 t_0, t_0; \nu) - W_0(x - x_* - c_* t_*; t_*, \nu), \phi_j(x - x_* - c_* t_*; t_*, \nu) \rangle \Big|_{(x_*, t_*, c_*) = (x_0, t_0, c_0)} \\
&\quad = t_0 \langle [\partial_x W_0](x - x_0 - c_0 t_0, t_0; \nu), \phi_j(x - x_0 - c_0 t_0; t_0, \nu) \rangle.
\end{aligned}$$

Since  $\partial_x W_0(x, t; \nu)$ ,  $\phi_0(x; 2\nu t)$ , and  $\phi_1(x; 2\nu t)$  are even functions, and  $\partial_t W_0(x, t; \nu)$  and  $\phi_2(x; 2\nu t)$  are odd functions centered about  $x = n\pi$ ,  $n \in \mathbb{Z}$ , we have that

$$\begin{aligned}
0 &= \langle [\partial_t W](y, t_0; \nu), \phi_0(y; t_0, \nu) \rangle \\
&= \langle [\partial_t W](y, t_0; \nu), \phi_1(y; t_0, \nu) \rangle \\
&= \langle [\partial_x W](y, t_0; \nu), \phi_2(y; t_0, \nu) \rangle \\
&= \langle 1, \phi_2(y; t_0, \nu) \rangle
\end{aligned}$$

where  $y := x - x_0 - c_0 t_0$ . In fact,

$$0 = \langle 1, \phi_j(y; t_0, \nu) \rangle \quad \forall j \neq 0$$

since, integrating the eigenfunction equation (2.11) from  $y = -\pi$  to  $\pi$  and using periodicity we get

$$0 = \lambda_n \int_{-\pi}^{\pi} \phi_n(y; t_0, \nu) dy,$$

where  $\lambda_n = 0$  only for  $n = 0$ . Furthermore, using

$$\exp\left[\frac{ax^2}{\epsilon^2}\right] = 1 + \frac{ax^2}{\epsilon^2} + \frac{a^2x^4}{2\epsilon^4} + \mathcal{O}(\epsilon^3) \quad \forall x \in I_f(\epsilon),$$

the asymptotic expansions for the derivatives of  $W_0(x, t; \nu)$ , equations (2.6),

$$\begin{aligned} \partial_x W_0(x, t; \nu) &= \frac{1}{t} \left[ 1 - \frac{\pi^2}{2\nu t} \operatorname{sech}^2\left(\frac{\pi x}{2\nu t}\right) + \mathcal{O}\left(\frac{1}{t} e^{-1/\nu t}\right) \right] \\ \partial_t W_0(x, t; \nu) &= \frac{1}{t^2} \left[ -x + \pi \tanh\left(\frac{\pi x}{2\nu t}\right) + \frac{\pi^2 x}{2\nu t} \operatorname{sech}^2\left(\frac{\pi x}{2\nu t}\right) + \mathcal{O}\left(\frac{1}{t} e^{-1/\nu t}\right) \right] \end{aligned} \quad (3.7)$$

and the integrals in Table 1, we get that

$$\begin{aligned} \langle [\partial_x W_0](y, t_0; \nu), \phi_0(y; t_0, \nu) \rangle &= -\frac{1}{t_0} \sqrt{\frac{4\pi^3}{3\epsilon^2}} \left[ 1 - \frac{\epsilon^2(24 - \pi^2)}{12\pi^2} + \mathcal{O}(\epsilon^3) \right] \\ \langle 1, \phi_0(y; t_0, \nu) \rangle &= \sqrt{\frac{3\epsilon^2}{\pi}} [1 + \mathcal{O}(\epsilon^3)] = \frac{3\epsilon^2}{2\pi^2} \sqrt{\frac{4\pi^3}{3\epsilon^2}} [1 + \mathcal{O}(\epsilon^3)] \\ \langle [\partial_x W_0](y, t_0; \nu), \phi_1(y; t_0, \nu) \rangle &= -\frac{1}{t_0} \sqrt{\frac{4\pi^3}{3\epsilon^2}} [1 + \mathcal{O}(\epsilon^{3/2})] \\ \langle [\partial_t W_0](y, t_0; \nu), \phi_2(y; t_0, \nu) \rangle &= -\frac{2}{t_0^2} \sqrt{\frac{3}{2\pi^2}} \left[ \frac{\pi^3}{3} + \mathcal{O}(\epsilon) \right]. \end{aligned} \quad (3.8)$$

Finally, using the fact that  $v \in H_{per}^1$  and integrating by parts we have

$$\begin{aligned} \frac{d}{dx_*} \langle v_0(x, t_0; x_0, c_0; \nu), \phi_j(x - x_* - c_* t_*; t_*, \nu) \rangle &= -\langle v_0(x, t_0; x_0, c_0; \nu), \partial_x \phi_j(x - x_* - c_* t_*; t_*, \nu) \rangle \\ &= \langle \partial_x v_0(x, t_0; x_0, c_0; \nu), \phi_j(x - x_* - c_* t_*; t_*, \nu) \rangle \\ &\leq \|\partial_x v_0\|_{L_{per}^2} \leq \|v_0\|_{H_{per}^1} \end{aligned}$$

and similarly for the  $c_*$  and  $t_*$  derivatives. Thus

$$\begin{aligned} &\left( \begin{array}{ccc} \left| \frac{d\mathcal{F}}{dx_*} \right| & \left| \frac{d\mathcal{F}}{dc_*} \right| & \left| \frac{d\mathcal{F}}{dt_*} \right| \\ \left| \right| & \left| \right| & \left| \right| \end{array} \right) \Big|_{(x_*, t_*, c_*; v_0) = (x_0, t_0, c_0; 0)} \\ &= \begin{pmatrix} \langle [\partial_x W_0](y), \phi_0(y) \rangle & \langle t_0 [\partial_x W_0](y) - 1, \phi_0(y) \rangle & \langle c_0 [\partial_x W_0](y) - [\partial_t W_0](y), \phi_0(y) \rangle \\ \langle [\partial_x W_0](y), \phi_1(y) \rangle & \langle t_0 [\partial_x W_0](y) - 1, \phi_1(y) \rangle & \langle c_0 [\partial_x W_0](y) - [\partial_t W_0](y), \phi_1(y) \rangle \\ \langle [\partial_x W_0](y), \phi_2(y) \rangle & \langle t_0 [\partial_x W_0](y) - 1, \phi_2(y) \rangle & \langle c_0 [\partial_x W_0](y) - [\partial_t W_0](y), \phi_2(y) \rangle \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{t_0} \sqrt{\frac{4\pi^3}{3\epsilon^2}} \left[ 1 - \frac{\epsilon^2(24 - \pi^2)}{12\pi^2} + \mathcal{O}(\epsilon^3) \right] & -\sqrt{\frac{4\pi^3}{3\epsilon^2}} \left[ 1 - \frac{\epsilon^2(6 - \pi^2)}{12\pi^2} + \mathcal{O}(\epsilon^3) \right] & -\frac{c_0}{t_0} \sqrt{\frac{4\pi^3}{3\epsilon^2}} [1 + \mathcal{O}(\epsilon^2)] \\ -\frac{1}{t_0} \sqrt{\frac{4\pi^3}{3\epsilon^2}} [1 + \mathcal{O}(\epsilon^{3/2})] & -\sqrt{\frac{4\pi^3}{3\epsilon^2}} [1 + \mathcal{O}(\epsilon^{3/2})] & -\frac{c_0}{t_0} \sqrt{\frac{4\pi^3}{3\epsilon^2}} [1 + \mathcal{O}(\epsilon^{3/2})] \\ 0 & 0 & \frac{2}{t_0^2} \sqrt{\frac{3}{2\pi^2}} \left[ \frac{\pi^3}{3} + \mathcal{O}(\epsilon) \right] \end{pmatrix} \\ &=: A(\epsilon) \end{aligned}$$

which is invertible since  $\det(A(\epsilon)) = -\frac{\pi^3}{t_0^3} \sqrt{\frac{8}{3}} [1 + \mathcal{O}(\epsilon)]$  which, for all  $\epsilon$  sufficiently small, is not equal to zero. We observe, in particular, that  $\det(A(\epsilon)) = \mathcal{O}(1)$ , which implies that the difference  $\|v_* - v_0\|$  is small for all  $\epsilon \ll 1$ .  $\blacksquare$

Thus it remains to prove Proposition 3.1, which we do in the remainder of this section.

### 3.1 Overview and formal asymptotics

In this section we give a formal asymptotic analysis argument to provide intuition for our proof of Proposition 3.1 and the form of the eigenfunctions (3.5). The rigorous proof makes up the majority of this work and is given in Sections 3.2-3.4. We focus on the  $n = 1, 2$  cases since all of the technical difficulties arise in these cases. Let  $x \in [-\pi, \pi]$ ; then, using estimates (2.6), the definition  $\epsilon^2 := 2\nu t$ , and formally dropping the higher order  $\mathcal{O}(e^{-1/\nu t})$  terms, the eigenfunction problem (3.3) is

$$\epsilon^2 \partial_{xx} \tilde{\phi}_n - \left[ 1 - \frac{\pi^2}{\epsilon^2} \operatorname{sech}^2 \left( \frac{\pi x}{\epsilon} \right) + \frac{1}{\epsilon^2} \left( x - \pi \tanh \left( \frac{\pi x}{\epsilon} \right) \right)^2 \right] \tilde{\phi}_n = 2t\lambda_n \tilde{\phi}_n. \quad (3.9)$$

Let  $\hat{\lambda}_n = 2t\lambda_n$ ; rescaling space as  $\zeta := x/\epsilon$  (which, for reasons which will become clear shortly, we call the “slow scale”) regularizes the problem, so that (3.9) becomes

$$\partial_{\zeta\zeta} \hat{\phi}_n - \left[ 1 - \frac{\pi^2}{\epsilon^2} \operatorname{sech}^2 \left( \frac{\pi\zeta}{\epsilon} \right) + \left( \zeta - \frac{\pi}{\epsilon} \tanh \left( \frac{\pi\zeta}{\epsilon} \right) \right)^2 \right] \hat{\phi}_n = \hat{\lambda}_n \hat{\phi}_n. \quad (3.10)$$

The functions  $\tanh(\cdot)$  and  $\operatorname{sech}(\cdot)$  have highly localized derivatives with

$$\operatorname{sech}(y) = \mathcal{O}(e^{-y}) \quad \text{and} \quad \tanh(\pm y) = \pm 1 + \mathcal{O}(e^{-y}) \quad \text{for } |y| \sim \infty.$$

Thus, for  $|\zeta| \in [\sqrt{\epsilon}, \pi/\epsilon]$ , the terms  $\frac{1}{\epsilon} \operatorname{sech}(\pi\zeta/\epsilon)$  and  $\frac{1}{\epsilon} [\pm 1 - \tanh(\pi\zeta/\epsilon)]$  are  $\mathcal{O}(\frac{1}{\epsilon} e^{-1/\sqrt{\epsilon}})$ . Then formally taking the limit  $\epsilon \rightarrow 0$  of (3.10) results in the limiting eigenvalue problem

$$\begin{aligned} \partial_{\zeta\zeta} \hat{\phi}_n - [1 + (\zeta + \pi/\epsilon)^2] \hat{\phi}_n &= \hat{\lambda}_n \hat{\phi}_n, & \text{for } \zeta < 0 & \quad \text{and} \\ \partial_{\zeta\zeta} \hat{\phi}_n - [1 + (\zeta - \pi/\epsilon)^2] \hat{\phi}_n &= \hat{\lambda}_n \hat{\phi}_n, & \text{for } \zeta > 0. \end{aligned}$$

We re-center the problem by defining  $\xi := \zeta - \pi/\epsilon$  and the fact that  $\tilde{\phi}_n(x - 2\pi) = \tilde{\phi}_n(x)$  to get

$$\partial_{\xi\xi} \hat{\phi}_n - [1 + \xi^2] \hat{\phi}_n = \hat{\lambda}_n \hat{\phi}_n \quad (3.11)$$

for  $\xi \in [-\pi/\epsilon + \sqrt{\epsilon}, \pi/\epsilon - \sqrt{\epsilon}]$  (which corresponds with  $x \in I_s(\epsilon)$  in Proposition 3.1). Equation (3.11) has explicit eigenvalues  $\hat{\lambda}_n = -2n$  with associated eigenfunctions

$$\hat{\phi}_n(\xi) = H_{n-1}(\xi) e^{-\xi^2/2}$$

where  $H_n(\xi)$  are the physicist’s Hermite polynomials, the first few of which are

$$H_0(y) = 1, \quad H_1(y) = 2y, \quad H_2(y) = 2(2y^2 - 1), \quad H_3(y) = 4y(2y^2 - 3).$$

The slow variables, however, do not capture the behavior of the eigenfunctions for  $|\xi| \ll \sqrt{\epsilon}$  where the terms  $\frac{1}{\epsilon} \operatorname{sech}(\pi\xi/\epsilon)$  and  $\frac{1}{\epsilon} [\pm 1 - \tanh(\pi\xi/\epsilon)]$  are non-negligible. On the other hand, introducing the faster space scale  $z := x/\epsilon^2$  (which we henceforth refer to as the “fast scale”), equation (3.3) becomes

$$\partial_{zz} \check{\phi}_n - [\epsilon^2 + \pi^2 - 2\pi^2 \operatorname{sech}^2(\pi z) + \epsilon^4 z^2 - 2\pi\epsilon^2 z \tanh(\pi z)] \check{\phi}_n = \epsilon^2 \hat{\lambda}_n \check{\phi}_n. \quad (3.12)$$

Hence, for  $z \in [-1/\sqrt{\epsilon}, 1/\sqrt{\epsilon}]$  (which corresponds with  $x \in I_f(\epsilon)$  in Proposition 3.1), the terms  $\epsilon^2 z$  are  $\mathcal{O}(\epsilon^{3/2})$ . Again formally taking the limit  $\epsilon \rightarrow 0$  results in the limiting eigenvalue problem

$$\partial_{zz} \check{\phi}_n + \pi^2 [2\operatorname{sech}^2(\pi z) - 1] \check{\phi}_n = 0. \quad (3.13)$$

Equation (3.13) has two linearly independent solutions

$$P(z) = \operatorname{sech}(\pi z) \quad \text{and} \quad Q(z) = \sinh(\pi z) + \pi z \operatorname{sech}(\pi z).$$

We set  $\check{\phi}_2(z; \hat{\lambda}_n) = Q(z)$ , anticipating that the fast eigenfunction does not depend, to leading order, on the eigenvalues  $\hat{\lambda}_n$ . As we will show below, however, the matching occurs on the terms which exponentially grow like  $e^{\pi z}$ ; thus, since  $\text{sech}(\pi z)$  is exponentially decaying, for  $\check{\phi}_1$  we need to include the  $\mathcal{O}(\epsilon^2)$  correction so that  $\check{\phi}_1(z; \hat{\lambda}_n) = P(z) + \epsilon^2 P_1(z; \hat{\lambda}_n)$  where

$$P_1(z; \hat{\lambda}_n) = \frac{\hat{\lambda}_n}{\pi^2} \cosh(\pi z) + \left( \frac{z^2}{2} + c \right) \text{sech}(\pi z)$$

solves

$$\partial_z^2 P_1(z; \hat{\lambda}_n) + \pi^2 [2\text{sech}^2(\pi z) - 1] P_1(z; \hat{\lambda}_n) = \left[ 1 + \hat{\lambda}_n - 2\pi z \tanh(\pi z) \right] P(z; \hat{\lambda}_n).$$

$P_1(x)$  now includes the exponentially growing term  $\cosh(\pi z)$ . The fast variables are complementary to the slow variables in the sense that now they do not capture the behavior of the eigenfunctions for  $|z| \gg 1/\sqrt{\epsilon}$  where the terms  $\epsilon^2 z$  and  $\epsilon^4 z^2$  are non-negligible.

Our decomposition of the interval  $[-\epsilon^{3/2}, 2\pi - \epsilon^{3/2}] = I_s(\epsilon) \cup I_f(\epsilon)$  now becomes clear. For  $x \in I_s(\epsilon)$ , we expect the slow-variable eigenfunctions  $\hat{\phi}$  to give a good approximation to  $\check{\phi}$ , whereas for  $x \in I_f(\epsilon)$  we expect the fast-variable eigenfunctions  $\check{\phi}$  to give a good approximation. See Figure 3.

