Higher Weierstrass Points of Klein's Quartic

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Abstract

Let M be a compact Riemann surface of genus $g \geq 2$. A higher-order Weierstrass point, or n-Weierstrass point, is a generalized Weierstrass point for the pluricanonical series $|nK|$. Many connections are known between the fixed points of the automorphisms of compact Riemann surfaces M and their Weierstrass points. We will apply these methods to Klein's quartic curve, $\mathfrak{X}$, the unique curve of genus 3 with 168 automorphisms. We will show the extent to which the fixed points of the automorphisms of $\mathfrak{X}$ are higher-order Weierstrass points and determine their weights.

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1 Introduction

Let M be a compact Riemann surface of genus $g \geq 2$. Let $|D|$ be a base-point free complete linear series on M with $r = \dim|D|$ and $d = \deg|D|$. A point $x \in M$ where a divisor $G \in |D|$ has order greater than r is called a generalized Weierstrass point for that linear series. If K is the canonical divisor, n is an integer ≥ 2, and $|D| = |nK|$ this is called a higher-order Weierstrass point, or n-Weierstrass point. In this notation an ordinary Weierstrass point corresponds to the case $n = 1$. The weight of a generalized Weierstrass point x, denoted $w(D, x)$, is an integer assigned to x depending on the sequence of zeros at x of the divisors in $|D|$.

There are well known connections between ordinary Weierstrass points and fixed points for non-trivial automorphisms. Specifically, the automorphisms of M fix as a set the Weierstrass points. Hurwitz [10] used this fact to prove that M can only have finitely many automorphisms. Kuribayashi and Komiya [12] use the Weierstrass points of some genus 3 curves to explicitly construct their automorphism groups. Lewittes [14] proved:

Theorem 1.1. *Let M be a compact Riemann surface of genus ≥ 2 and T a non-trivial automorphism of M. If T has at least 5 fixed points, then all of the fixed points of T are Weierstrass points of M.*

It is known [17] that the set of all higher-order Weierstrass points ($n \to \infty$) is dense in M. Since a curve can have only finitely many fixed points of automorphisms, studying them cannot give us complete knowledge of the higher-order Weierstrass points. They do however give us examples of what Silverman and Voloch [21] call the multiple Weierstrass points of a curve (points which are n-Weierstrass points for infinitely many n), of which they show a given curve can have only finitely many. We consider Klein's quartic curve, which has the maximal number of automorphisms for its genus, giving us more information on the the higher-order and multiple Weierstrass points than a general curve.

Accola [1] and [2], Duma [4], Farkas and Kra [5], and Guerrero [8] looked at conditions for fixed points of automorphisms to be higher-order Weierstrass points, and in some cases considered their weights. Horiuchi and Tanimoto [9] combined and expanded some of these results, especially for automorphisms of small orders.

In general, not many properties of higher-order Weierstrass points are known, especially for ones which are not fixed points of automorphisms. Silverman [20] considered the higher-order Weierstrass points on hyperelliptic curves, specifically over number fields, and the extensions that the Weierstrass points generate. It has also been proved [13] and again in [3] that a general curve of genus $g \geq 3$ has only weight 1 higher-order Weierstrass points.

Even in the case of ordinary Weierstrass points there are still many questions being answered. Magaard and Völklein [15] consider the transitivity of the the action of the automorphism groups on ordinary Weierstrass points of Hurwitz curves, of which Klein's quartic is one. Girard [7] investigates the structure of the groups generated by the ordinary Weierstrass points of plane quartics with 8 hyperflexes. Towse [22], [23] and [24] bounds the weights of specific cases of ordinary Weierstrass points. Rohrlich [19] discusses finding Weierstrass points of modular curves $\Gamma \setminus \mathcal{H}^*$ of genus $g \geq 2$, such as Klein's quartic curve, $\mathfrak{X} = X(7)$ [11], as roots of modular forms of weight $g(g+1)$ for Γ. Continuing the case of modular curves, higher-order Weierstrass points are the roots of automorphic forms of weight $(2n-1)^2 g(g-1)/2$ [6, 3.1]; this aspect will be considered in a paper in preparation.

