On Automorphisms and the Higher-Order Weierstrass points of Klein's Quartic Curve

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Weierstrass points

Let g_d^r be a base-point free complete linear system on M. Let $x \in M$. An integer n, $0 \leq n \leq d$, is called a **g_d^r-nongap at x** if there is a divisor $D \in g_d^r$ with $d \equiv nx + E$, where E is integral and doesn't contain x. Let n_i denote the i^{th} nongap. There are $r+1$ g_d^r-nongaps at x, $0 = n_0 < n_1 < \cdots < n_r \leq d$. The complement of this set in the integers between 0 and d is called the **g_d^r-gaps at x** and is denoted $g_1, \ldots, g_{d-r}$.

A **generalized Weierstrass point at g_d^r** is a point $x \in M$ where the r^{th} g_d^r-nongap, n_r is greater than r. A **Weierstrass point** is the special case where $g_d^r = |K|$ and a **higher-order Weierstrass point** is where $g_d^r = |nK|$.

The **weight of a generalized Weierstrass point** for $g_d^r = |G|$ at $x \in M$, $w(G, x)$, is defined to be the sum

$$\sum_{j=0}^{r} (n_j - j).$$

Results about Higher-Order Weierstrass points

Proposition 1 (Accola1). *Let $T \in Aut(M)$ be of prime order n. If T has 3 or more fixed points, then every fixed point of T is an n-Weierstrass point.*

Proposition 2 (Farkas and Kra). *Let $T \in Aut(M)$ be of prime order n. If T has 3 or more fixed points, then every fixed point of T is a q-q-Weierstrass point for $q \geq 2$, $q \equiv 1 (mod\, n)$.*

Proposition 3 (Guerrero). *Let $T \in Aut(M)$ of prime order n fixing two points, p_1 and p_2. Assume $M/\langle T \rangle$ has genus 1. Let φ be the projection $M \to M/\langle T \rangle$ and $\varphi(p_i) = a_i$, $i = 1, 2$. Then p_1 and p_2 are q-Weierstrass points for $q \geq 2$ if and only if $a_1 - a_2$ is a nonzero rational point on $M/\langle T \rangle$.*

Proposition 4 (Duma). *Let $T \in Aut(M)$ be of order 2. If $\nu(T) \geq 3$, then every fixed point of T is a q-Weierstrass point for $q \geq 2$.*

Weights of the Higher-Order Weierstrass points

Proposition 5 (Accola1). *Let T be an automorphism of M of order n, $\sum_{j=0}^{n-1} T^j M = W$ the orbit space, $\varphi : M \to W$ the natural map and $x \in M$ a fixed point of T. If W has genus 0, then $nK \equiv n(2g-2)x$.*

Proposition 6 (Accola1). *Suppose for $x \in M$ we have $mK \equiv m(2g-2)x$. Then*
(1) If $m = 1$, $l = 2, 3, \ldots,$ we have $w(lK, x) = g + w(K, x)$.
(2) If $m \geq 2$, and $l = 1, 2, \ldots$ we have $w(lmK, x) = w((lm+1)K, x) = g + w(K, x)$.
(3) If $m \geq 3$, $l = 1, 2, \ldots$ and $k = 2, 3, \ldots, m-1$ we have $w((lm+k)K, x) = w(kK, x)$.
(4) If $m \geq 4$, $l = 1, 2, \ldots$ and $k = 2, 3, \ldots, m-1$ we have $w((lm-k+1)K, x) = w(kK, x)$.

Bounds on the Weights

Theorem 1 (Accola1)**.** *Let* $|D| = g_d^r$ *be a complete linear series without fixed points of index* $i(D)$*. If* $x \in M$ *then*

$$w(D, x) \leq (p - i(D))(p - i(D) + 1)/2.$$

If we have equality, then M *is hyperelliptic.*

Theorem 2 (Accola1)**.** *Let* $|G| = g_d^r$ *be complete and base-point free with* $i(G) = 0$*. Suppose that* $x \in M$ *is not an ordinary Weierstrass point. Then*

$$w(G, x) \leq [(g + 1)^2/4].$$

If we have equality then the surface is hyperelliptic.