We formally construct eigenfunctions  $\check{\phi}_n(x)$  for (3.9) by pasting a slow and a fast solution together; due to symmetry considerations, we glue  $\hat{\phi}_n((x - \pi)/\epsilon)$  with  $\check{\phi}_1(x/\epsilon^2; \hat{\lambda}_n)$  for  $n$  odd and to  $\check{\phi}_2(z; \hat{\lambda}_n)$  for  $n$  even. The formal asymptotic analysis procedure is as follows. We add the formal eigenfunctions for (3.10) and (3.12) with relative scaling  $C_n$ . We determine  $C_n$  by requiring  $\hat{\phi}_n((x - \pi)/\epsilon) = C_n \check{\phi}_n(x/\epsilon^2)$  in the overlap region  $|x| \approx \epsilon^{3/2}$ . We then subtract the overlap at the matching point  $x = \epsilon^{3/2}$ ; we define the overlap function  $\check{\phi}_n := \hat{\phi}_n(\sqrt{\epsilon} - \pi/\epsilon) = C_n \check{\phi}_n(1/\sqrt{\epsilon})$ . We consider  $x \in [0, \pi]$ ; the analysis for  $x \in [-\pi, 0]$  is completely analogous by symmetry. The resulting eigenfunctions are of the form

$$\begin{aligned} \check{\phi}_1(x; t, \nu) &= e^{-(x-\pi)^2/2\epsilon^2} + C_1 \left[ 1 + \frac{x^2}{2\epsilon^2} + \epsilon^2 c \right] \text{sech} \left( \frac{\pi x}{\epsilon^2} \right) - C_1 \frac{\epsilon^2}{\pi^2} \cosh \left( \frac{\pi x}{\epsilon^2} \right) - \check{\phi}_1 \\ \check{\phi}_2(x; t, \nu) &= \frac{x - \pi}{\epsilon} e^{-(x-\pi)^2/2\epsilon^2} + C_2 \sinh \left( \frac{\pi x}{\epsilon^2} \right) + C_2 \frac{\pi x}{\epsilon^2} \text{sech} \left( \frac{\pi x}{\epsilon^2} \right) - \check{\phi}_2 \end{aligned}$$

We define the spatial variable

$$\eta := \frac{x}{\epsilon^{3/2}} = \frac{\zeta}{\sqrt{\epsilon}} = \sqrt{\epsilon} z$$

which captures the behavior of  $\check{\phi}_n$  in the overlap region. Then, for  $0 < \eta = \mathcal{O}(1)$ , the matching conditions  $C_n \hat{\phi}_n(x/\epsilon) = \check{\phi}_n(x/\epsilon^2)$  are

$$\begin{aligned} e^{-\pi^2/2\epsilon^2} e^{\eta\pi/\sqrt{\epsilon}} e^{-\epsilon\eta^2/2} &= C_1 \left( 1 + \frac{\epsilon\eta^2}{2} \right) \frac{2}{e^{\pi\eta/\sqrt{\epsilon}} + e^{-\pi\eta/\sqrt{\epsilon}}} - C_1 \frac{\epsilon^2}{2\pi^2} \left( e^{\pi\eta/\sqrt{\epsilon}} + e^{-\pi\eta/\sqrt{\epsilon}} \right) \\ \frac{(\pi + \epsilon\sqrt{\epsilon}\eta)}{\epsilon} e^{-\pi^2/2\epsilon^2} e^{\eta\pi/\sqrt{\epsilon}} e^{-\epsilon\eta^2/2} &= \frac{1}{2} \left( e^{\pi\eta/\sqrt{\epsilon}} - e^{-\pi\eta/\sqrt{\epsilon}} \right) + C_2 \frac{\pi\eta}{\sqrt{\epsilon}} \frac{2}{e^{\pi\eta/\sqrt{\epsilon}} + e^{-\pi\eta/\sqrt{\epsilon}}}. \end{aligned}$$

which to leading order becomes

$$e^{-\pi^2/2\epsilon^2} e^{\eta\pi/\sqrt{\epsilon}} = -C_1 \frac{\epsilon^2}{2\pi^2} e^{\pi\eta/\sqrt{\epsilon}} \quad \text{and} \quad \frac{\pi}{\epsilon} e^{-\pi^2/2\epsilon^2} e^{\eta\pi/\sqrt{\epsilon}} = C_2 \frac{1}{2} e^{\pi\eta/\sqrt{\epsilon}}$$

and is satisfied by  $C_1 = \frac{-2\pi^2}{\epsilon^2} e^{-\pi^2/2\epsilon^2}$  and  $C_2 = \frac{2\pi}{\epsilon} e^{-\pi^2/2\epsilon^2}$  with overlap

$$\check{\phi}_1 = e^{-\pi^2/2\epsilon^2} e^{\pi x/\epsilon^2} \quad \text{and} \quad \check{\phi}_2 = \frac{\pi}{\epsilon} e^{-\pi^2/2\epsilon^2} e^{\pi x/\epsilon^2}$$

We emphasize that the matching for both eigenfunctions was done using the coefficients in front of the exponentially growing terms  $e^{\eta\pi/\sqrt{\epsilon}}$  and is why we needed to include the first order correction term in  $\check{\phi}_1(z)$ . Putting everything together, and subtracting the overlap we get

$$\begin{aligned} \check{\phi}_1(x; t, \nu) &= e^{-(x-\pi)^2/2\epsilon^2} - e^{-\pi^2/2\epsilon^2} \left\{ \frac{2\pi^2}{\epsilon^2} \left[ 1 + \frac{x^2}{2\epsilon^2} + \epsilon^2 c \right] \text{sech} \left( \frac{\pi x}{\epsilon^2} \right) - 2 \cosh \left( \frac{\pi x}{\epsilon^2} \right) \right\} - e^{-\pi^2/2\epsilon^2} e^{\pi x/\epsilon^2} \\ \check{\phi}_2(x; t, \nu) &= \frac{1}{\epsilon} \left[ (x - \pi) e^{-(x-\pi)^2/2\epsilon^2} + 2\pi e^{-\pi^2/2\epsilon^2} \sinh \left( \frac{\pi x}{\epsilon^2} \right) - \pi e^{-\pi^2/2\epsilon^2} e^{\pi x/\epsilon^2} \right]. \end{aligned}$$



The analysis for  $x \in [-\pi, 0]$  is completely analogous and the results can be extended to  $x \in \mathbb{R}$  by periodicity. The asymptotic results agree with (3.5). A schematic of the resulting eigenfunctions  $\tilde{\phi}_1$  through  $\tilde{\phi}_4$  is shown in Figure 3.

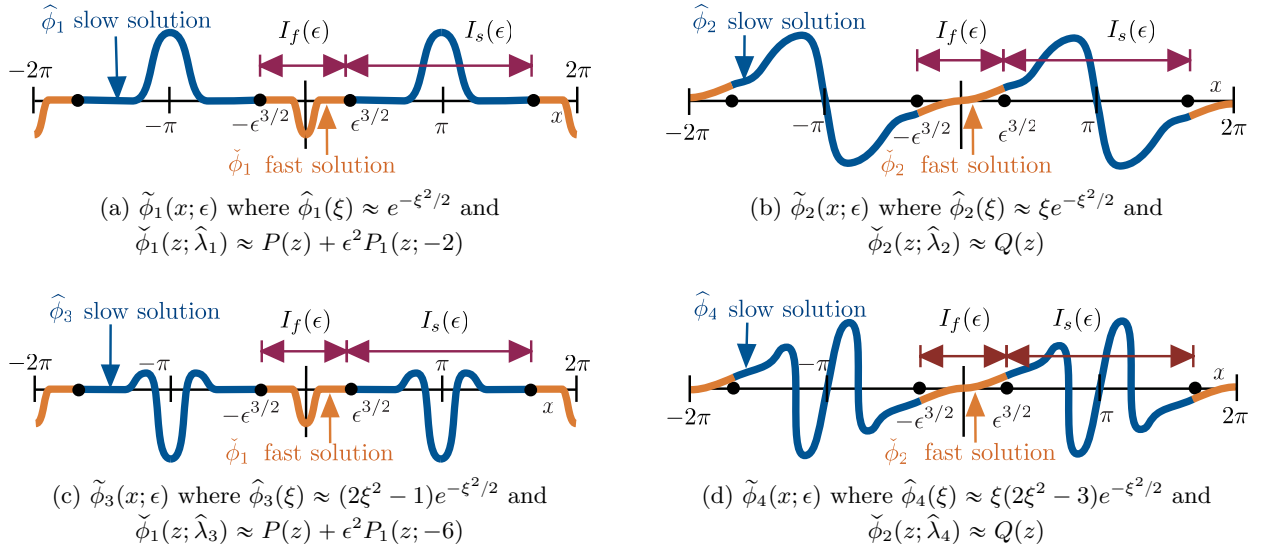


Figure 3: Eigenfunctions for (3.3) constructed by gluing a slow solution  $\hat{\phi}_n$  to a fast solution  $\check{\phi}_j$ . Due to symmetry considerations, we glue  $\hat{\phi}_n$  to  $\check{\phi}_1$  for  $n$  odd and to  $\check{\phi}_2$  for  $n$  even. Figures not drawn to scale; in fact, the magnitude of  $\check{\phi}_j$  is exponentially small relative to the magnitude of  $\hat{\phi}_n$ .

We make a few observations. First, to leading order, the eigenvalues  $\lambda_n = \hat{\lambda}_n/2t = -n/t$  are given by the slow eigenvalue problem (3.10). Secondly, the contribution to  $\tilde{\phi}_n(x)$  from the fast eigenfunctions  $\check{\phi}_n(x/\epsilon^2)$  is exponentially smaller than the contribution from the slow eigenfunctions  $\hat{\phi}_n(x/\epsilon)$ . However, as we have already remarked, undoing transformation (3.2), which is exponentially localized in  $x \in I_f(\epsilon)$ , the behavior of eigenfunctions (3.6) for (2.11) in  $x \in I_f(\epsilon)$  becomes relevant. Thus it is essential that we carefully construct the eigenfunctions in both the slow and the fast variables.

In Sections 3.2-3.4 we make the above formal arguments rigorous by computing the eigenfunctions for (3.3). In Sections 3.2 and 3.3 we rigorously compute the eigenfunction in each of the spatial regimes,  $I_s(\epsilon)$  and  $I_f(\epsilon)$  respectively, using the spatial scaling motivated by the arguments above. We then rigorously match these solutions at the overlap point  $x = \pm\epsilon^{3/2}$  in Section 3.4.

### 3.2 Slow variables

In this section we compute the eigenfunctions for (3.3) for  $x \in I_s(\epsilon)$ . Motivated by the formal asymptotic analysis in Section 3.1 we define the slow variable  $\xi := (x - \pi)/\epsilon$ . We call the eigenfunctions in these coordinates  $\hat{\phi}_n(\xi)$ ; they are defined for  $\xi \in [-\pi/\epsilon + \epsilon^{1/2}, \pi/\epsilon - \epsilon^{1/2}] =: \hat{I}_s(\epsilon)$  and satisfy

$$\partial_{\xi\xi}\hat{\phi}_n - \left[ \widehat{W}_\xi(\xi; \epsilon) + \widehat{W}^2(\xi; \epsilon) \right] \hat{\phi}_n = \hat{\lambda}_n \hat{\phi}_n \quad (3.14)$$

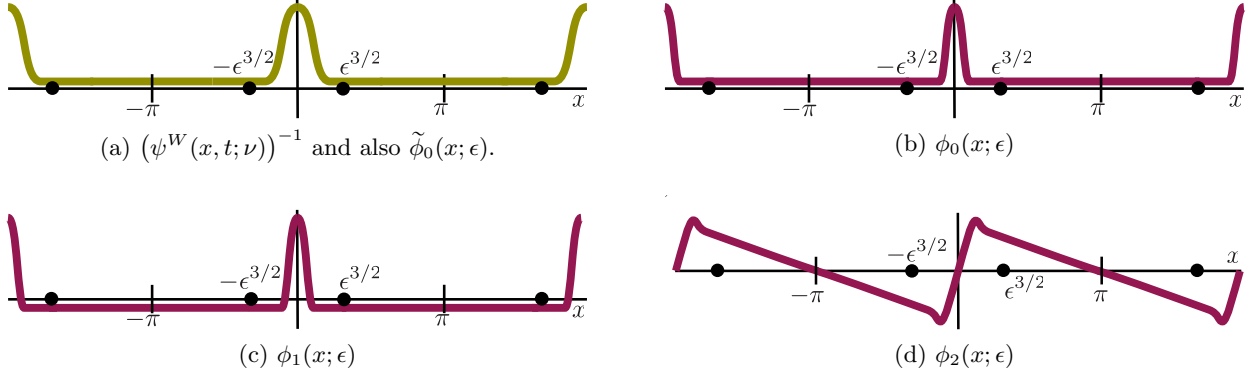


Figure 4: (4a) The function in (3.2) used to compute the eigenfunctions for (2.11) from the eigenfunctions for (3.3):  $\phi_j(x; \epsilon) = [\psi^W(x, t; \nu)]^{-1} \tilde{\phi}_j(x; \epsilon)$ . Additionally, by comparing the formula for the transformation (3.2) to the formula for eigenfunction  $\phi_0(x)$  (3.1), we get  $[\psi^W(x, t; \nu)]^{-2} = [\psi^W(x, t; \nu)]^{-1} \tilde{\phi}_0(x; \epsilon)$ ; thus, Figure 4a also represents the un-transformed eigenfunction  $\tilde{\phi}_0(x; \epsilon)$ . (4b) Eigenfunction  $\phi_0(x; \epsilon) = [\psi^W(x, t; \nu)]^{-2}$ , which we explicitly found in (3.1). (4c) and (4d) The eigenfunctions  $\phi_1(x; \epsilon)$  and  $\phi_2(x; \epsilon)$ , respectively. Due to the fact that  $[\psi^W(x, t; \nu)]^{-1}$  is exponentially localized at the origin, the behavior of the fast solutions  $\tilde{\phi}_j(z; \epsilon)$  is magnified, making the behavior and influence of the fast solutions  $\tilde{\phi}_j(z; \epsilon)$  visible. Figures not drawn to scale.

where for any  $t \in \mathbb{R}^+$

$$\begin{aligned} \widehat{W}(\xi; \epsilon) &:= \frac{t}{\epsilon} W_0(\epsilon\xi + \pi, t; \nu) = \left[ \xi - \frac{2\pi}{\epsilon} \frac{\sum_{n \in \mathbb{Z}} n \widehat{\text{exp}}_n(\xi; \epsilon)}{\sum_{n \in \mathbb{Z}} \widehat{\text{exp}}_n(\xi; \epsilon)} \right], \\ \widehat{W}_\xi(\xi; \epsilon) &:= t [\partial_x W_0](\epsilon\xi + \pi, t; \nu) = \left[ 1 - \frac{4\pi^2}{\epsilon^2} \left( \frac{\sum_{n \in \mathbb{Z}} n^2 \widehat{\text{exp}}_n(\xi; \epsilon)}{\sum_{n \in \mathbb{Z}} \widehat{\text{exp}}_n(\xi; \epsilon)} - \left( \frac{\sum_{n \in \mathbb{Z}} n \widehat{\text{exp}}_n(\xi; \epsilon)}{\sum_{n \in \mathbb{Z}} \widehat{\text{exp}}_n(\xi; \epsilon)} \right)^2 \right) \right], \\ \text{and } \widehat{\text{exp}}_n(\xi; \epsilon) &:= \begin{cases} \exp[-2n\pi(n\pi - \epsilon\xi)/\epsilon^2] & : n \geq 0 \\ \exp[2n\pi(-n\pi + \epsilon\xi)/\epsilon^2] & : n \leq 0 \end{cases} \end{aligned} \quad (3.15)$$

The form of  $\widehat{\text{exp}}_n(\xi; \epsilon)$  follows from the same type of computations as for (2.7) in Proposition 2.1

$$\exp\left[\frac{-(\epsilon\xi + 2\pi - 2n\pi)^2}{2\epsilon^2}\right] = \exp\left[\frac{-(\epsilon\xi - 2\pi(n-1))^2}{2\epsilon^2}\right] = \exp\left[\frac{-\xi^2}{2}\right] \exp\left[\frac{-2\pi(-\epsilon\xi(n-1) + (n-1)^2)}{\epsilon^2}\right],$$

factoring out the dominant mode  $\exp[-\xi^2/2]$  from the numerator and denominator and shifting  $n$ . We remark that even though  $\widehat{W}_\xi(\xi; \epsilon)$  is determined by an appropriate transformation of  $\partial_x W_0(x, t; \nu)$ , it is also true that  $\partial_\xi \widehat{W}(\xi; \epsilon) = \widehat{W}_\xi(\xi; \epsilon)$ ; hence our notation.