In Section 2 we prove the following extensions of Theorem 1.1:

Theorem 1.2. *Let T be an involution of a compact Riemann surface M with canonical divisor K_M, $s \geq 2$ the number of fixed points A_l of T, $1 \leq l \leq s$, then*

$$w(nK_M, A_l) = \frac{1}{8}(-2+s)s.$$

Theorem 1.3. *Let T be an automorphism of prime order $t \geq 3$ on a compact Riemann surface M of genus $g \geq 3$ with $s = 2$ fixed points. If $n \equiv 0$ or $1 \pmod{t}$ then the fixed points of T are not n-Weierstrass points.*

Using these and previous results [1], we will show in Theorem 3.1 that, of the three types of fixed points of Klein's quartic curve, two types are higher-order Weierstrass points (in fact multiple

Weierstrass points), while the third type is not. We will also determine the weights of the higher-order Weierstrass points.

2 Weights of Higher Order Weierstrass Points

We begin by making some more precise definitions regarding generalized Weierstrass points and their weights.

Definition 2.1. Let $|D|$ be a base-point free complete linear series on M. Let $x \in M$. An integer n, $0 \leq n \leq d$, is called a $|D|$-nongap at x if there is a divisor $G \in |D|$ with $G \sim nx + E$, where E is integral and doesn't contain x.

Definition 2.2. Let n_i denote the i^{th} nongap. There are $r+1$ $|D|$-nongaps at x, $0 = n_0 < n_1 < \cdots < n_r \leq d$. The complement of this set in the integers between 0 and d is called the $|D|$-gaps at x and is denoted $g_1, \ldots, g_{d-r}$.

Definition 2.3. The weight of a generalized Weierstrass point for $|D|$ at $x \in M$, $w(D, x)$, is defined to be the sum

$$\sum_{j=0}^{r} (n_j - j).$$

Thus if we know the gap sequence for a Weierstrass point we know its weight, or if we know the weight, we know something about the gap sequence (for instance, if the weight is 1, we know the gap sequence precisely), but unfortunately not the gap sequence in general; however, we can still bound the weights of the n-Weierstrass points. The following theorem was proved for ordinary Weierstrass points by Hurwitz, and generalized to higher-order Weierstrass points by Accola [1].

Theorem 2.1. *Let $|D|$ be a complete linear series without fixed points of index $i(D)$. If $x \in M$ then*

$$w(D, x) \leq (g - i(D))(g - i(D) + 1)/2.$$

If we have equality, then M is hyperelliptic.

In the special case where the pluricanonical divisor, mK, is lineary equivalent to a multiple of the point $x \in M$ in question, [1] gives us very precise results with regards to the weights of the higher-order Weierstrass points:

Theorem 2.2. *[1, Cor. 3.10] Suppose for $x \in M$ we have $mK \equiv m(2g-2)x$. Then*
(1) If $m = 1$, $l = 2, 3, \ldots,$ we have $w(lK, x) = g + w(K, x)$.
(2) If $m \geq 2$, and $l = 1, 2, \ldots$ we have $w(lmK, x) = w((lm+1)K, x) = g + w(K, x)$.
(3) If $m \geq 3$, $l = 1, 2, \ldots$ and $k = 2, 3, \ldots, m-1$ we have $w((lm+k)K, x) = w(kK, x)$.
(4) If $m \geq 4$, $l = 1, 2, \ldots$ and $k = 2, 3, \ldots, m-1$ we have $w((lm-k+1)K, x) = w(kK, x)$.

Corollary 2.1. *Let $mK \equiv m(2g-2)x$ as above. Then $w(nK, x) = w(vK, x)$ for n, $v \geq 2$ and $n \equiv v \pmod{m}$.*

Proof. If $m = 1$ then for all $n \geq 2$ we have $w(nK, x) = g + w(K, x)$ by (1). If $m \geq 2$ and $n \equiv v \equiv 0 \,\text{or}\, 1 \pmod{m}$ then the result follows from (2). If $m \geq 3$ the remaining cases follow from (3). □

This will be useful for some automorphisms, more precisely:

Proposition 2.1. *[1, Cor. 4.5] Let T be an automorphism of M of order t, $\sum_{j=0}^{t-1} T^j M = W$ the orbit space, $\varphi : M \to W$ the natural map and $x \in M$ a fixed point of T. If W has genus 0, then $tK \equiv t(2g-2)x$.*

It is often the case however, that W has genus $h \neq 0$, leaving us unable to use the above results. In that case, let us assume that T has $s \geq 2$ fixed points, $A_1, \ldots, A_s \in M$, with images $a_1, \ldots, a_s \in W$ under φ. Then there exists a meromorphic function $y \in \mathcal{M}(M) \setminus \varphi^*(\mathcal{M}(W))$ such that $y^t \in \varphi^*(\mathcal{M}(W))$, $\mathcal{M}(M) = \varphi^*(\mathcal{M}(W))[y]$ and $y \circ T = \tau y$ where $\tau = e^{2\pi i/t}$ (see [2, §6.14]) for details). We can write the divisor of y as

$$(y) = \sum_{j=1}^{s} f_j A_j + \varphi^{-1}(E_0)$$

with $0 < f_j \leq t-1$, $\sum f_j \equiv 0 \pmod{t}$ and E_0 a divisor on W. Also, let $|D|$ be a non-special, $< T >$-invariant linear system, with $r = \dim|D|$ and $d = \deg|D|$, and $D_0 \in |D|$ satisfying $TD_0 = D_0$. Then D_0 can be written in the form

$$D_0 = \sum_{j=1}^{s} g_j A_j + \varphi^{-1}(G_0)$$

for some divisor G_0 on W, $0 \leq g_j < t$ and $d \equiv \sum g_j \pmod{t}$. In particular, for $|D| = |nK|$, $n \geq 2$, we have [16, V.1]

$$D_0 = \sum_{j=0}^{s} \overline{n(t-1)} A_j + \varphi^{-1}(nK_W + [\frac{n(t-1)}{t}](a_1 + \cdots + a_s))$$

where K_W is the canonical divisor for W, $\overline{n(t-1)}$ is the smallest non-negative residue of $n(t-1) \pmod{t}$ and $[\cdot]$ is the floor function. (Thus $n(t-1) = \overline{n(t-1)} + [\frac{n(t-1)}{t}]$.)

Theorem 2.3. *[2, §6.19] With notation as above, for $1 \leq l \leq s$,*

$$w(D, A_l) = \sum_{m=0}^{t-1} \frac{1}{2}(2t_{ml} + d_{ml}t + (d_m - 1)t)(d_m - d_{ml})$$

with $t_{ml} = \overline{mf_l + g_l}$, $d_m = (d - \sum_{j=1}^{s} t_{mj})/t - h + 1$ and d_{ml} is the number of integers in the set $\{0, \ldots, r\}$ congruent to $t_{ml} \pmod{t}$.

Theorem 2.4. *Let T be an involution, then*

$$w(nK_M, A_l) = \frac{1}{8}(-2 + s)s.$$

Proof. We begin by considering the Riemann-Hurwitz formula for the natural covering map $\varphi : M \to W$

$$2g - 2 = 2(2h - 2) + s(2 - 1),$$

which gives $s \equiv 0 \pmod 2$ and $h = (2g + 2 - s)/4$. Since $t = 2$, we have $f_j = 1$ for all $j = 1, \ldots, s$. As we saw above, since $|D| = |nK_M|$, the g_j's are also all equal, and in fact $g_j = 0$ if n is even and $g_j = 1$ if n is odd.

We consider four cases, depending on the parities of g and n. In the case where $g \equiv n \equiv 0 \pmod 2$, we have $g_j = 0$ and by definition (for $j = 1, \ldots, s$)

$$\begin{array}{ll} t_{0j} & = 0 \qquad t_{1j} = 1 \\ d_0 & = \frac{d}{2} - h + 1 \\ d_1 & = \frac{d-s}{2} - h + 1. \end{array}$$

Also, $r + 1 = (2n - 1)(g - 1)$ is odd, so

$$d_{0j} = \frac{r}{2} + 1 \qquad d_{1j} = \frac{r}{2}.$$

This gives

$$w(nK_M, A_l) = \frac{1}{8}(-2 + s)s$$

after simplification. The other 3 cases are similar. □

Theorem 2.5. *Let T be an automorphism of prime order $t \geq 3$ on a compact Riemann surface M of genus $g \geq 3$ with $s = 2$ fixed points. If $n \equiv 0$ or $1 \pmod t$ then the fixed points of T are not n-Weierstrass points.*

Proof. As above, we consider the Riemann-Hurwitz formula for the natural covering map $\varphi : M \to W$

$$2g - 2 = t(2h - 2) + 2(t - 1),$$

which gives $t|g$ and $h = g/t$. Recall that, for K the canonical divisor for M, we have $d = \deg|nK| = n(2g - 2)$ and $r = \dim|nK| = (2n - 1)(g - 1) - 1 = d - g$.