Motivated by the formal analysis we re-write (3.14) as

$$\partial_{\xi\xi} \widehat{\phi}_n - \left[1 + \xi^2 + \widehat{\mathcal{N}}(\xi; \epsilon)\right] \widehat{\phi}_n = (-2n + \widehat{\Lambda}_n) \widehat{\phi}_n$$

with  $\widehat{\Lambda}_n := \widehat{\lambda}_n + 2n$  and  $\widehat{\mathcal{N}}(\xi; \epsilon) := \widehat{W}_x(\xi; \epsilon) + \widehat{W}^2(\xi; \epsilon) - (1 + \xi^2)$ , which is equivalent to the first order system

$$\partial_\xi \widehat{U}_n = \widehat{\mathcal{A}}_n(\xi) \widehat{U}_n + \widehat{\mathcal{N}}_n(\widehat{U}_n, \xi; \epsilon, \widehat{\Lambda}_n) \quad (3.16)$$

where  $\widehat{U}_n := (\widehat{\phi}_n, \widehat{\psi}_n)^T$  with  $\widehat{\psi}_n := \partial_\xi \widehat{\phi}_n$ ,

$$\widehat{\mathcal{A}}_n := \begin{pmatrix} 0 & 1 \\ 1 + \xi^2 - 2n & 0 \end{pmatrix}, \quad \text{and } \widehat{\mathcal{N}}_n(\widehat{\phi}_n, \widehat{\psi}_n, \xi; \epsilon, \widehat{\Lambda}_n) := \begin{pmatrix} 0 \\ (\widehat{\mathcal{N}}(\xi; \epsilon) + \widehat{\Lambda}_n) \widehat{\phi}_n \end{pmatrix}.$$

**Lemma 3.3** Fix  $\hat{\epsilon}_1 > 0$ . There exists  $0 < \hat{C}(\hat{\epsilon}_1) < \infty$  such that for all  $\epsilon \leq \hat{\epsilon}_1$  and  $\xi \in \hat{I}_s(\epsilon)$ ,

$$\left| \hat{\mathcal{N}}(\xi; \epsilon) \right| \leq \frac{\hat{C}(\hat{\epsilon}_1)}{\epsilon^2} \exp[-\pi^2/\epsilon^2] \exp[-(\pi - \epsilon\xi)^2/\epsilon^2] \exp[\xi^2] \quad (3.17a)$$

$$\leq \frac{\hat{C}(\hat{\epsilon}_1)}{\epsilon^2} \exp[-2\pi/\sqrt{\epsilon}]. \quad (3.17b)$$

**Proof.** Define  $r := \exp[-2\pi(\pi - \epsilon|\xi|)/\epsilon^2]$ . Then, due to (3.15),  $0 < \widehat{\text{exp}}_n(\xi; \epsilon) \leq r^{|n|}$  with  $r \leq \exp[-2\pi/\sqrt{\epsilon}] < 1$ ; furthermore, since  $\widehat{\text{exp}}_0(\xi; \epsilon) = 1$  for all  $\xi$  and  $\epsilon$ ,  $\sum_{n \in \mathbb{Z}} \widehat{\text{exp}}_n(\xi; \epsilon) \geq 1$ . Thus there exists  $0 < \hat{C}(\hat{\epsilon}_1) < \infty$  such that for all  $\epsilon \leq \hat{\epsilon}_1$

$$\begin{aligned} \left| \hat{\mathcal{N}}(\xi; \epsilon) \right| &= \left| \widehat{W}_x(\xi; \epsilon) + \widehat{W}^2(\xi; \epsilon) - (1 + \xi^2) \right| \\ &= \left| \frac{8\pi^2}{\epsilon^2} \left( \frac{\sum_{n \in \mathbb{Z}} n \widehat{\text{exp}}_n(\xi; \epsilon)}{\sum_{n \in \mathbb{Z}} \widehat{\text{exp}}_n(\xi; \epsilon)} \right)^2 - \frac{4\pi^2}{\epsilon^2} \frac{\sum_{n \in \mathbb{Z}} n^2 \widehat{\text{exp}}_n(\xi; \epsilon)}{\sum_{n \in \mathbb{Z}} \widehat{\text{exp}}_n(\xi; \epsilon)} - \frac{4\pi\xi}{\epsilon} \frac{\sum_{n \in \mathbb{Z}} n \widehat{\text{exp}}_n(\xi; \epsilon)}{\sum_{n \in \mathbb{Z}} \widehat{\text{exp}}_n(\xi; \epsilon)} \right| \\ &\leq \frac{4\pi}{\epsilon^2} \left[ 2 \left( \sum_{n \in \mathbb{Z}} |n| r^{|n|} \right)^2 + \sum_{n \in \mathbb{Z}} n^2 r^{|n|} + \epsilon|\xi| \sum_{n \in \mathbb{Z}} |n| r^{|n|} \right] \\ &\leq \frac{4\pi}{\epsilon^2} \left[ 2 \left( \frac{2r}{(1-r)^2} \right)^2 + \frac{2r(1+r)}{(1-r)^3} + \epsilon|\xi| \frac{2r}{(1-r)^2} \right] \\ &\leq \frac{\hat{C}(\hat{\epsilon}_1)r}{\epsilon^2} = \frac{\hat{C}(\hat{\epsilon}_1)}{\epsilon^2} \exp[-2\pi^2/\epsilon^2] \exp[2\pi\xi/\epsilon] \\ &= \frac{\hat{C}(\hat{\epsilon}_1)}{\epsilon^2} \exp[-\pi^2/\epsilon^2] \exp[-(\pi - \epsilon\xi)^2/\epsilon^2] \exp[\xi^2] \\ &\leq \frac{\hat{C}(\hat{\epsilon}_1)}{\epsilon^2} \exp[-2\pi/\sqrt{\epsilon}], \end{aligned}$$

using the fact that  $\epsilon|\xi| \leq \pi - \epsilon^{3/2}$ . ■

For  $n \in \{1, 2, 3, 4\}$  the leading-order evolution equation  $\partial_\xi \hat{V}_n = \hat{\mathcal{A}}_n(\xi) \hat{V}_n$  has the two linearly independent solutions  $\hat{V}_{n,j}(\xi)$ ,  $j \in \{1, 2\}$ , where

$$\begin{aligned} \hat{V}_{1,1}(\xi) &:= \begin{pmatrix} e^{-\xi^2/2} \\ -\xi e^{-\xi^2/2} \end{pmatrix} & \hat{V}_{1,2}(\xi) &:= \frac{1}{2} \begin{pmatrix} \sqrt{\pi} e^{-\xi^2/2} \text{erfi}(\xi) \\ \left[ -\sqrt{\pi} \xi e^{-\xi^2} \text{erfi}(\xi) + 2 \right] e^{\xi^2/2} \end{pmatrix} \\ \hat{V}_{2,1}(\xi) &:= \begin{pmatrix} \xi e^{-\xi^2/2} \\ (1 - \xi^2) e^{-\xi^2/2} \end{pmatrix} & \hat{V}_{2,2}(\xi) &:= \begin{pmatrix} \left[ 1 - \sqrt{\pi} \xi e^{-\xi^2} \text{erfi}(\xi) \right] e^{\xi^2/2} \\ \left[ -\xi + \sqrt{\pi}(\xi^2 - 1) e^{-\xi^2} \text{erfi}(\xi) \right] e^{\xi^2/2} \end{pmatrix} \\ \hat{V}_{3,1}(\xi) &:= \begin{pmatrix} (2\xi^2 - 1) e^{-\xi^2/2} \\ \xi(5 - 2\xi^2) e^{-\xi^2/2} \end{pmatrix} & \hat{V}_{3,2}(\xi) &:= \frac{1}{4} \begin{pmatrix} \left[ 2\xi + \sqrt{\pi}(1 - 2\xi^2) e^{-\xi^2} \text{erfi}(\xi) \right] e^{\xi^2/2} \\ \left[ 4 - 2\xi^2 + \sqrt{\pi}(2\xi^2 - 5) \xi e^{-\xi^2} \text{erfi}(\xi) \right] e^{\xi^2/2} \end{pmatrix} \\ \hat{V}_{4,1}(\xi) &:= \begin{pmatrix} \xi(2\xi^2 - 3) e^{-\xi^2/2} \\ (-2\xi^4 + 9\xi^2 - 3) e^{-\xi^2/2} \end{pmatrix} & \hat{V}_{4,2}(\xi) &:= \frac{1}{6} \begin{pmatrix} \left[ 2 - 2\xi^2 + \sqrt{\pi} \xi(2\xi^2 - 3) e^{-\xi^2} \text{erfi}(\xi) \right] e^{\xi^2/2} \\ \left[ 2\xi(\xi^2 - 4) + \sqrt{\pi}(-2\xi^4 + 9\xi^2 - 3) e^{-\xi^2} \text{erfi}(\xi) \right] e^{\xi^2/2} \end{pmatrix}, \end{aligned}$$

as can be verified by explicit computation. We solve (3.16) for  $\xi \in \hat{I}_s(\epsilon) := [-\pi/\epsilon + \sqrt{\epsilon}, \pi/\epsilon - \sqrt{\epsilon}]$ . We expect  $\hat{\phi}_n(\xi)$  is close to the formal eigenfunction  $H_{n-1}(\xi) e^{-\xi^2/2}$ ; thus, owing to symmetry considerations, we assume that  $\hat{U}_n(0) \in \text{span} \left\{ \hat{V}_{n,1}(0) \right\}$ . We then parametrize the corresponding solution to (3.16) at the matching point  $x = \pm \epsilon^{3/2}$ , which corresponds with  $\xi = \mp(\pi/\epsilon - \sqrt{\epsilon}) =: \mp \xi_0$ .

**Proposition 3.4** Define for every  $\epsilon$  the norm  $\|u(\cdot)\|_\epsilon = \sup_{\xi \in \hat{I}_s(\epsilon)} |u(\xi)|$ ; also define

$$\check{\Lambda}_1 := \frac{1}{\xi_0} e^{\xi_0^2} \hat{\Lambda}_1, \quad \check{\Lambda}_2 := \frac{1}{\xi_0^3} e^{\xi_0^2} \hat{\Lambda}_2, \quad \check{\Lambda}_3 := \frac{1}{\xi_0^5} e^{\xi_0^2} \hat{\Lambda}_3, \quad \text{and} \quad \check{\Lambda}_4 := \frac{1}{\xi_0^7} e^{\xi_0^2} \hat{\Lambda}_4.$$

Then there exist constants  $\widehat{\epsilon}_0, \widehat{\rho}_1, \widehat{\rho}_2 > 0$  such that for all  $0 \leq \epsilon \leq \widehat{\epsilon}_0$  the set of all solutions to (3.16) with  $\|u(\cdot)\|_\epsilon \leq \widehat{\rho}_1$ ,  $\widehat{U}_n(0) = \widehat{d}_n \widehat{V}_{n,1}(0)$  and  $|d_n|, |\check{\Lambda}_n| \leq \widehat{\rho}_2$  are given by

$$\begin{aligned}
\widehat{\phi}_1(\xi; \epsilon, \check{\Lambda}_1) &= \widehat{d}_1 \left[ 1 + \mathcal{O}(\epsilon^{-2} e^{-2\pi/\sqrt{\epsilon}} \ln \epsilon + |\check{\Lambda}_1|) \right] e^{-\xi^2/2} \\
\widehat{\psi}_1(\xi; \epsilon, \check{\Lambda}_1) &= -\widehat{d}_1 \left[ 1 + \mathcal{O}(\epsilon^{-2} e^{-2\pi/\sqrt{\epsilon}} \ln \epsilon + |\check{\Lambda}_1|) \right] \xi e^{-\xi^2/2} \\
\widehat{\phi}_2(\xi; \epsilon, \check{\Lambda}_2) &= \widehat{d}_2 \left[ 1 + \mathcal{O}(\epsilon^{-2} e^{-2\pi/\sqrt{\epsilon}} \ln \epsilon + |\check{\Lambda}_2|) \right] \xi e^{-\xi^2/2} \\
\widehat{\psi}_2(\xi; \epsilon, \check{\Lambda}_2) &= -\widehat{d}_2 \left[ 1 + \mathcal{O}(\epsilon^{-2} e^{-2\pi/\sqrt{\epsilon}} \ln \epsilon + |\check{\Lambda}_2|) \right] (\xi^2 - 1) e^{-\xi^2/2}, \\
\widehat{\phi}_3(\xi; \epsilon, \check{\Lambda}_3) &= \widehat{d}_3 \left[ 1 + \mathcal{O}(\epsilon^{-2} e^{-2\pi/\sqrt{\epsilon}} \ln \epsilon + |\check{\Lambda}_3|) \right] (2\xi^2 - 1) e^{-\xi^2/2} \\
\widehat{\psi}_3(\xi; \epsilon, \check{\Lambda}_3) &= -\widehat{d}_3 \left[ 1 + \mathcal{O}(\epsilon^{-2} e^{-2\pi/\sqrt{\epsilon}} \ln \epsilon + |\check{\Lambda}_3|) \right] \xi (2\xi^2 - 5) e^{-\xi^2/2} \\
\widehat{\phi}_4(\xi; \epsilon, \check{\Lambda}_4) &= \widehat{d}_4 \left[ 1 + \mathcal{O}(\epsilon^{-2} e^{-2\pi/\sqrt{\epsilon}} \ln \epsilon + |\check{\Lambda}_4|) \right] \xi (2\xi^2 - 3) e^{-\xi^2/2} \\
\widehat{\psi}_4(\xi; \epsilon, \check{\Lambda}_4) &= -\widehat{d}_4 \left[ 1 + \mathcal{O}(\epsilon^{-2} e^{-2\pi/\sqrt{\epsilon}} \ln \epsilon + |\check{\Lambda}_4|) \right] (2\xi^4 - 9\xi^2 + 3) e^{-\xi^2/2}
\end{aligned} \tag{3.18}$$

where the coefficients in front of  $\check{\Lambda}_n$  at the matching point  $\xi = \xi_0$  are

$$\begin{aligned}
\widehat{\phi}_1 : \frac{\sqrt{\pi}}{4} [1 + \mathcal{O}(\epsilon^2)] \check{\Lambda}_1 & \quad \widehat{\psi}_1 : -\frac{\sqrt{\pi}}{4} [1 + \mathcal{O}(\epsilon^2)] \check{\Lambda}_1 \\
\widehat{\phi}_2 : \frac{\sqrt{\pi}}{8} [1 + \mathcal{O}(\epsilon^2)] \check{\Lambda}_2 & \quad \widehat{\psi}_2 : -\frac{\sqrt{\pi}}{8} [1 + \mathcal{O}(\epsilon^2)] \check{\Lambda}_2 \\
\widehat{\phi}_3 : \frac{\sqrt{\pi}}{8} [1 + \mathcal{O}(\epsilon^2)] \check{\Lambda}_1 & \quad \widehat{\psi}_3 : -\frac{\sqrt{\pi}}{8} [1 + \mathcal{O}(\epsilon^2)] \check{\Lambda}_1 \\
\widehat{\phi}_4 : \frac{3\sqrt{\pi}}{16} [1 + \mathcal{O}(\epsilon^2)] \check{\Lambda}_2 & \quad \widehat{\psi}_4 : -\frac{3\sqrt{\pi}}{16} [1 + \mathcal{O}(\epsilon^2)] \check{\Lambda}_2.
\end{aligned} \tag{3.19}$$

Furthermore,

$$\begin{aligned}
\widehat{\phi}_1(-\xi; \cdot) = \widehat{\phi}_1(\xi; \cdot), \quad \widehat{\phi}_2(-\xi; \cdot) = -\widehat{\phi}_2(\xi; \cdot), \quad \widehat{\phi}_3(-\xi; \cdot) = \widehat{\phi}_3(\xi; \cdot), \quad \widehat{\phi}_4(-\xi; \cdot) = -\widehat{\phi}_4(\xi; \cdot) \\
\widehat{\psi}_1(-\xi; \cdot) = -\widehat{\psi}_1(\xi; \cdot), \quad \widehat{\psi}_2(-\xi; \cdot) = \widehat{\psi}_2(\xi; \cdot), \quad \widehat{\psi}_3(-\xi; \cdot) = -\widehat{\psi}_3(\xi; \cdot), \quad \widehat{\psi}_4(-\xi; \cdot) = \widehat{\psi}_4(\xi; \cdot).
\end{aligned}$$

We remark that the definition of  $\check{\Lambda}_n$  implies that the eigenvalues for (3.3) are exponentially close to the eigenvalues for (3.11). This is consistent with our numerical simulations; we will show why this is a valid assumption in Section 3.4. Note further that (3.18) shows that the eigenfunctions  $\widehat{\phi}_n$  are close to the formal eigenfunctions  $H_{n-1}(\xi) e^{-\xi^2/2}$  as expected from the formal calculations in Section 3.1.