Next we consider the possible values for the $f_j^\prime s$. By definition we know that $0 < f_j \leq t - 1$ and $\sum f_j \equiv 0 \pmod t$. Since $s = 2$ this means $f_1 + f_2 = t$, so if $f_1 = b$ for $b \in \{1, 2, \ldots, t-1\}$, $f_2 = t - b$.

Case $n \equiv 0 \pmod t$:
Since $0 \leq g_j \leq t - 1$ and $g_j \equiv n(t - 1) \pmod t$, we have $g_j = 0$ for $j = 1, 2$. This gives $t_{01} = t_{02} = 0$. For $1 \leq m \leq t - 1$, by definition $t_{m1} = \overline{mf_1}$, and $t_{m2} = \overline{mf_2} = \overline{m(t - f_1)}$, so $t_{mj} \neq 0$ and $t_{m1} + t_{m2} = t$, since t is prime. Thus

$$d_0 = \frac{d}{t} - h + 1 = \frac{r}{t} + 1, \qquad d_m = \frac{d - t}{t} - h + 1 = \frac{r}{t}.$$

Next, notice that $r = d - g \equiv 0 \pmod t$, so by definition,

$$d_{0j} = \frac{r}{t} + 1, \qquad d_{mj} = \frac{r}{t}$$

for $j = 1, 2$ and $1 \le m \le t-1$. Thus

$$w(nK_M, A_l) = 0$$

for each of the fixed points A_l of T.

Case $n \equiv 1 \pmod{t}$:
Since $0 \le g_j \le t-1$ and $g_j \equiv n(t-1) \pmod{t}$, we have $g_j = t-1$ for $j = 1, 2$. This gives $t_{01} = t_{02} = t-1$. For $1 \le m \le t-1$, we have $t_{m1} + t_{m2} \equiv (mf_1 + t - 1) + (m(t - f_1) + t - 1) \equiv t - 2 \pmod{t}$. By definition $t_{mj} = mf_j + (t-1)$; since t is prime, $mf_j \not\equiv 0 \pmod{t}$, hence $t_{mj} \neq t-1$. So $0 \le t_{mj} \le t-2$, which implies $t_{m1} + t_{m2} = t-2$. Thus for $1 \le m \le t-1$

$$d_0 = \frac{d - 2(t-1)}{t} - h + 1 = \frac{r+2}{t} - 1, \qquad d_m = \frac{d - (t-2)}{t} - h + 1 = \frac{r+2}{t}.$$

Next, notice that $r + 1 \equiv t - 1 \pmod{t}$, so the set $\{0, \dots, r\}$ has one fewer entry equivalent to $t-1$ modulo t than the other congruence classes, so

$$d_{0j} = \frac{r+2}{t} - 1, \qquad d_{mj} = \frac{r+2}{t}$$

for $j = 1, 2$ and $1 \le m \le t-1$. Thus

$$w(nK_M, A_l) = 0$$

for each of the fixed points A_l of T. □

Corollary 2.2. *Let T be an automorphism of order 3 on a compact Riemann surface M of genus $g = 3$ with $s = 2$ fixed points. Then the fixed points of T are not n-Weierstrass points for any $n \ge 2$.*

Proof. By the Theorem, the only case we have to check is if $n \equiv 2 \pmod{3}$. Then $g_j = 1$ for $j = 1, 2$. This implies that $t_{01} = t_{02} = 1$ and $t_{m1} + t_{m2} = 2$ for $m = 1, 2$. Thus

$$d_m = \frac{d-2}{3} - h + 1 = \frac{r+1}{3}$$

for $m = 0, 1, 2$. Next we notice that $r + 1 \equiv 0 \pmod{3}$, so

$$d_{mj} = \frac{r+1}{3}$$

for $j = 1, 2$ and $m = 0, 1, 2$. Hence for each of the fixed points A_l of T we have

$$w(nK, A_l) = 0.$$

□

3 Klein's quartic curve and its automorphisms

Klein's quartic, $\mathfrak{X}$, (also known as the modular curve $X(7)$) is the genus 3 curve canonically modelled by the equation

$$XY^3 + YZ^3 + ZX^3 = 0.$$

$\mathfrak{X}$ has 168 automorphisms, the maximum for a curve of genus 3. Let F be the map of $\mathfrak{X}$ onto its orbits under the 168 automorphisms. Then the fixed points of the automorphisms correspond to the branch points of F. It turns out that F has three types of branch points: 24 points of multiplicity 7, 56 points of multiplicity 3, and 84 points of multiplicity 2. After Klein [11], we will call these *a-points*, *b-points* and *c-points* respectively. Klein identifies these points on the curve as the inflection points (*a-points*), contact points of the bitangents (*b-points*) and sextatic points (*c-points*). On the modular curve $X(7)$ we find that the *a-points* are the cusps, and the *b-points* and *c-points* are the elliptic points coming from i and $\omega = e^{2\pi i/3}$ respectively.

Being a curve of genus 3, $\mathfrak{X}$ has at most $(g-1)g(g+1) = 24$ ordinary Weierstrass points, counted with weight. It is not hard to show that $\mathfrak{X}$ achieves this upper bound (see [5, VII], [18] or [19] for details). Thus the 24 *a-points* of $\mathfrak{X}$ are the ordinary Weierstrass points, each with weight 1.

Following [12, §4], we can explicitly describe the 24 Weierstrass points and use them to define the automorphisms of $\mathfrak{X}$. The Weierstrass points are:

$$A_0 = (0,0,1),^T \;\; A_1 = (0,1,0),^T \;\; A_\infty = (1,0,0),^T \;\; P_{ij} = (x_{ij}, y_{ij}, 1),^T$$

for $1 \leq i \leq 3$, $1 \leq j \leq 7$. Here the y_{ij}'s are the roots of $(y^7)^3 + 289(y^7)^2 - 57y^7 - 1 = 0$, specifically, if ω_1, ω_2 and ω_3 are the real roots, then $y_{ij} = \zeta^{j-1}\omega_i$ where $\zeta = e^{2\pi i/7}$. The x_{ij}'s are $x_{ij} = -\zeta^{3j-j}\omega_i/(\omega_i+1)$. Then the group of 168 automorphisms of Klein's quartic is generated by three elements, σ, τ and λ, which act in the following ways:

Element 1: $\sigma(A_k) = A_k$ for $k \in \{0, 1, \infty\}$ and for $i = 1,2,3$, $\sigma(P_{ij}) = P_{i(j+1)}$, with $j+1$ taken modulo 7. Thus σ is an automorphism of order 7 with three fixed points.

Element 2: $\tau(A_0) = A_1$, $\tau(A_1) = A_\infty$, $\tau(A_\infty) = A_0$ and τ permutes the set $\{P_{ij}\}$. τ generates a cyclic group of automorphisms of order 3.

Element 3: $\lambda(A_\infty) = P_{11}$. This is an involution, and in fact if we let $\lambda = \lambda_{11}$, we can write $\lambda_{ij} = \sigma^k\tau^m\lambda$ for suitable k and m.

We find that σ and its conjugates each fix 3 *a-points*. Similarly, the fixed points of the other automorphisms of $\mathfrak{X}$ are as follows [11]: τ and its conjugates each fix 2 *b-points*, λ and its conjugates each fix 4 *c-points*.[1]

Let $w_a(n)$ be the weight of the *a-points* as n-Weierstrass points, $w_b(n)$ the weight of the *b-points* and $w_c(n)$ the weight of the *c-points*. Also, let $w_0(n)$ be the combined weight as n-Weierstrass

[1] Note that this easily gives coordinates for *b-points* by computing the eigenvectors of the matrices in [12]. The algebraic coordinates for the *c-points* are more easily computed using the birationally equivalent model for $\mathfrak{X}$, $w^7 = z(z-1)^2$ in [5, VII.3] as roots of the polynomial $1 + 510z - 14631z^2 + 80090z^3 - 218058z^4 + 316290z^5 - 253239z^6 + 131562z^7 - 70998z^8 + 37950z^9 - 8955z^{10} - 522z^{11} + z^{12}$, from the Wronskian for $\mathcal{L}(2K)$.

points of the general points of $\mathfrak{X}$ that are on separate orbits for the automorphism group, i.e. so that $168w_0(n)$ is the total weight of all the general points of $\mathfrak{X}$ as n-Weierstrass points. Then

$$168w_0(n) + 84w_c(n) + 56w_b(n) + 24w_a(n) = (2n-1)^2 \cdot 12 \quad (1)$$

using the formula $(2n-1)^2(g-1)^2g$ for the total weight of all n-Weierstrass points in [2, §2.6] and [3].