**Proof.** All solutions to (3.16) with initial data  $\widehat{U}_n(0) = \widehat{d}_n \widehat{V}_{n,1}(0)$  satisfy the fixed point equation

$$\widehat{U}_n(\xi) = \widehat{d}_n \widehat{V}_{n,1}(\xi) + \widehat{V}_{n,1}(\xi) \int_0^\xi \langle \widehat{W}_{n,1}(\tau), \widehat{\mathcal{N}}_n(\widehat{U}_n(\tau), \tau; \epsilon, \widehat{\Lambda}_n) \rangle d\tau + \widehat{V}_{n,2}(\xi) \int_0^\xi \langle \widehat{W}_{n,2}(\tau), \widehat{\mathcal{N}}_n(\widehat{U}_n(\tau), \tau; \epsilon, \widehat{\Lambda}_n) \rangle d\tau \tag{3.20}$$

where

$$\begin{aligned}
\widehat{W}_{1,1}(\xi) &:= \frac{1}{2} \begin{pmatrix} [-\sqrt{\pi}\xi e^{-\xi^2} \operatorname{erfi}(\xi) + 2] e^{\xi^2/2} \\ -\sqrt{\pi} e^{-\xi^2/2} \operatorname{erfi}(\xi) \end{pmatrix} & \widehat{W}_{1,2}(\xi) &:= \begin{pmatrix} \xi e^{-\xi^2/2} \\ e^{-\xi^2/2} \end{pmatrix} \\
\widehat{W}_{2,1}(\xi) &:= \begin{pmatrix} [\xi + \sqrt{\pi}(1 - \xi^2) e^{-\xi^2} \operatorname{erfi}(\xi)] e^{\xi^2/2} \\ [1 - \sqrt{\pi}\xi e^{-\xi^2} \operatorname{erfi}(\xi)] e^{\xi^2/2} \end{pmatrix} & \widehat{W}_{2,2}(\xi) &:= \begin{pmatrix} (1 - \xi^2) e^{-\xi^2/2} \\ -\xi e^{-\xi^2/2} \end{pmatrix}, \\
\widehat{W}_{3,1}(\xi) &:= \frac{1}{4} \begin{pmatrix} [2\xi^2 - 4 + \sqrt{\pi}(5 - 2\xi^2)\xi e^{-\xi^2} \operatorname{erfi}(\xi)] e^{\xi^2/2} \\ [2\xi + \sqrt{\pi}(1 - 2\xi^2) e^{-\xi^2} \operatorname{erfi}(\xi)] e^{\xi^2/2} \end{pmatrix} & \widehat{W}_{3,2}(\xi) &:= \begin{pmatrix} \xi(5 - 2\xi^2) e^{-\xi^2/2} \\ (1 - 2\xi^2) e^{-\xi^2/2} \end{pmatrix} \\
\widehat{W}_{4,1}(\xi) &:= \frac{1}{6} \begin{pmatrix} [2\xi(\xi^2 - 4) + \sqrt{\pi}(-2\xi^4 + 9\xi^2 - 3) e^{-\xi^2} \operatorname{erfi}(\xi)] e^{\xi^2/2} \\ [2\xi^2 - 2 + \sqrt{\pi}\xi(3 - 2\xi^2) e^{-\xi^2} \operatorname{erfi}(\xi)] e^{\xi^2/2} \end{pmatrix} & \widehat{W}_{4,2}(\xi) &:= \begin{pmatrix} (2\xi^4 - 9\xi^2 + 3) e^{-\xi^2/2} \\ \xi(2\xi^2 - 3) e^{-\xi^2/2} \end{pmatrix},
\end{aligned}$$

are two linearly independent solutions to the associated adjoint equation  $\widehat{W}'_n = -\widehat{\mathcal{A}}_n^*(\xi)\widehat{W}_n$ , which have been normalized so that  $\langle \widehat{V}_{n,i}, \widehat{W}_{n,j} \rangle_{\mathbb{R}^2} = \delta_{ij}$ . Equation (3.20) is linear and defined for  $\xi \in \mathbb{R}$ ; thus solutions exist and are bounded on any finite interval. However, they may not be uniformly bounded in  $\epsilon$  since the interval of integration  $\widehat{I}_s(\epsilon)$  grows like  $1/\epsilon$ . Our first goal, therefore, is to show that the constant bounding the higher order terms in (3.18) does not grow with  $\widehat{I}_s(\epsilon)$ . Motivated by the formal analysis we use the ansatz  $\widehat{\phi}_n(\xi) = H_{n-1}(\xi)e^{-\xi^2/2}\widehat{u}_n(\xi)$  and  $\widehat{\psi}_n(\xi) = \frac{d}{d\xi} [H_{n-1}(\xi)e^{-\xi^2/2}]\widehat{v}_n(\xi)$  to solve (3.20). We focus on  $n = 1, 2$ , since all of the technical difficulties arise in these cases; the  $n = 3, 4$  cases can be proven completely analogously. The resulting evolution equations for  $\widehat{u}_n$  and  $\widehat{v}_n$  are

$$\begin{aligned}
\widehat{u}_1(\xi; \epsilon, \check{\Lambda}_1) &= \widehat{d}_1 - \frac{\sqrt{\pi}}{2} \int_0^\xi e^{-\tau^2} \operatorname{erfi}(\tau) \left( \widehat{\mathcal{N}}(\tau; \epsilon) + \widehat{\Lambda}_1 \right) \widehat{u}_1(\tau; \epsilon, \check{\Lambda}_1) d\tau \\
&\quad + \frac{\sqrt{\pi}}{2} \operatorname{erfi}(\xi) \int_0^\xi e^{-\tau^2} \left( \widehat{\mathcal{N}}(\tau; \epsilon) + \widehat{\Lambda}_1 \right) \widehat{u}_1(\tau; \epsilon, \check{\Lambda}_1) d\tau \\
&=: \widehat{\mathcal{F}}_{1,u}(\widehat{u}_1; \epsilon, \widehat{d}_1, \widehat{\Lambda}_1)
\end{aligned} \tag{3.21a}$$

$$\begin{aligned}
\widehat{v}_1(\xi; \epsilon, \check{\Lambda}_1) &= \widehat{d}_1 - \frac{\sqrt{\pi}}{2} \int_0^\xi e^{-\tau^2} \operatorname{erfi}(\tau) \left( \widehat{\mathcal{N}}(\tau; \epsilon) + \widehat{\Lambda}_1 \right) \widehat{u}_1(\tau; \epsilon, \check{\Lambda}_1) d\tau \\
&\quad - \frac{1}{2\xi} \left[ 2e^{\xi^2} - \sqrt{\pi}\xi \operatorname{erfi}(\xi) \right] \int_0^\xi e^{-\tau^2} \left( \widehat{\mathcal{N}}(\tau; \epsilon) + \widehat{\Lambda}_1 \right) \widehat{u}_1(\tau; \epsilon, \check{\Lambda}_1) d\tau \\
&=: \widehat{\mathcal{F}}_{1,v}(\widehat{u}_1; \epsilon, \widehat{d}_1, \widehat{\Lambda}_1)
\end{aligned} \tag{3.21b}$$

$$\begin{aligned}
\widehat{u}_2(\xi; \epsilon, \check{\Lambda}_2) &= \widehat{d}_2 + \int_0^\xi \tau \left[ 1 - \sqrt{\pi}\tau e^{-\tau^2} \operatorname{erfi}(\tau) \right] \left( \widehat{\mathcal{N}}(\tau; \epsilon) + \widehat{\Lambda}_2 \right) \widehat{u}_2(\tau; \epsilon, \check{\Lambda}_2) d\tau \\
&\quad - \frac{1}{\xi} \left[ e^{\xi^2} - \sqrt{\pi}\xi \operatorname{erfi}(\xi) \right] \int_0^\xi \tau^2 e^{-\tau^2} \left( \widehat{\mathcal{N}}(\tau; \epsilon) + \widehat{\Lambda}_2 \right) \widehat{u}_2(\tau; \epsilon, \check{\Lambda}_2) d\tau \\
&=: \widehat{\mathcal{F}}_{2,u}(\widehat{u}_2; \epsilon, \widehat{d}_2, \widehat{\Lambda}_2)
\end{aligned} \tag{3.21c}$$

All terms in (3.21a)-(3.21c) are well defined for all  $\xi$  since for  $\xi$  small we have

$$\int_0^\xi e^{-\tau^2} d\tau = \xi - \frac{\xi^3}{3} + \mathcal{O}(\xi^5) \quad \text{and} \quad \int_0^\xi \tau^2 e^{-\tau^2} d\tau = \frac{\xi^3}{3} + \mathcal{O}(\xi^5).$$

For  $\psi_2(\xi)$  we fix  $\xi_1 > 1$  and make the ansatz

$$\psi_2(\xi; \epsilon, \check{\Lambda}_2) = \begin{cases} e^{-\xi^2/2} \check{v}_2(\xi; \epsilon, \check{\Lambda}_2) & : |\xi| \leq \xi_1 \\ e^{-\xi^2/2} \left[ \check{v}_2(|\xi_1|; \epsilon, \check{\Lambda}_2) + (1 - \xi^2) \widehat{v}_2(\xi; \epsilon, \check{\Lambda}_2) \right] & : |\xi| \geq \xi_1 \end{cases}$$

where  $\check{v}_2$  is defined for  $|\xi| \leq \xi_1$  and  $\hat{v}_2$  is defined for  $|\xi| \geq \xi_1$  and

$$\begin{aligned}
\check{v}_2(\xi; \epsilon, \check{\Lambda}_2) &= \hat{d}_2(1 - \xi^2) + (1 - \xi^2) \int_0^\xi \tau [1 - \sqrt{\pi}\tau e^{-\tau} \operatorname{erfi}(\tau)] (\hat{\mathcal{N}}(\tau; \epsilon) + \hat{\Lambda}_2) \hat{u}_2(\tau; \epsilon, \check{\Lambda}_2) d\tau \\
&\quad + \left[ \xi e^{\xi^2} - \sqrt{\pi}(\xi^2 - 1) \operatorname{erfi}(\xi) \right] \int_0^\xi \tau^2 e^{-\tau^2} (\hat{\mathcal{N}}(\tau; \epsilon) + \hat{\Lambda}_2) \hat{u}_2(\tau; \epsilon, \check{\Lambda}_2) d\tau \\
\hat{v}_2(\xi; \epsilon, \check{\Lambda}_2) &= \hat{d}_2 \frac{\xi_1^2 - \xi^2}{1 - \xi^2} + \int_{\xi_1}^\xi \tau [1 - \sqrt{\pi}\tau e^{-\tau} \operatorname{erfi}(\tau)] (\hat{\mathcal{N}}(\tau; \epsilon) + \hat{\Lambda}_2) \hat{u}_2(\tau; \epsilon, \check{\Lambda}_2) d\tau \\
&\quad - \frac{1}{\xi^2 - 1} \left[ \xi e^{\xi^2} - \sqrt{\pi}(\xi^2 - 1) \operatorname{erfi}(\xi) \right] \int_{\xi_1}^\xi \tau^2 e^{-\tau^2} (\hat{\mathcal{N}}(\tau; \epsilon) + \hat{\Lambda}_2) \hat{u}_2(\tau; \epsilon, \check{\Lambda}_2) d\tau \\
&= : \hat{\mathcal{F}}_{2,v}(\hat{u}_2; \epsilon, \hat{d}_2, \hat{\Lambda}_2).
\end{aligned} \tag{3.21d}$$

Now  $\check{v}_2(\xi)$  is clearly uniformly bounded with

$$\check{v}_2(\xi; \epsilon, \check{\Lambda}_2) = \hat{d}_2(1 - \xi^2) + \mathcal{O}(\epsilon^{-2} e^{-2\pi/\sqrt{\epsilon}} \ln \epsilon + |\check{\Lambda}_2|) \quad \text{for } |\xi| \leq \xi_1$$

and  $\hat{\mathcal{F}}_{2,v}$  is well-defined for all  $\xi \geq \xi_1$ . Define  $\hat{\mathcal{D}}_\epsilon(\rho) := \{u \in \mathcal{C}^0(\hat{I}_s(\epsilon)) : \|u\|_\epsilon \leq \rho\}$ . Our goal is to show there exists  $\hat{\rho}_1, \hat{\rho}_2, \hat{\epsilon}_0 \ll 1$  small enough such that

$$\hat{\mathcal{F}}_{n,j}(\hat{u}; \epsilon, \hat{d}_n, \check{\Lambda}_n) : \hat{\mathcal{D}}_\epsilon(\hat{\rho}_1) \times \{\epsilon \leq \hat{\epsilon}_0\} \times \{|\hat{d}_n|, |\check{\Lambda}_n| \leq \hat{\rho}_2\} \rightarrow \hat{\mathcal{D}}_\epsilon(\hat{\rho}_1) \quad \text{with } j \in \{u, v\},$$

whence  $\hat{u}_n(\xi; \epsilon, \check{\Lambda}_n)$  and  $\hat{v}_n(\xi; \epsilon, \check{\Lambda}_n)$  will be uniformly bounded in  $\hat{I}_s(\epsilon)$ . Using (3.17a) to bound the nonlinearity when multiplied by an exponentially small integrand  $\sim e^{-\tau^2}$  and (3.17b) to bound the nonlinearity when multiplied by an algebraic integrand  $\sim e^{-\tau^2} \operatorname{erfi}(\tau)$ , and Claim 3.5 below, there exists a  $0 < C_2(\hat{\epsilon}_1) < \infty$  such that for all  $\hat{u}_1 \in \hat{\mathcal{D}}_\epsilon(\rho)$  and  $\epsilon \leq \hat{\epsilon}_2$ ,

$$\begin{aligned}
\|\hat{\mathcal{F}}_{1,u}(\hat{u}_1; \epsilon, \hat{d}_1, \xi_0 e^{-\xi_0^2} \check{\Lambda}_1)\|_\epsilon &\leq |\hat{d}_1| + \frac{\sqrt{\pi}\rho}{2} \left[ \left( \frac{\hat{C}(\hat{\epsilon}_1)}{\epsilon^2} e^{-2\pi/\sqrt{\epsilon}} + \frac{1}{\epsilon} e^{-\pi^2/\epsilon^2} e^{2\pi/\sqrt{\epsilon}} \check{\Lambda}_1 \right) \int_0^{\xi_0} e^{-\tau^2} \operatorname{erfi}(\tau) d\tau \right. \\
&\quad \left. + \frac{\hat{C}(\hat{\epsilon}_1)}{\epsilon^2} e^{-\pi^2/\epsilon^2} \operatorname{erfi}(\xi_0) \int_0^{\xi_0} e^{-(\pi-\epsilon\tau)^2/\epsilon^2} d\tau + \xi_0 e^{-\xi_0^2} \check{\Lambda}_1 \operatorname{erfi}(\xi_0) \int_0^{\xi_0} e^{-\tau^2} d\tau \right] \\
&\leq |\hat{d}_1| + \frac{\sqrt{\pi}\rho \hat{C}_2(\hat{\epsilon}_1)}{2} \left[ \left( \frac{\hat{C}(\hat{\epsilon}_1)}{\epsilon^2} e^{-2\pi/\sqrt{\epsilon}} + \frac{1}{\epsilon} e^{-\pi^2/\epsilon^2} e^{2\pi/\sqrt{\epsilon}} \check{\Lambda}_1 \right) \ln \epsilon + \frac{\hat{C}(\hat{\epsilon}_1)}{\epsilon} e^{-2\pi/\sqrt{\epsilon}} + \check{\Lambda}_1 \right].
\end{aligned}$$

It is now straightforward to show that there exist constants  $\hat{\rho}_1, \hat{\rho}_2 > 0$  and  $0 < \hat{\epsilon}_0 \leq \hat{\epsilon}_2$  such that  $\hat{\mathcal{F}}_n(\hat{u}_n; \epsilon, \hat{d}_n, e^{-\xi_0^2} \check{\Lambda}_n) \in \hat{\mathcal{D}}_\epsilon(\hat{\rho}_1)$  for all  $\hat{u}_n \in \hat{\mathcal{D}}_\epsilon(\hat{\rho}_1)$ ,  $|\hat{d}_n|, |\check{\Lambda}_n| \leq \hat{\rho}_1$ , and  $\epsilon \leq \hat{\epsilon}_0$ . We remark that the coefficients in  $\check{\Lambda}_1$  is  $\mathcal{O}(1)$  as a consequence of our choice of scaling of  $\hat{\Lambda}_1$ .

A completely analogous argument holds for  $\hat{\mathcal{F}}_{1,v}$ ,  $\hat{\mathcal{F}}_{2,u}$ , and  $\hat{\mathcal{F}}_{2,v}$ , with the following modification

- (i) For  $\hat{\mathcal{F}}_{2,v}$  we use the function space  $\hat{\mathcal{D}}_\epsilon(\rho) := \{u \in \mathcal{C}^0([\xi_1, \xi_0]) : \|u\|_\epsilon \leq \rho\}$ .
- (ii) For  $\hat{\mathcal{F}}_{2,u}$ , in order to get the specific form of the  $\mathcal{O}(\epsilon^{-2} e^{-2\pi/\sqrt{\epsilon}} \ln \epsilon + |\check{\Lambda}_2|)$  we need

$$\operatorname{argmax}_{\xi \in \hat{I}_s(\epsilon)} \left| \int_0^\xi \tau [1 - \sqrt{\pi}\tau e^{-\tau^2} \operatorname{erfi}(\tau)] d\tau \right| = \operatorname{argmax}_{\xi \in \hat{I}_s(\epsilon)} \left| \frac{1}{\xi} e^{\xi^2} [1 - \sqrt{\pi}\xi e^{-\xi^2} \operatorname{erfi}(\xi)] \int_0^\xi \tau^2 e^{-\tau^2} d\tau \right| = \pm \xi_0.$$

In other words, we need to keep the minus signs and still show that the argmax occurs at the end of the interval  $\hat{I}_s(\epsilon)$ . But this is true for all  $\epsilon$  small enough by using the asymptotic expansions shown in Table 2 to get

$$\begin{aligned}
\lim_{\xi \rightarrow \infty} \int_0^\xi \tau [1 - \sqrt{\pi}\tau e^{-\tau^2} \operatorname{erfi}(\tau)] &= \lim_{\xi \rightarrow \infty} \left[ \frac{1}{2} \ln \left( \frac{1}{\xi} \right) + \mathcal{O}(1) \right] \rightarrow -\infty \\
\lim_{\xi \rightarrow \infty} \frac{1}{\xi} [e^{\xi^2} - \sqrt{\pi}\xi \operatorname{erfi}(\xi)] \int_0^\xi \tau^2 e^{-\tau^2} d\tau &= \lim_{\xi \rightarrow \infty} e^{\xi^2} \frac{1}{\xi^3} \left[ -\frac{\sqrt{\pi}}{8} + \mathcal{O}(1/\xi^2) \right] \rightarrow -\infty,
\end{aligned}$$

and noting that the expressions are bounded on any bounded interval.

---

$\operatorname{erfi}(\xi)$	$e^{\xi^2} \frac{1}{\xi\sqrt{\pi}} \left[ 1 + \frac{1}{2\xi^2} + \mathcal{O}\left(\frac{1}{\xi^4}\right) \right]$
$\int_0^\xi e^{-\tau^2} d\tau$	$\frac{\sqrt{\pi}}{2} - e^{-\xi^2} \frac{1}{2\xi} \left[ 1 - \frac{1}{2\xi^2} + \mathcal{O}\left(\frac{1}{\xi^4}\right) \right]$
$\int_0^\xi \tau^2 e^{-\tau^2} d\tau$	$\frac{\sqrt{\pi}}{4} - e^{-\xi^2} \frac{\xi}{4} \left[ 2 + \frac{1}{\xi^2} + \mathcal{O}\left(\frac{1}{\xi^4}\right) \right]$
$\int_0^\xi \tau^4 e^{-\tau^2} d\tau$	$\frac{3\sqrt{\pi}}{8} - e^{-\xi^2} \frac{\xi^3}{4} \left[ 2 + \frac{3}{\xi^2} + \mathcal{O}\left(\frac{1}{\xi^4}\right) \right]$
$\int_0^\xi \tau^6 e^{-\tau^2} d\tau$	$\frac{15\sqrt{\pi}}{16} - e^{-\xi^2} \frac{\xi^5}{4} \left[ 2 + \frac{5}{\xi^2} + \mathcal{O}\left(\frac{1}{\xi^4}\right) \right]$
$\sqrt{\pi} \int_0^\xi e^{-\tau^2} \operatorname{erfi}(\tau) d\tau$	$-\ln\left(\frac{1}{\xi}\right) - \frac{1}{2}\psi^{(0)}\left(\frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{\xi^2}\right)$
$\int_0^\xi \tau [1 - \sqrt{\pi}\tau e^{-\tau^2} \operatorname{erfi}(\tau)] d\tau$	$\frac{1}{2} \ln\left(\frac{1}{\xi}\right) + \frac{1}{4}\psi^{(0)}\left(-\frac{1}{2}\right) + \mathcal{O}\left(\frac{1}{\xi^2}\right)$
$\int_0^\xi \tau^3 [1 - \sqrt{\pi}\tau e^{-\tau^2} \operatorname{erfi}(\tau)] d\tau$	$-\frac{\xi^2}{4} + \frac{3}{4} \ln\left(\frac{1}{\xi}\right) + \frac{3}{8}\psi^{(0)}\left(-\frac{3}{2}\right) + \mathcal{O}\left(\frac{1}{\xi^2}\right)$
$\int_0^\xi \tau^5 [1 - \sqrt{\pi}\tau e^{-\tau^2} \operatorname{erfi}(\tau)] d\tau$	$-\frac{\xi^2(\xi^2+3)}{8} + \frac{15}{8} \ln\left(\frac{1}{\xi}\right) + \frac{15}{16}\psi^{(0)}\left(-\frac{5}{2}\right) + \mathcal{O}\left(\frac{1}{\xi^2}\right)$

---

Table 2: *The asymptotic behavior of all terms in (3.20) for  $\xi \gg 1$  and  $n \in \{1, 2, 3, 4\}$ . The integrals and asymptotic expansions were computed using Mathematica.  $\psi^{(0)}(x)$  is the digamma function, where  $\psi^{(0)}(1/2) = -\gamma - \ln(4)$ ,  $\psi^{(0)}(-1/2) = 2 - \gamma - \ln(4)$ ,  $\psi^{(0)}(-3/2) = \frac{8}{3} - \gamma - \ln(4)$ ,  $\psi^{(0)}(-5/2) = \frac{45}{15} - \gamma - \ln(4)$ , and  $\gamma = \lim_{n \rightarrow \infty} \left( \sum_{j=1}^n \frac{1}{j} - \ln n \right)$  is the Euler-Mascheroni constant.*

(iii) A similar issue as (ii) arises in  $\widehat{\mathcal{F}}_{2,v}$ ; a completely analogous argument gives the desired result.

Using the uniform bounds on  $\widehat{u}_n$  we get estimates (3.18). Plugging these estimates back into (3.21), again using Claim 3.5 and the asymptotic expansions shown in Table 2, we can explicitly integrate the terms multiplying  $\check{\Lambda}_n$  to leading order at  $\xi = \xi_0$  since  $\widehat{d}_n$  is a constant. We obtain (3.19).

The symmetries then follow from the symmetry of the nonlinear term  $\widehat{N}(\xi; \epsilon)$  which is an even function in  $\xi$  since  $W(x; \epsilon)$  is odd and  $W_x(x; \epsilon)$  is even in  $x$ , as we noted in Section 2.1. Hence, for all even functions  $\widehat{u}_n(\xi)$ ,  $\widehat{\mathcal{F}}_n(\widehat{u}_n; \cdot)$  is even. Thus  $\widehat{u}_n(\xi)$  and  $\widehat{v}_n(\xi)$  are even and the symmetries for  $\widehat{\phi}_n$  and  $\widehat{\psi}_n$  follow from the symmetries of  $H_n(\xi)e^{-\xi^2/2}$ .  $\blacksquare$

It remains to prove the following claim.

**Claim 3.5** *Fix  $\widehat{\epsilon}_1$  as in Lemma 3.3. Then there exists  $0 < \widehat{C}_2(\widehat{\epsilon}_1) < \infty$  such that*

$$\int_0^{\xi_0} e^{-\tau^2} \operatorname{erfi}(\tau) d\tau \leq \widehat{C}_2(\widehat{\epsilon}_1) \ln \epsilon \quad \text{and} \quad \operatorname{erfi}(\xi_0) \int_0^{\xi_0} e^{-\tau^2} d\tau \leq \widehat{C}_2(\widehat{\epsilon}_1) \epsilon e^{\pi^2/\epsilon^2} e^{-2\pi/\sqrt{\epsilon}}$$

and, moreover, such that

$$\operatorname{erfi}(\xi_0) \int_0^{\xi_0} e^{-(\pi-\epsilon\tau)^2/\epsilon^2} d\tau \leq \widehat{C}_2(\widehat{\epsilon}_1) \epsilon e^{\pi^2/\epsilon^2} e^{-2\pi/\sqrt{\epsilon}}.$$

**Proof.** The claim follows from the asymptotic expansions in Table 2, the facts that

$$\int_0^{\xi_0} e^{-(\pi-\epsilon\tau)^2/\epsilon^2} d\tau \leq \int_{-\infty}^{\infty} e^{-(\pi-\epsilon\tau)^2/\epsilon^2} d\tau = \int_{-\infty}^{\infty} e^{-\tau^2} d\tau = \sqrt{\pi}$$

due to symmetry, and the small argument approximation  $\int_0^{\sqrt{\epsilon}} e^{-\tau^2} d\tau = \sqrt{\epsilon} [1 + \mathcal{O}(\epsilon)]$ .  $\blacksquare$

### 3.3 Fast variables

In this section we compute the eigenfunctions for (3.3) for  $x \in I_f(\epsilon) := [-\epsilon^{3/2}, \epsilon^{3/2}]$ . Motivated by the formal asymptotic analysis in Section 3.1 we define the fast variable  $z := x/\epsilon^2$ . We call the eigenfunctions in these

coordinates  $\check{\phi}_n(z)$ ; they are defined for  $z \in [-1/\sqrt{\epsilon}, 1/\sqrt{\epsilon}] =: \check{I}_f(\epsilon)$  and satisfy

$$\partial_{zz}\check{\phi}_n - \left[ \check{W}_z(z; \epsilon) + \check{W}^2(z; \epsilon) \right] \check{\phi}_n = \epsilon^2 \hat{\lambda}_n \check{\phi}_n \quad (3.22)$$

where for any  $t \in \mathbb{R}^+$

$$\begin{aligned} \check{W}(z; \epsilon) &:= tW_0(\epsilon^2 z, t; \nu), \\ \check{W}_z(z; \epsilon) &:= t\epsilon^2 [\partial_x W_0](\epsilon^2 z, t; \nu), \end{aligned}$$

We remark that even though  $\check{W}_z(z; \epsilon)$  is obtained through an appropriate transformation of  $\partial_x W_0(x, t; \nu)$ , it is also true that  $\check{W}_z(z; \epsilon) = \partial_z \check{W}(z; \epsilon)$ ; hence our notation.

Motivated by the formal analysis we re-write (3.22) as

$$\partial_{zz}\check{\phi}_n - \left[ \pi^2 - 2\pi^2 \operatorname{sech}^2(\pi z) + \check{\mathcal{N}}(z; \epsilon) \right] \check{\phi}_n = \epsilon^2 \hat{\lambda}_n \check{\phi}_n$$

with  $\check{\mathcal{N}}(z; \epsilon) := \check{W}_z(z; \epsilon) + \check{W}^2(z; \epsilon) - \pi^2[1 - 2\operatorname{sech}^2(\pi z)]$ , which is equivalent to the first order system

$$\partial_z \check{U}_n = \check{\mathcal{A}}_n(z) \check{U}_n + \check{\mathcal{N}}_n(\check{U}_n, z; \epsilon, \hat{\Lambda}_n) \quad (3.23)$$

where  $\check{U}_n := (\check{\phi}_n, \check{\psi}_n)^T$  with  $\check{\psi}_n := \partial_z \check{\phi}_n$ ,  $\hat{\lambda}_n = -2n + \hat{\Lambda}_n$  from Section 3.2,

$$\check{\mathcal{A}}_n := \begin{pmatrix} 0 & 1 \\ \pi^2[1 - 2\operatorname{sech}^2(\pi z)] & 0 \end{pmatrix}, \quad \text{and} \quad \check{\mathcal{N}}_n(\check{\phi}_n, \check{\psi}_n, z; \epsilon, \hat{\Lambda}_n) := \begin{pmatrix} 0 \\ (\check{\mathcal{N}}(z; \epsilon) + \epsilon^2 \hat{\lambda}_n) \check{\phi}_n \end{pmatrix}.$$

**Lemma 3.6** Define  $\check{\mathcal{N}}_{\text{alg}}(z; \epsilon) := \epsilon^2[1 - 2\pi z \tanh(\pi z)] + \epsilon^4 z^2$  and  $\check{\mathcal{N}}_{\text{exp}}(z; \epsilon) := \check{\mathcal{N}}(z; \epsilon) - \check{\mathcal{N}}_{\text{alg}}(z; \epsilon)$ . Then there exists  $\check{\epsilon}_1 > 0$  and  $0 < \check{C}(\check{\epsilon}_1) < \infty$  such that for all  $\epsilon \leq \check{\epsilon}_1$  and  $z \in \check{I}_f(\epsilon)$ ,

$$\left| \check{\mathcal{N}}_{\text{exp}}(z; \epsilon) \right| \leq \check{C}(\check{\epsilon}_1) e^{-1/\epsilon^2}$$

Thus, for all  $\epsilon \leq \check{\epsilon}_1$ ,  $\check{\mathcal{N}}(z; \epsilon)$  is exponentially close to  $\check{\mathcal{N}}_{\text{alg}}(z; \epsilon)$ . In particular, there exists a constant  $0 < \check{C}_1(\check{\epsilon}_1) < \infty$  such that for all  $\epsilon \leq \check{\epsilon}_1$  and  $z \in \check{I}_f(\epsilon)$

$$\left| \check{\mathcal{N}}(z; \epsilon) \right| \leq \check{C}_1(\check{\epsilon}_1) \epsilon^{3/2}$$

**Proof.** The result follows from the definitions of  $\check{W}$  and  $\check{W}_z$  in terms of  $W$  and estimates (2.6).  $\blacksquare$

The leading order evolution equation  $\partial_z \check{V} = \check{\mathcal{A}}(z) \check{V}$  has the two linearly independent solutions  $\check{V}_j(z)$ ,  $j \in \{1, 2\}$ , where

$$\check{V}_1(z) := \begin{pmatrix} -\operatorname{sech}(\pi z) \\ \pi \operatorname{sech}(\pi z) \tanh(\pi z) \end{pmatrix} \quad \text{and} \quad \check{V}_2(z) := \frac{1}{2\pi} \begin{pmatrix} \sinh(\pi z) + \pi z \operatorname{sech}(\pi z) \\ \pi [\cosh(\pi z) + \operatorname{sech}(\pi z) - \pi z \operatorname{sech}(\pi z) \tanh(\pi z)] \end{pmatrix},$$

as can be verified by explicit computation. Observe that the leading order terms no longer depends on  $n$ . Due to symmetry considerations we construct purely even or purely odd eigenfunctions; thus we assume that either  $\check{U}_n(0) \in \operatorname{span} \{ \check{V}_1(0) \}$  or  $\operatorname{span} \{ \check{V}_2(0) \}$ . We then parametrize the corresponding solution to (3.16) at the matching point  $x = \pm \epsilon^{3/2}$ , which corresponds with  $z = \pm 1/\sqrt{\epsilon} =: \pm z_0$ .

**Proposition 3.7** Define for every  $\epsilon$  the norm  $\|u(\cdot)\|_\epsilon = \sup_{z \in \check{I}_f(\epsilon)} |u(z)|$ . Then for each for  $n \in \mathbb{N}$  there exist constants  $\epsilon_0, \check{\rho}_1, \check{\rho}_2 > 0$  such that for all  $0 \leq \epsilon \leq \epsilon_0$  the set of all solutions to (3.23) with  $\hat{\lambda}_n = -2n + \hat{\Lambda}_n$ , and which satisfy  $\|u(\cdot)\|_\epsilon \leq \check{\rho}_1$ , with  $|d_n|, |\hat{\Lambda}_n| \leq \check{\rho}_2$  and  $\check{U}_n(0) = \check{d}_n \check{V}_1(0)$  are given by

$$\begin{aligned} \check{\phi}_1(z; \epsilon, \hat{\lambda}_n) &= \check{d}_n \left[ -\operatorname{sech}^2(\pi z) \left( 1 + \frac{\epsilon^2 z^2}{2} + \frac{n\epsilon^2}{\pi^2} \right) + \frac{n\epsilon^2}{\pi^2} + \mathcal{O}_n(\epsilon^{5/2} + \epsilon^2 |\hat{\Lambda}_n|) \right] \cosh(\pi z), \\ \check{\psi}_1(z; \epsilon, \hat{\lambda}_n) &= \check{d}_n \pi \left[ \operatorname{sech}^2(\pi z) \left( 1 - \frac{\epsilon^2 z}{\pi} \coth(\pi z) + \frac{\epsilon^2 z^2}{2} + \frac{n\epsilon^2}{\pi^2} \right) + \frac{n\epsilon^2}{\pi^2} + \mathcal{O}_n(\epsilon^{5/2} + \epsilon^2 |\hat{\Lambda}_n|) \right] \sinh(\pi z) \end{aligned} \quad (3.24a)$$



and for  $\check{U}_n(0) = \check{d}_n \check{V}_2(0)$  are given by

$$\begin{aligned}\check{\phi}_2(z; \epsilon, \hat{\lambda}_n) &= \check{d}_n \frac{1}{2\pi} \left[ 1 + \mathcal{O}_n(\epsilon + \epsilon^{3/2} |\hat{\Lambda}_n|) \right] [\sinh(\pi z) + \pi z \operatorname{sech}(\pi z)], \\ \check{\psi}_2(z; \epsilon, \hat{\lambda}_n) &= \check{d}_n \frac{1}{2} \left[ 1 + \mathcal{O}_n(\epsilon + \epsilon^{3/2} |\hat{\Lambda}_n|) \right] [\cosh(\pi z) + \operatorname{sech}(\pi z) - \pi z \operatorname{sech}(\pi z) \tanh(\pi z)].\end{aligned}\quad (3.24b)$$

Furthermore,  $\check{\phi}_1(-z) = \check{\phi}_1(z)$ ,  $\check{\psi}_1(-z) = -\check{\psi}_1(z)$ ,  $\check{\phi}_2(-z) = -\check{\phi}_2(z)$ , and  $\check{\psi}_2(-z) = \check{\psi}_2(z)$ .