Theorem 3.1. *The fixed points of the automorphisms of Klein's quartic curve appear as higher-order Weierstrass points as follows:*

1. *$a-$points: Let $n \geq 2$, then*
   - *(a) If $n \equiv 0 \,\text{or}\, 1 \pmod 7$, then $w_a(n) = 4$*
   - *(b) If $n \equiv 2 \,\text{or}\, 6 \pmod 7$, then $w_a(n) = 1$.*
   - *(c) If $n \equiv 3 \,\text{or}\, 5 \pmod 7$, then $w_a(n) = 2$.*
   - *(d) If $n \equiv 4 \pmod 7$, then $w_a(n) = 0$.*
2. *$b-$points: $w_b(n) = 0$ for all n.*
3. *$c-$points: $w_c(n) = 1$ for all $n \geq 2$.*

Proof. (1) As noted above, the *a-points* are the ordinary Weierstrass points for $\mathfrak{X}$, each with weight 1. We also know that σ and its conjugates, which are of order 7, each fix three *a-points*. If we apply the Riemann-Hurwitz formula, we find that the orbit space of $\mathfrak{X}$ under each of these automorphisms has genus 0. Thus Proposition 2.1 above implies that $7K \sim 28x$ for x any *a-point*, hence we can apply Theorem 2.2 with $m = 7$. We find that, if $n \equiv 0 \pmod 7$ and $v \equiv 1 \pmod 7$, n, $v > 1$, Corollary 2.1 and Theorem 2.2(2) give us

$$w_a(n) = w_a(v) = 3 + w_a(1) = 4.$$

If $n \equiv 2 \pmod 7$ and $v \equiv 6 \pmod 7$, n, $v > 1$, Corollary 2.1 and Theorem 2.2(4) with $k = 2$ give us

$$w_a(n) = w_a(v) = w_a(2).$$

Finally, if $n \equiv 3 \pmod 7$ and $v \equiv 5 \pmod 7$, n, $v > 1$, Corollary 2.1 and Theorem 2.2(4) with $k = 3$ give us

$$w_a(n) = w_a(v) = w_a(3).$$

Hence it is sufficient to determine the values $w_a(2)$, $w_a(3)$, and $w_a(4)$ to prove (1).

We now consider equation (1). For $w_a(2)$ we have

$$168w_0(2) + 84w_c(2) + 56w_b(2) + 24w_a(2) = 108,$$

the only integral solution to which is $w_0(2) = w_b(2) = 0$, $w_c(2) = w_a(2) = 1$.

By Theorem 2.1, since $\mathfrak{X}$ is not hyperelliptic and $i(nK) = 0$ for $n \geq 2$, we know that $w(nK, x) < 6$ for all $x \in \mathfrak{X}$. Considering equation (1) modulo 7, we can then determine the remaining values of $w_a(n)$ [2, §6.1]:

$$w_a(3) = w_a(5) = 2, \quad w_a(4) = 0.$$

(2)We now consider the *b-points.* We know that they are neither ordinary nor 2-Weierstrass points. We also know that τ and its conjugates are of order 3 and each fix 2 *b-points.* Thus we can apply Corollary 2.2, which tells us that $w_b(n) = 0$ for all $n \geq 2$.

(3)Finally, we consider the *c-points.* We have seen that they are 2-Weierstrass points of weight 1. To determine the situation for the *c-points* as n-Weierstrass points for $n \geq 3$, apply Proposition 2.4 with $s = 4$, hence for all values of $n \geq 2$ we have $w_c(n) = 1/8(-2+4)4 = 1$. □

Remark: Olsen's result [17] that the set of all higher-order Weierstrass points is dense tells us that eventually any M must have higher-order Weierstrass points which are not fixed points of automorphisms. Notice that our results force $w_0(n) \geq 1$ for all $n \geq 3$.

We gather the results below (when $n > 8$ the table repeats as in Corollary 2.1):

n	*a-points*	*b-points*	*c-points*	Other points	Total $(2n-1)^2 12$
1	1×24	0	0	0	24
2	1×24	0	1×84	0	108
3	2×24	0	1×84	168	300
4	0	0	1×84	504	588
5	2×24	0	1×84	840	972
6	1×24	0	1×84	1344	1452
7	4×24	0	1×84	1848	2028
8	4×24	0	1×84	2520	2700

Table 1: The Higher-Order Weierstrass points for Klein's Quartic Curve

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