We remark that for all  $0 < N < \infty$ , it is possible to choose  $\epsilon_0$  and  $\hat{\rho}_2$  small enough (where  $\hat{\rho}_2$  was chosen in the proof of Proposition 3.4) such that  $|\hat{\Lambda}_n| \leq \hat{\rho}_2$  whenever  $\check{\Lambda}_n \leq \hat{\rho}_2$  for all  $n \leq N$ . We also remark that, unlike in the analogous proposition for the slow variables, Proposition 3.7, where we computed a different eigenfunction associated with each eigenvalue  $\hat{\lambda}_n \approx -2n$ , here we have only two functions  $\check{\phi}_1$  and  $\check{\phi}_2$ , which now take  $\hat{\lambda}_n$  as a parameter. This difference is in accord with the formal analysis which indicates that, at least to leading order, we expect that the fast eigenfunctions to solve the eigenvalue-independent equation

$$\check{\phi}_{zz} + \pi^2 [2\operatorname{sech}^2(\pi z) - 1] \check{\phi} = 0.$$

**Proof.** The argument is completely analogous to Proposition 3.4 so we abbreviate the proof. The symmetries follow from the same argument as in Proposition 3.4. For the other claims we set up the fixed point equation on the space of bounded functions

$$\check{\mathcal{D}}_\epsilon(\rho) := \{u \in \mathcal{C}^0(\check{I}_f(\epsilon)) : \|u\|_\epsilon \leq \rho\}.$$

using the Variation of Parameters formula, the normalized adjoint eigenfunctions

$$\widehat{W}_1(z) := \frac{1}{2\pi} \begin{pmatrix} -\pi [\cosh(\pi z) + \operatorname{sech}(\pi z) - \pi z \operatorname{sech}(\pi z) \tanh(\pi z)] \\ \sinh(\pi z) + \pi z \operatorname{sech}(\pi z) \end{pmatrix} \quad \text{and} \quad \widehat{W}_2(z) := \begin{pmatrix} \pi \operatorname{sech}(\pi z) \tanh(\pi z) \\ \operatorname{sech}(\pi z) \end{pmatrix}$$

and the ansatz

$$\begin{aligned}\check{\phi}_1(z; \epsilon, \hat{\lambda}_n) &= \cosh(\pi z) \check{u}_1(z; \epsilon, \hat{\lambda}_n), & \check{\phi}_2(z; \epsilon, \hat{\lambda}_n) &= \frac{1}{2\pi} [\sinh(\pi z) + \pi z \operatorname{sech}(\pi z)] \check{u}_2(z; \epsilon, \hat{\lambda}_n), \\ \check{\psi}_1(z; \epsilon, \hat{\lambda}_n) &= \pi \sinh(\pi z) \check{v}_1(z; \epsilon, \hat{\lambda}_n), & \check{\psi}_2(z; \epsilon, \hat{\lambda}_n) &= \frac{1}{2} [\cosh(\pi z) + \operatorname{sech}(\pi z) - \pi z \operatorname{sech}(\pi z) \tanh(\pi z)] \check{v}_2(z; \epsilon, \hat{\lambda}_n).\end{aligned}$$

We emphasize that  $\hat{u}_1$  exponentially grows in  $z$ , rather than exponentially decaying as the linear eigenfunction  $\operatorname{sech}(\pi z)$  might suggest. This ansatz is motivated by the formal asymptotic analysis. Owing to Claim 3.8 below the following expressions are well defined and bounded on any bounded interval

$$\begin{aligned}\check{u}_1(z; \epsilon, \hat{\lambda}_n) &= -\check{d}_1 \operatorname{sech}^2(\pi z) \\ &+ \frac{1}{2\pi} \left[ -\operatorname{sech}^2(\pi z) \int_0^z [\sinh(\pi \tau) \cosh(\pi \tau) + \pi \tau] \left( \check{\mathcal{N}}(\tau; \epsilon) + \epsilon^2 \hat{\lambda}_n \right) \check{u}_1(\tau; \epsilon, \hat{\lambda}_n) d\tau \right. \\ &\quad \left. + [\tanh(\pi z) + \pi z \operatorname{sech}^2(\pi z)] \int_0^z \left( \check{\mathcal{N}}(\tau; \epsilon) + \epsilon^2 \hat{\lambda}_n \right) \check{u}_1(\tau; \epsilon, \hat{\lambda}_n) d\tau \right] \\ &=: \check{\mathcal{F}}_{1,u}(\check{u}_1; \epsilon, \check{d}_1, -2n + \hat{\Lambda}_n)\end{aligned}\quad (3.25a)$$

$$\begin{aligned}\check{v}_1(z; \epsilon, \hat{\lambda}_n) &= \check{d}_1 \operatorname{sech}^2(\pi z) \\ &+ \frac{1}{2\pi} \left[ \operatorname{sech}^2(\pi z) \int_0^z [\sinh(\pi \tau) \cosh(\pi \tau) + \pi \tau] \left( \check{\mathcal{N}}(\tau; \epsilon) + \epsilon^2 \hat{\lambda}_n \right) \check{v}_1(\tau; \epsilon, \hat{\lambda}_n) d\tau \right. \\ &\quad \left. + \left[ \coth(\pi z) - \pi z \operatorname{sech}^2(\pi z) + \frac{1}{\cosh(\pi z) \sinh(\pi z)} \right] \int_0^z \left( \check{\mathcal{N}}(\tau; \epsilon) + \epsilon^2 \hat{\lambda}_n \right) \check{v}_1(\tau; \epsilon, \hat{\lambda}_n) d\tau \right] \\ &=: \check{\mathcal{F}}_{1,v}(\check{v}_1; \epsilon, \check{d}_1, -2n + \hat{\Lambda}_n)\end{aligned}\quad (3.25b)$$

$$\begin{aligned}
\check{u}_2(z; \epsilon, \hat{\lambda}_n) &= \check{d}_2 \\
&+ \frac{1}{2\pi} \left[ -\frac{1}{\cosh(\pi z) \sinh(\pi z) + \pi z} \int_0^z [\sinh(\pi\tau) + \pi\tau \operatorname{sech}(\pi\tau)]^2 \left( \check{\mathcal{N}}(\tau; \epsilon) + \epsilon^2 \hat{\lambda}_n \right) \check{u}_2(\tau; \epsilon, \hat{\lambda}_n) d\tau \right. \\
&\quad \left. + \int_0^z \operatorname{sech}(\pi\tau) [\sinh(\pi\tau) + \pi\tau \operatorname{sech}(\pi\tau)] \left( \check{\mathcal{N}}(\tau; \epsilon) + \epsilon^2 \hat{\lambda}_n \right) \check{u}_2(\tau; \epsilon, \hat{\lambda}_n) d\tau \right] \\
&=: \check{\mathcal{F}}_{2,u}(\check{u}_2; \epsilon, \check{d}_2, -2n + \hat{\Lambda}_n)
\end{aligned} \tag{3.25c}$$

$$\begin{aligned}
\check{v}_2(z; \epsilon, \hat{\lambda}_n) &= \check{d}_2 \\
&+ \frac{1}{2\pi} \left[ \frac{\tanh(\pi z)}{\cosh^2(\pi z) + 1 - \pi z \tanh(\pi z)} \int_0^z [\sinh(\pi\tau) + \pi\tau \operatorname{sech}(\pi\tau)]^2 \left( \check{\mathcal{N}}(\tau; \epsilon) + \epsilon^2 \hat{\lambda}_n \right) \check{u}_2(\tau; \epsilon, \hat{\lambda}_n) d\tau \right. \\
&\quad \left. + \int_0^z \operatorname{sech}(\pi\tau) [\sinh(\pi\tau) + \pi\tau \operatorname{sech}(\pi\tau)] \left( \check{\mathcal{N}}(\tau; \epsilon) + \epsilon^2 \hat{\lambda}_n \right) \check{u}_2(\tau; \epsilon, \hat{\lambda}_n) d\tau \right] \\
&=: \check{\mathcal{F}}_{2,v}(\check{u}_2; \epsilon, \check{d}_2, -2n + \hat{\Lambda}_n).
\end{aligned} \tag{3.25d}$$

Thus  $(\check{\phi}_n, \check{\psi}_n)$  satisfies (3.23) if, and only if,  $\check{u}_n$  and  $\check{v}_n$  satisfy (3.25). Using Lemma 3.6 and Claim 3.8 below we find that for all  $\check{u}_n \in \check{\mathcal{D}}_\epsilon(\rho)$ ,  $z \in \check{I}_f(\epsilon)$  there exists  $0 < \check{C}_2(\check{\epsilon}_1) < \infty$  such that

$$\begin{aligned}
&\|\check{\mathcal{F}}_{1,u}(\check{u}_1; \epsilon, \check{d}_1, -2n + \hat{\Lambda}_n)\|_\epsilon \\
&\leq |\check{d}_1| + \frac{\rho}{2\pi} \left( \check{C}_1(\check{\epsilon}_1) \epsilon^{3/2} + \epsilon^2(-2n + \hat{\Lambda}_n) \right) \\
&\quad \times \left\| \operatorname{sech}^2(\pi z) \int_0^z [\sinh(\pi\tau) \cosh(\pi\tau) + \pi\tau] d\tau + [\tanh(\pi z) + \pi z \operatorname{sech}^2(\pi z)] \int_0^z d\tau \right\|_\epsilon \\
&\leq |\check{d}_1| + \frac{\rho}{2\pi} \left( \check{C}_1(\check{\epsilon}_1) \epsilon + \epsilon^{3/2}(-2n + \hat{\Lambda}_n) \right) \check{C}_2(\check{\epsilon}_1) (\sqrt{\epsilon} + 1)
\end{aligned}$$

It is now straightforward to show that there exists constants  $\check{\rho}_1, \check{\rho}_2 > 0$  and  $0 < \check{\epsilon}_0 \leq \check{\epsilon}_1$  such that  $\check{\mathcal{F}}_{1,u}(\check{u}_n; \epsilon, \check{d}_n, \hat{\Lambda}_n) \in \check{\mathcal{D}}_\epsilon(\check{\rho}_1)$  for all  $\check{u}_n \in \check{\mathcal{D}}_\epsilon(\check{\rho}_1)$ ,  $|\check{d}_n|, |\hat{\Lambda}_n| \leq \check{\rho}_1$ , and  $\epsilon \leq \check{\epsilon}_0$ . A completely analogous argument holds for  $\check{\mathcal{F}}_{1,v}, \check{\mathcal{F}}_{2,u}$ , and  $\check{\mathcal{F}}_{2,v}$ . Using this uniform bound on  $\check{u}_n$  in (3.25) and again Claim 3.8 we get the expansions<sup>2</sup>

$$\begin{aligned}
\check{u}_1(z; \epsilon, -2n + \hat{\Lambda}_n) &= \check{d}_n \left[ -\operatorname{sech}^2(\pi z) + \mathcal{O}_n(\epsilon + \epsilon^{3/2} |\hat{\Lambda}_n|) \right], & \check{u}_2(z; \epsilon, -2n + \hat{\Lambda}_n) &= \check{d}_n \frac{1}{2\pi} \left[ 1 + \mathcal{O}_n(\epsilon + \epsilon^{3/2} |\hat{\Lambda}_n|) \right], \\
\check{v}_1(z; \epsilon, -2n + \hat{\Lambda}_n) &= \check{d}_n \pi \left[ -\operatorname{sech}^2(\pi z) + \mathcal{O}_n(\epsilon + \epsilon^{3/2} |\hat{\Lambda}_n|) \right], & \check{v}_2(z; \epsilon, -2n + \hat{\Lambda}_n) &= \check{d}_n \frac{1}{2} \left[ 1 + \mathcal{O}_n(\epsilon + \epsilon^{3/2} |\hat{\Lambda}_n|) \right].
\end{aligned}$$

We observe that the leading order terms for  $\check{u}_1(z)$  and  $\check{v}_1(z)$  at the matching point  $z = \pm z_0$  are the  $\mathcal{O}(\epsilon)$  terms since  $\operatorname{sech}^2(\pi z_0) = \mathcal{O}(e^{-2\pi/\sqrt{\epsilon}})$ . Thus we compute the next order terms by plugging the expansion for  $\check{u}_1(z)$  back into (3.25a) and integrating explicitly using the form of  $\check{N}_{\text{alg}}$  and

$$\begin{aligned}
&\int_0^z [\sinh(\pi\tau) \cosh(\pi\tau) + \pi\tau] \operatorname{sech}^2(\pi\tau) d\tau = z \tanh(\pi z) \\
&\int_0^z [\sinh(\pi\tau) \cosh(\pi\tau) + \pi\tau] \operatorname{sech}^2(\pi\tau) 2\pi\tau \tanh(\pi\tau) d\tau = \pi z^2 \tanh^2(\pi z) \\
&\int_0^z [\sinh(\pi\tau) \cosh(\pi\tau) + \pi\tau] \operatorname{sech}^2(\pi\tau) \tau^2 d\tau \\
&= \frac{1}{3\pi^3} \left( 6\pi z \operatorname{Li}_2(-e^{-2\pi z}) + 3\operatorname{Li}_3(-e^{-2\pi z}) - 2\pi^3 z^3 - 6\pi^2 z^2 \ln(1 + e^{-2\pi z}) + 3\pi^3 z^3 \tanh(\pi z) + \frac{9\zeta(3)}{4} \right) \\
&\int_0^z \operatorname{sech}^2(\pi\tau) d\tau = \frac{1}{\pi} \tanh(\pi z) \\
&\int_0^z \operatorname{sech}^2(\pi\tau) 2\pi\tau \tanh(\pi\tau) d\tau = \frac{1}{\pi} (\tanh(\pi z) - \pi z \operatorname{sech}^2(\pi z)) \\
&\int_0^z \operatorname{sech}^2(\pi\tau) \tau^2 d\tau = \frac{1}{\pi^3} \left( \operatorname{Li}_2(-e^{-2\pi z}) - \pi^2 z^2 - 2\pi z \ln(1 + e^{-2\pi z}) + \pi^2 z^2 \tanh(\pi z) + \frac{\pi^2}{12} \right)
\end{aligned}$$

where  $\zeta(z)$  is the Riemann zeta function. We get (3.24). ■

	$z \rightarrow 0$	$z \rightarrow \infty$
$\text{Li}_2(-e^{-2\pi z})$	$-\frac{\pi^2}{12} + 2\pi z \ln(2) - \pi^2 z^2 + \frac{\pi^3 z^3}{3} + \mathcal{O}(z^4)$	$e^{-2\pi z}[-1 + \mathcal{O}(e^{-2\pi z})]$
$\text{Li}_3(-e^{-2\pi z})$	$-\frac{3\zeta(3)}{4} + \frac{\pi^3 z}{6} - \pi^2 z^2 \ln(4) + \frac{2\pi^3 z^3}{3} + \mathcal{O}(z^4)$	$e^{-2\pi z}[-1 + \mathcal{O}(e^{-2\pi z})]$
$\ln(e^{-2\pi z} + 1)$	$\ln(2) - \pi z + \frac{\pi^2 z^2}{2} + \mathcal{O}(z^3)$	$e^{-2\pi z}[1 + \mathcal{O}(e^{-2\pi z})]$
$\cosh(\pi z)$	$1 + \frac{\pi^2 z^2}{2} + \mathcal{O}(z^4)$	$\frac{1}{2}e^{\pi z} (1 + \mathcal{O}(e^{-2\pi z}))$
$\sinh(\pi z)$	$\pi z + \frac{\pi^3 z^3}{6} + \mathcal{O}(z^5)$	$\frac{1}{2}e^{\pi z} (1 + \mathcal{O}(e^{-2\pi z}))$
$\tanh(\pi z)$	$\pi z - \frac{\pi^3 z^3}{3} + \mathcal{O}(z^5)$	$1 + \mathcal{O}(e^{-2\pi z})$
$\text{sech}(\pi z)$	$1 - \frac{\pi^2 z^2}{2} + \mathcal{O}(z^4)$	$e^{-\pi z} (2 + \mathcal{O}(e^{-2\pi z}))$
$\text{csch}(\pi z)$	$\frac{1}{\pi z} - \frac{\pi z}{6} + \mathcal{O}(z^3)$	$e^{-\pi z} (2 + \mathcal{O}(e^{-2\pi z}))$
$\text{coth}(\pi z)$	$\frac{1}{\pi z} + \frac{\pi z}{3} + \mathcal{O}(z^3)$	$1 + \mathcal{O}(e^{-2\pi z})$

Table 3: *The asymptotic behavior of relevant functions for the integrals in (3.25).  $\text{Li}_n(x)$  is the polylogarithm function and  $\zeta(z)$  is the Riemann zeta function. Expansions computed using Mathematica.*

It remains to prove the following claim.

**Claim 3.8** *All integrals in (3.25) are well defined and bounded on any bounded interval. Furthermore, there exists  $\check{\epsilon}_2 > 0$  such that the maximum of each of the following integrals for  $|z| \leq z_0$  occurs at  $z = \pm z_0 := \pm 1/\sqrt{\epsilon}$  for all  $\epsilon \leq \check{\epsilon}_2$*

$$(i) \max_{|z| \leq z_0} \left| \left[ \tanh(\pi z) + \pi z \text{sech}^2(\pi z) \right] \int_0^z d\tau \right| = z_0 + \mathcal{O}(z_0^2 e^{-2\pi z_0})$$

$$(ii) \max_{|z| \leq z_0} \left| \left[ \coth(\pi z) - \pi z \text{sech}^2(\pi z) + \frac{1}{\cosh(\pi z) \sinh(\pi z)} \right] \int_0^z d\tau \right| = z_0 + \mathcal{O}(z_0^2 e^{-2\pi z_0})$$

$$(iii) \max_{|z| \leq z_0} \left| \int_0^z \tanh(\pi \tau) d\tau \right| = z_0 - \frac{\ln 2}{\pi} + \mathcal{O}(e^{-2\pi z_0})$$

and so that the following integrals are bounded uniformly in  $z_0$

$$(iv) \left| \text{sech}^2(\pi z) \int_0^z [\sinh(\pi \tau) \cosh(\pi \tau) + \pi \tau] d\tau \right|$$

$$(v) \left| \frac{1}{\cosh(\pi z) \sinh(\pi z) + \pi z} \int_0^z [\sinh^2(\pi \tau) + \pi \tau \tanh(\pi \tau)] d\tau \right|$$

$$(vi) \left| \frac{\tanh(\pi z)}{\cosh^2(\pi z) + 1 - \pi z \tanh(\pi z)} \int_0^z [\sinh^2(\pi \tau) + \pi \tau \tanh(\pi \tau)] d\tau \right|$$

**Proof.** To show that the integrals are well defined we need to check that they are finite for all  $z$  bounded. This is clear for (i), (iii) and (iv) since each of these expressions at  $z = 0$  equals zero. For (vi) we observe that  $\cosh^2(\pi z) + 1 - \pi z \tanh(\pi z)$  is never zero since  $\cosh^2(0) + 1 - 0 \cdot \tanh(0) = 2 > 0$  and at  $\pi z = 2$

$$\begin{aligned} \frac{d}{dz} \left[ \cosh^2(\pi z) + 1 - \pi z \tanh(\pi z) \right] \Big|_{\pi z=2} &= \pi \left[ 2 \cosh(\pi z) \sinh(\pi z) - \tanh(\pi z) - \pi z \text{sech}^2(\pi z) \right] \Big|_{\pi z=2} \\ &= \frac{\pi}{4} \left[ \sinh(4\pi z) - 4\pi z \right] \text{sech}^2(\pi z) \Big|_{\pi z=2} > 0 \end{aligned}$$

since  $\sinh(x) - x \geq 0$  for all  $x \geq 0$  (this can be seen since  $\sinh(0) = 0$  and  $\frac{d}{dx} \sinh(x) = \cosh(x) \geq 1$ ). Thus it remains to consider (ii) and (v), which may develop a singularity at  $z = 0$ .

(ii) We explicitly evaluate the integral to obtain  $f_2(z) := \left[ \coth(\pi z) - \pi z \text{sech}^2(\pi z) + \text{sech}(\pi z) \text{csch}(\pi z) \right] z$ . Using

<sup>2</sup>The notation  $\mathcal{O}_n$  refers to the fact that the constant may depend on  $n$ .

the asymptotic expansions in Table 3 we get

$$\lim_{z \rightarrow 0} f_2(z) = \lim_{z \rightarrow 0} \left[ \frac{2}{\pi} + \mathcal{O}(z^2) \right] = \frac{2}{\pi}$$

(v) We explicitly integrate to obtain

$$\begin{aligned} f_5(z) &= : \frac{1}{\cosh(\pi z) \sinh(\pi z) + \pi z} \int_0^z [\sinh^2(\pi \tau) + \pi \tau \tanh(\pi \tau)] d\tau \\ &= \frac{1}{\cosh(\pi z) \sinh(\pi z) + \pi z} \left[ \frac{\sinh(2\pi z)}{4\pi} - \frac{z}{2} - \frac{\pi}{24} - \frac{\text{Li}_2(-e^{-2\pi z})}{2\pi} + \frac{\pi z^2}{2} + z \ln(e^{-2\pi z} + 1) \right] \end{aligned}$$

where  $\text{Li}_2(x)$  is the polylogarithm function. Using the expansions in Table 3 we find

$$\lim_{z \rightarrow 0} f_5(z) = \lim_{z \rightarrow 0} \frac{1}{2\pi z + \mathcal{O}(z^3)} \left[ \frac{2\pi^2 z^3}{3} + \mathcal{O}(z^5) \right] = \lim_{z \rightarrow 0} \frac{1}{1 + \mathcal{O}(z^2)} \left[ \frac{\pi z^2}{3} + \mathcal{O}(z^4) \right] = 0.$$

Away from  $z = 0$  we use the fact that each of the six expressions is even; thus, without loss of generality, we assume  $z \geq 0$ . For (i)-(iii) we first show that the maximum occurs at  $z = z_0$  and then use the large argument asymptotic expansion of the integral to evaluate the maximum. For (iv) - (vi) we explicitly compute the expression in the limit  $z \rightarrow \infty$  and it is bounded; thus, since we've already shown that each expression is bounded for  $z = 0$  and they are continuous, they are bounded for all  $z > 0$ .

(i) We explicitly evaluate the integral to obtain  $f_1(z) := [\tanh(\pi z) + \pi z \text{sech}^2(\pi z)] z$ . Then

$$\lim_{z \rightarrow \infty} \frac{f_1(z)}{z} = 1$$

so that  $f_1(z) \sim z$  as  $z \rightarrow \infty$ ; thus, since  $z_0 = 1/\sqrt{\epsilon} \xrightarrow{\epsilon \rightarrow 0} \infty$ , there exists  $\check{\epsilon}_2$  such that

$$\max_{0 \leq z \leq z_0} f_1(z) = f_1(z_0) = z_0 (1 + \mathcal{O}(z_0 e^{-2\pi z_0}))$$

for all  $\epsilon \leq \check{\epsilon}_2$ , where  $f_1(z_0)$  was determined using the asymptotic expansions in Table 3.

(ii) Follows exactly as (i).

(iii) The fact that the maximum occurs at  $z = z_0$  is clear since  $\tanh(\pi z)$  is monotone increasing. We integrate explicitly and use the asymptotic expansion for  $\ln(1 + e^{-2\pi z_0})$  for  $z_0 \gg 1$  shown in Table 3 to get the asymptotic expansion.

(iv) We explicitly integrate to obtain

$$\begin{aligned} f_4(z) &:= \text{sech}^2(\pi z) \int_0^z [\sinh(\pi \tau) \cosh(\pi \tau) + \pi \tau] d\tau = \text{sech}^2 \pi z \left[ \frac{\cosh^2(\pi z) - 1}{2\pi} + \frac{\pi z^2}{2} \right] \\ &= \frac{1}{2\pi} + \frac{1}{2\pi} [-1 + \pi^2 z^2] \text{sech}^2 \pi z. \end{aligned}$$

It is now clear that  $\lim_{z \rightarrow \infty} f_4(z) = 0$ .

(v) Using the expansions in Table 3 and  $f_5(z)$  defined above, we find

$$\lim_{z \rightarrow \infty} f_5(z) = \lim_{z \rightarrow \infty} \frac{4e^{-2\pi z}}{1 + \mathcal{O}(ze^{-2\pi z})} \left[ \frac{e^{2\pi z}}{8\pi} + \mathcal{O}(1) \right] = \lim_{z \rightarrow \infty} \frac{1}{1 + \mathcal{O}(ze^{-2\pi z})} \left[ \frac{1}{2\pi} + \mathcal{O}(e^{-2\pi z}) \right] = \frac{1}{2\pi}.$$

(vi) Follows exactly as (v). ■

At the matching point  $z = 1/\sqrt{\epsilon}$ , we will need the following improved estimates on  $\check{\phi}_1$  and  $\check{\psi}_1$ , which can be obtained by substituting (3.24) back into (3.25) one more time.

**Proposition 3.9** *Let  $\epsilon_0, \check{\rho}_1, \check{\rho}_2 > 0$  be as in Proposition 3.7. Then the set of all solutions to (3.23) with  $\hat{\lambda}_n = -2n + \hat{\Lambda}_n$ ,  $\|u(z)\|_\epsilon \leq \check{\rho}_1$ ,  $|d_n|, |\hat{\Lambda}_n| \leq \check{\rho}_2$  and  $\check{U}_n(0) = \check{d}_n \check{V}_1(0)$  are given at the matching point  $z_0 = 1/\sqrt{\epsilon}$  by*

$$\begin{aligned}\check{\phi}_1(z_0; \epsilon, \hat{\lambda}_n) &= \check{d}_n \left[ \frac{n\epsilon^2}{\pi^2} - \frac{n\epsilon^3}{2\pi^2} + \mathcal{O}_n(\epsilon^{7/2} + \epsilon^2|\hat{\Lambda}_n|) \right] \cosh(\pi z_0), \\ \check{\psi}_1(z_0; \epsilon, \hat{\lambda}_n) &= \check{d}_n \pi \left[ \frac{n\epsilon^2}{\pi^2} - \frac{n\epsilon^3}{2\pi^2} + \mathcal{O}_n(\epsilon^{7/2} + \epsilon^2|\hat{\Lambda}_n|) \right] \sinh(\pi z_0)\end{aligned}\quad (3.26)$$

### 3.4 Gluing

Using the approximations to the eigenfunctions in the slow and fast variables, from Propositions 3.4 and 3.7 respectively, we show that there exists a unique global eigenfunction for (3.3)

$$\lambda_n \check{\phi}_n := \nu \partial_{xx} \check{\phi}_n - \frac{1}{2} \left[ \partial_x W_0(x, t; \nu) + \frac{1}{2\nu} W_0^2(x, t; \nu) \right] \check{\phi}_n \quad (3.27)$$

which can be constructed by gluing a fast eigenfunction to a slow eigenfunction at the overlap point  $x = \epsilon^{3/2}$ . Due to symmetry considerations, we glue  $\hat{\phi}_n$  to  $\check{\phi}_1$  for  $n$  odd and to  $\check{\phi}_2$  for  $n$  even. The matching conditions can be understood as follows. We need both that the functions  $\hat{\phi}_n$  and  $\check{\phi}_n$  are the same at the matching point as well as their slopes

$$\frac{d\hat{\phi}_n((x - \pi)/\epsilon, \cdot)}{dx} = \frac{1}{\epsilon} \hat{\psi}_n((x - \pi)/\epsilon; \cdot) \quad \text{and} \quad \frac{d\check{\phi}_n(x/\epsilon^2; \cdot)}{dx} = \frac{1}{\epsilon^2} \check{\psi}_n(x/\epsilon^2; \cdot).$$

Since (3.27) is linear, any scalar multiple of  $\hat{\phi}_n(\xi; \epsilon)$  and  $\check{\phi}_n(x; \epsilon)$  is an eigenfunction in the appropriate scaling regime; thus, instead of matching the slopes directly we impose the condition that the ratio of the fast eigenfunction and its derivatives is equal to the ratio of the slow eigenfunction and its derivative at the matching point:

$$f_{n,1}(\check{\Lambda}_n; \epsilon) := \epsilon^2 \left[ \frac{\hat{\psi}_n((x - \pi)/\epsilon; \epsilon, \check{\Lambda}_n)}{\epsilon \hat{\phi}_n((x - \pi)/\epsilon; \epsilon, \check{\Lambda}_1)} - \frac{\check{\psi}_{\text{mod}(n,2)+1}(x/\epsilon^2; \epsilon, \hat{\lambda}_n)}{\epsilon^2 \check{\phi}_{\text{mod}(n,2)+1}(x/\epsilon^2; \epsilon, \hat{\lambda}_1)} \right] \Big|_{x=\epsilon^{3/2}} = 0 \quad (3.28a)$$

where

$$\hat{\lambda}_1 := -2 + \xi_0 e^{-\xi_0^2} \check{\Lambda}_1, \quad \hat{\lambda}_2 := -4 + \xi_0^3 e^{-\xi_0^2} \check{\Lambda}_2, \quad \hat{\lambda}_3 := -6 + \xi_0^5 e^{-\xi_0^2} \check{\Lambda}_3, \quad \text{and} \quad \hat{\lambda}_4 := -8 + \xi_0^7 e^{-\xi_0^2} \check{\Lambda}_4.$$

The factor  $\epsilon^2$  in front regularizes the problem and can be thought of as taking the  $z$ , rather than  $x$ , derivatives. We observe that (3.28a) has no explicit dependence on the magnitude of the eigenfunctions. Using the Implicit Function Theorem we will show that there exists a unique fixed point to (3.28a) near  $\epsilon = \check{\Lambda}_n = 0$ . For this  $\check{\Lambda}_n$ , we ensure that the magnitude of the slow and fast eigenfunction at the same at the matching point by showing that there exists a unique  $C_n$  such that

$$f_{n,2}(C_n, \check{\Lambda}_n(\epsilon); \epsilon) := \left[ \hat{\phi}_n((x - \pi)/\epsilon; \epsilon, \check{\Lambda}_n) - C_n \check{\phi}_{\text{mod}(n,2)+1}(x/\epsilon^2; \epsilon, \hat{\lambda}_n) \right] \Big|_{x=\epsilon^{3/2}} = 0 \quad (3.28b)$$

which we will again show is true using the Implicit Function Theorem. We start with condition (3.28a). Using the expansions (3.18) and (3.24) at the matching point  $x = \epsilon^{3/2}$  (equivalently,  $\xi = (-\pi + \epsilon^{3/2})/\epsilon$  and  $z = 1/\sqrt{\epsilon}$ ) with coefficients in front of  $\check{\Lambda}_n$  given by (3.19) we get

$$\begin{aligned}\frac{1}{\pi} f_{1,1}(\check{\Lambda}_1; \epsilon) &= \frac{\left[ 1 - \frac{\sqrt{\pi}}{4} \check{\Lambda}_1 + \mathcal{O}(\epsilon^{-2} e^{-2\pi/\sqrt{\epsilon}} \ln \epsilon + \epsilon^2 |\check{\Lambda}_1|) \right] \left[ 1 + \mathcal{O}(\epsilon^{3/2}) \right]}{\left[ 1 + \frac{\sqrt{\pi}}{4} \check{\Lambda}_1 + \mathcal{O}(\epsilon^{-2} e^{-2\pi/\sqrt{\epsilon}} \ln \epsilon + \epsilon^2 |\check{\Lambda}_1|) \right]} \\ &\quad - \frac{\left[ 1 + \mathcal{O}(\epsilon^{3/2} + \frac{1}{\epsilon} e^{-(-\pi + \epsilon^{3/2})^2/2\epsilon^2} |\check{\Lambda}_1|) \right] \left[ 1 + \mathcal{O}(e^{-2\pi/\sqrt{\epsilon}}) \right]}{\left[ 1 + \mathcal{O}(\epsilon^{3/2} + \frac{1}{\epsilon} e^{-(-\pi + \epsilon^{3/2})^2/2\epsilon^2} |\check{\Lambda}_1|) \right]}.\end{aligned}$$

$$\begin{aligned}
\frac{1}{\pi} f_{2,1}(\check{\Lambda}_2; \epsilon) &= \frac{\left[1 - \frac{\sqrt{\pi}}{8} \check{\Lambda}_2 + \mathcal{O}(\epsilon^{-2} e^{-2\pi/\sqrt{\epsilon}} \ln \epsilon + \epsilon^2 |\check{\Lambda}_2|)\right] \left[1 + \mathcal{O}(\epsilon^{3/2})\right]}{\left[1 + \frac{\sqrt{\pi}}{8} \check{\Lambda}_2 + \mathcal{O}(\epsilon^{-2} e^{-2\pi/\sqrt{\epsilon}} \ln \epsilon + \epsilon^2 |\check{\Lambda}_2|)\right] \left[1 + \mathcal{O}(\epsilon^{3/2})\right]} \\
&\quad - \frac{\left[1 + \mathcal{O}(\epsilon + \frac{1}{\sqrt{\epsilon}} e^{-(\pi + \epsilon^{3/2})^2/2\epsilon^2} |\check{\Lambda}_2|)\right] \left[1 + \mathcal{O}(\frac{1}{\sqrt{\epsilon}} e^{-2\pi/\sqrt{\epsilon}})\right]}{\left[1 + \mathcal{O}(\epsilon + \frac{1}{\sqrt{\epsilon}} e^{-(\pi + \epsilon^{3/2})^2/2\epsilon^2} |\check{\Lambda}_2|)\right] \left[1 + \mathcal{O}(\frac{1}{\sqrt{\epsilon}} e^{-2\pi/\sqrt{\epsilon}})\right]} \\
\frac{1}{\pi} f_{3,1}(\check{\Lambda}_3; \epsilon) &= \frac{\left[1 - \frac{\sqrt{\pi}}{8} \check{\Lambda}_3 + \mathcal{O}(\epsilon^{-2} e^{-2\pi/\sqrt{\epsilon}} \ln \epsilon + \epsilon^2 |\check{\Lambda}_3|)\right] \left[1 + \mathcal{O}(\epsilon^{3/2})\right]}{\left[1 + \frac{\sqrt{\pi}}{8} \check{\Lambda}_3 + \mathcal{O}(\epsilon^{-2} e^{-2\pi/\sqrt{\epsilon}} \ln \epsilon + \epsilon^2 |\check{\Lambda}_3|)\right] \left[1 + \mathcal{O}(\epsilon^{3/2})\right]} \\
&\quad - \frac{\left[1 + \mathcal{O}(\epsilon^{3/2} + \frac{1}{\epsilon^3} e^{-(\pi + \epsilon^{3/2})^2/2\epsilon^2} |\check{\Lambda}_3|)\right] \left[1 + \mathcal{O}(e^{-2\pi/\sqrt{\epsilon}})\right]}{\left[1 + \mathcal{O}(\epsilon^{3/2} + \frac{1}{\epsilon^3} e^{-(\pi + \epsilon^{3/2})^2/2\epsilon^2} |\check{\Lambda}_3|)\right]} \\
\frac{1}{\pi} f_{4,1}(\check{\Lambda}_4; \epsilon) &= \frac{\left[1 - \frac{3\sqrt{\pi}}{16} \check{\Lambda}_4 + \mathcal{O}(\epsilon^{-2} e^{-2\pi/\sqrt{\epsilon}} \ln \epsilon + \epsilon^2 |\check{\Lambda}_4|)\right] \left[1 + \mathcal{O}(\epsilon^{3/2})\right]}{\left[1 + \frac{3\sqrt{\pi}}{16} \check{\Lambda}_4 + \mathcal{O}(\epsilon^{-2} e^{-2\pi/\sqrt{\epsilon}} \ln \epsilon + \epsilon^2 |\check{\Lambda}_4|)\right] \left[1 + \mathcal{O}(\epsilon^{3/2})\right]} \\
&\quad - \frac{\left[1 + \mathcal{O}(\epsilon + \frac{1}{\epsilon^2 \sqrt{\epsilon}} e^{-(\pi + \epsilon^{3/2})^2/2\epsilon^2} |\check{\Lambda}_4|)\right] \left[1 + \mathcal{O}(\frac{1}{\sqrt{\epsilon}} e^{-2\pi/\sqrt{\epsilon}})\right]}{\left[1 + \mathcal{O}(\epsilon + \frac{1}{\epsilon^2 \sqrt{\epsilon}} e^{-(\pi + \epsilon^{3/2})^2/2\epsilon^2} |\check{\Lambda}_4|)\right] \left[1 + \mathcal{O}(\frac{1}{\sqrt{\epsilon}} e^{-2\pi/\sqrt{\epsilon}})\right]}
\end{aligned}$$

It is clear that  $f_{n,1}(0; 0) = 0$  and

$$\left. \frac{df_{n,1}}{d\check{\Lambda}_j} \right|_{(\check{\Lambda}_n; \epsilon) = (0; 0)} \neq 0$$

so that the hypotheses of the Implicit Function Theorem are satisfied. Expanding the unique function  $\check{\Lambda}_n(\epsilon)$  in orders of  $\epsilon$  we find

$$\check{\Lambda}_1 = \mathcal{O}(\epsilon^{3/2}), \quad \check{\Lambda}_2 = \mathcal{O}(\epsilon), \quad \check{\Lambda}_3 = \mathcal{O}(\epsilon^{3/2}), \quad \text{and} \quad \check{\Lambda}_4 = \mathcal{O}(\epsilon).$$

Next we solve (3.28b) using the expansions for  $\check{\Lambda}_n(\epsilon)$  and obtain the expressions

$$\begin{aligned}
e^{-\pi/\sqrt{\epsilon}} f_{1,2}(C_1, \mathcal{O}(\epsilon^{3/2}); \epsilon) &:= \left[1 + \mathcal{O}(\epsilon^{3/2})\right] e^{-\pi^2/2\epsilon^2} e^{-\epsilon/2} - \frac{C_1 \epsilon^2}{\pi^2} \left[1 - \epsilon/2 + \mathcal{O}(\epsilon^{3/2})\right] \left[\frac{1}{2} + \mathcal{O}(e^{-2\pi/\sqrt{\epsilon}})\right] \\
e^{-\pi/\sqrt{\epsilon}} f_{2,2}(C_2, \mathcal{O}(\epsilon); \epsilon) &:= \frac{\pi}{\epsilon} \left[1 + \mathcal{O}(\epsilon)\right] \left[-1 + \mathcal{O}(\epsilon^{3/2})\right] e^{-\pi^2/2\epsilon^2} e^{-\epsilon/2} - \frac{C_2}{2\pi} \left[1 + \mathcal{O}(\epsilon)\right] \left[\frac{1}{2} + \mathcal{O}\left(\frac{1}{\sqrt{\epsilon}} e^{-2\pi/\sqrt{\epsilon}}\right)\right] \\
e^{-\pi/\sqrt{\epsilon}} f_{3,2}(C_3, \mathcal{O}(\epsilon^{3/2}); \epsilon) &:= \frac{\pi^2}{\epsilon^2} \left[1 + \mathcal{O}(\epsilon^{3/2})\right] \left[2 + \mathcal{O}(\epsilon^{3/2})\right] e^{-\pi^2/2\epsilon^2} e^{-\epsilon/2} - \frac{3C_3 \epsilon^2}{\pi^2} \left[1 + \mathcal{O}(\epsilon)\right] \left[\frac{1}{2} + \mathcal{O}(e^{-2\pi/\sqrt{\epsilon}})\right] \\
e^{-\pi/\sqrt{\epsilon}} f_{4,2}(C_4, \mathcal{O}(\epsilon); \epsilon) &:= \frac{\pi^3}{\epsilon^3} \left[1 + \mathcal{O}(\epsilon)\right] \left[-2 + \mathcal{O}(\epsilon^{3/2})\right] e^{-\pi^2/2\epsilon^2} e^{-\epsilon/2} - \frac{C_4}{2\pi} \left[1 + \mathcal{O}(\epsilon)\right] \left[\frac{1}{2} + \mathcal{O}\left(\frac{1}{\sqrt{\epsilon}} e^{-2\pi/\sqrt{\epsilon}}\right)\right].
\end{aligned}$$

We define

$$2\pi^2 \check{C}_1 := \epsilon^2 e^{\pi^2/2\epsilon^2} e^{\epsilon/2} C_1, \quad -4\pi^2 \check{C}_2 := \epsilon e^{\pi^2/2\epsilon^2} e^{\epsilon/2} C_2, \quad \frac{4\pi^4}{3} \check{C}_3 := \epsilon^4 e^{\pi^2/2\epsilon^2} e^{\epsilon/2} C_3, \quad \text{and} \quad -8\pi^4 \check{C}_4 := \epsilon^3 e^{\pi^2/2\epsilon^2} e^{\epsilon/2} C_4$$

and

$$\begin{aligned}
\check{f}_{1,2}(\check{C}_1; \epsilon) &:= e^{(\pi - \epsilon^{3/2})^2/2\epsilon^2} f_{1,2} \left( \frac{2\pi^2}{\epsilon^2} e^{-\pi^2/2\epsilon^2} e^{-\epsilon/2} \check{C}_1, \mathcal{O}(\epsilon^{3/2}); \epsilon \right) \\
\check{f}_{2,2}(\check{C}_2; \epsilon) &:= \epsilon e^{(\pi - \epsilon^{3/2})^2/2\epsilon^2} f_{2,2} \left( -\frac{4\pi^2}{\epsilon} e^{-\pi^2/2\epsilon^2} e^{-\epsilon/2} \check{C}_2, \mathcal{O}(\epsilon); \epsilon \right) \\
\check{f}_{3,2}(\check{C}_3; \epsilon) &:= \epsilon^2 e^{(\pi - \epsilon^{3/2})^2/2\epsilon^2} f_{3,2} \left( \frac{4\pi^4}{3\epsilon^4} e^{-\pi^2/2\epsilon^2} e^{-\epsilon/2} \check{C}_3, \mathcal{O}(\epsilon^{3/2}); \epsilon \right) \\
\check{f}_{4,2}(\check{C}_4; \epsilon) &:= \epsilon^3 e^{(\pi - \epsilon^{3/2})^2/2\epsilon^2} f_{4,2} \left( -\frac{8\pi^4}{\epsilon^3} e^{-\pi^2/2\epsilon^2} e^{-\epsilon/2} \check{C}_4, \mathcal{O}(\epsilon); \epsilon \right).
\end{aligned}$$

Now it is clear that  $\check{f}_{n,2}(1;0) = 0$  and

$$\left. \frac{d\check{f}_{n,2}}{d\check{C}_n} \right|_{(\check{C}_n;\epsilon)=(1;0)} \neq 0$$

so that the hypotheses of the Implicit Function Theorem are again satisfied. Expanding the unique function  $\check{C}_n(\epsilon)$  in orders of  $\epsilon$  we find  $\check{C}_n(\epsilon) = 1 + \mathcal{O}(\epsilon)$ , and, in particular,  $\check{C}_1(\epsilon) = 1 + \epsilon/2 + \mathcal{O}(\epsilon^{3/2})$ .

Putting everything together, and recalling the definitions  $\epsilon := \sqrt{2\nu t}$ ,  $I_s(\epsilon) := [\epsilon^{3/2}, 2\pi - \epsilon^{3/2}]$ ,  $I_f(\epsilon) := [-\epsilon^{3/2}, \epsilon^{3/2}]$ , we get that

$$\begin{aligned} \lambda_1 &= \frac{1}{2t} \left( -2 + \mathcal{O}(\xi_0 e^{-\xi_0^2} \check{\Lambda}_1) \right) = -1/t + \mathcal{O}(\epsilon^{1/2} e^{-1/\epsilon^2}), \\ \lambda_2 &= \frac{1}{2t} \left( -4 + \mathcal{O}(\xi_0^3 e^{-\xi_0^2} \check{\Lambda}_2) \right) = -2/t + \mathcal{O}(\epsilon^{-2} e^{-1/\epsilon^2}), \\ \lambda_3 &= \frac{1}{2t} \left( -6 + \mathcal{O}(\xi_0^5 e^{-\xi_0^2} \check{\Lambda}_3) \right) = -3/t + \mathcal{O}(\epsilon^{-7/2} e^{-1/\epsilon^2}), \\ \lambda_4 &= \frac{1}{2t} \left( -8 + \mathcal{O}(\xi_0^7 e^{-\xi_0^2} \check{\Lambda}_4) \right) = -4/t + \mathcal{O}(\epsilon^{-4} e^{-1/\epsilon^2}) \end{aligned}$$

are eigenvalues for (3.27) with associated eigenfunctions

$$\tilde{\phi}_1 : \left\{ \begin{array}{ll} \sup_x \left| e^{(x-\pi)^2/2\epsilon^2} \tilde{\phi}_1(x; t, \nu) + 1 \right| \leq C(\epsilon_0) \epsilon^{3/2} & : x \in I_s(\epsilon) \\ \sup_x \left| \frac{\epsilon^2}{2\pi^2} e^{\pi^2/2\epsilon^2} \operatorname{sech}\left(\frac{\pi x}{\epsilon^2}\right) \tilde{\phi}_1(x; t, \nu) - \left[ \operatorname{sech}^2\left(\frac{\pi x}{\epsilon^2}\right) \left(1 + \frac{x^2}{2\epsilon^2} + \frac{\epsilon^2}{2\pi^2}\right) - \frac{\epsilon^2}{2\pi^2} \right] \right| \leq C(\epsilon_0) \epsilon^{3/2} & : x \in I_f(\epsilon) \end{array} \right\} \quad (3.29a)$$

$$\tilde{\phi}_2 : \left\{ \begin{array}{ll} \sup_x \left| \frac{\epsilon}{x-\pi} e^{(x-\pi)^2/2\epsilon^2} \tilde{\phi}_2(x; t, \nu) + 1 \right| \leq C(\epsilon_0) \epsilon & : x \in I_s(\epsilon) \\ \sup_x \left| \frac{\epsilon}{2\pi} e^{\pi^2/2\epsilon^2} \tilde{\phi}_2(x; t, \nu) - \left[ \sinh\left(\frac{\pi x}{\epsilon^2}\right) + \frac{\pi x}{\epsilon^2} \operatorname{sech}\left(\frac{\pi x}{\epsilon^2}\right) \right] \right| \leq C(\epsilon_0) \epsilon & : x \in I_f(\epsilon) \end{array} \right\} \quad (3.29b)$$

$$\tilde{\phi}_3 : \left\{ \begin{array}{ll} \sup_y \left| \frac{\epsilon^2}{2(x-\pi)^2 - \epsilon^2} e^{(x-\pi)^2/2\epsilon^2} \tilde{\phi}_3(x; t, \nu) + 1 \right| \leq C(\epsilon_0) \epsilon^{3/2} & : x \in I_s(\epsilon) \\ \sup_y \left| \frac{3\epsilon^4}{4\pi^4} e^{\pi^2/2\epsilon^2} \operatorname{sech}\left(\frac{\pi x}{\epsilon^2}\right) \tilde{\phi}_3(x; t, \nu) - \operatorname{sech}^2\left(\frac{\pi x}{\epsilon^2}\right) \right| \leq C(\epsilon_0) \epsilon & : x \in I_f(\epsilon) \end{array} \right\} \quad (3.29c)$$

$$\tilde{\phi}_4 : \left\{ \begin{array}{ll} \sup_y \left| \frac{\epsilon^3}{(x-\pi)[2(x-\pi)^2 - 3\epsilon^2]} e^{(x-\pi)^2/2\epsilon^2} \tilde{\phi}_4(x; t, \nu) + 1 \right| \leq C(\epsilon_0) \epsilon & : x \in I_s(\epsilon) \\ \sup_y \left| \frac{\epsilon^4}{4\pi^3} e^{\pi^2/2\epsilon^2} \operatorname{csch}\left(\frac{\pi x}{\epsilon^2}\right) \tilde{\phi}_4(x; t, \nu) - 1 \right| \leq C(\epsilon_0) \epsilon & : x \in I_f(\epsilon) \end{array} \right\} \quad (3.29d)$$

which are expansions (2.12) and (3.5). Proposition 3.1 now follows from following proposition and Sturm-Liouville theory for periodic boundary conditions (c.f. [9, Thms 2.1, 2.14]), which states that the eigenvalues are strictly ordered  $\lambda_0 > \lambda_1 \geq \lambda_2 > \lambda_3 \geq \lambda_4 > \dots$  and that an eigenfunction with exactly  $2n$  crossings of zero in  $x \in [-\pi, \pi)$  is the eigenfunction associated either with  $\lambda_{2n-1}$  or with  $\lambda_{2n}$ .

**Proposition 3.10** *Fix  $\epsilon_0 \ll 1$  such that the eigenfunctions  $\tilde{\phi}_j(x; \epsilon)$  are given as in (3.29) for all  $0 \leq \epsilon \leq \epsilon_0$ . Then  $\tilde{\phi}_1(x; \epsilon)$  and  $\tilde{\phi}_2(x; \epsilon)$  have exactly two zeros in the interval  $x \in [-\pi, \pi)$  and the eigenfunctions  $\tilde{\phi}_3(x; \epsilon)$  and  $\tilde{\phi}_4(x; \epsilon)$  have exactly four zeros in the interval  $x \in [\epsilon^{3/2}, 2\pi - \epsilon^{3/2})$  for all  $0 \leq \epsilon \leq \epsilon_0$ .*

**Proof.** The  $n = 2, 4$  cases are clear since  $\sinh(\pi x/\epsilon) = 0$  at  $x = 0 \in I_f(\epsilon)$ ,  $\frac{x-\pi}{\epsilon}$  has a single zero at  $x = \pi \in I_s(\epsilon)$ , and  $2\left(\frac{x-\pi}{\epsilon}\right)^2 - 3$  has two zeros at  $x = \pi \pm \epsilon\sqrt{3/2} \in I_s(\epsilon)$ , and by making  $\epsilon_0$  potentially smaller so that  $-1 + \mathcal{O}(\epsilon_0) < 0$ . The result for  $n = 1, 3$  is then a direct consequence of Sturm-Liouville theory since  $\lambda_0 > \lambda_1 > \lambda_2$  and  $\lambda_2 > \lambda_3 > \lambda_4$ .  $\blacksquare$

## 4 Discussion

In this work we have proposed a candidate metastable family for Burgers equation with periodic boundary conditions, which we denote  $W$ . The metastable family is parametrized by three parameters: the spatial location  $x_0$ , time  $t_0$ , and mean  $c_0$ . Our choice of metastable family was motivated by our numerical experiments, one example of which is shown in Figure 1. We furthermore proposed an explanation for the metastable behavior of  $W$  based on the spectrum of the operator  $\mathcal{L}$  which results from linearizing the Burgers equation about  $W$ . In particular, we showed that by appropriately varying the parameters  $x_0$ ,  $t_0$ , and  $c_0$ , perturbations to  $W$  can be made orthogonal to the first three eigenfunctions for  $\mathcal{L}$ ; we furthermore showed why this means that initial data “near” the metastable family converges to the family much faster than the evolution along the family. These results are summarized in Theorems 1 and 2. From a technical perspective, we derived the first five eigenvalues for  $\mathcal{L}$  using Sturm-Liouville theory and ideas from singular perturbation theory. In particular, we show that there are two relevant space regimes, which we call the “slow” and “fast” space scales; we construct the eigenfunctions in each regime separately and then rigorously glue the functions together using a Melnikov-like computation.

It is worth reiterating that our results show that the spectrum for  $\mathcal{L}$  is, to leading-order, independent of the viscosity  $\nu$ ; this result is particularly interesting since our analysis is not valid for the inviscid equation. Furthermore, our results are in contrast to [2], in which the authors proposed a metastable family for the Navier–Stokes equation with periodic boundary conditions, denoted  $\omega^b$ , and provided numerical evidence and analytical arguments which indicate that the real part of the least negative eigenvalue for the operator obtained from linearizing the Navier–Stokes equation about  $\omega^b$  is proportional to  $\sqrt{\nu}$ ; in other words, the metastable behavior of  $\omega^b$  does depend on the viscosity. On the other hand, in [4], Bedrossian, Masmoudi and Vicol show that the solution behavior for the Navier–Stokes equation in a neighborhood of the Couette flow depends on the time-regime: for small enough time scales the solution behavior is governed by the inviscid limit of Navier–Stokes, whereas viscous effects dominate after long enough times. Thus, our results raise the question about whether there is an even earlier time regime for the Navier–Stokes with periodic boundary conditions than that studied in [2], and a potentially different metastable family, in which convergence to a metastable family is independent of the viscosity.

**Acknowledgments** The authors wish to thank M. Beck for numerous discussions of metastability in Burgers equations and the Navier–Stokes equation and to thank C.K.R.T. Jones, T. Kaper, and B. Sandstede for discussions of Lin’s method and singularly perturbed eigenvalue problems. The work of CEW is supported in part by the NSF grant DMS-1311553.

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