# Dissertation Prospectus

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Klein's quartic curve, $\mathfrak{X}$, is the focus of this thesis. Also known as the modular curve $X(7)$, it is the genus 3 curve canonically modelled by the equation

$$XY^3 + YZ^3 + ZX^3 = 0.$$

$\mathfrak{X}$ has 168 automorphisms, the maximum for a curve of genus 3, making $\mathfrak{X}$ the smallest genus Hurwitz curve. Also unusual is the fact that $\mathfrak{X}$ has a split Jacobian [11]. This curve has a long tradition as the subject of much study due to its diverse applications in mathematics (see [10]) and computer science (see [7]).

We will examine three distinct aspects of this curve. The first is the most geometric, where we will look at subvarieties of moduli spaces (higher-order Weierstrass points). The second is more transcendental, using the special symmetries of the period lattice for this curve to define a genus 3 arithmetic-geometric mean. The final aspect stems from the second, where the special properties of $\mathfrak{X}$, in particular the elliptic curves contained in the Jacobian [11], leads to a need for a greater understanding of the genus 1 arithmetic-geometric mean.

We now address the first aspect in more detail. Let $M$ be a compact Riemann surface of genus $g \geq 2$ with canonical divisor $K$. A point $x \in M$ where a divisor $G \in |K|$ has order greater than $g-1$ is called a Weierstrass point for that linear series. The moduli space of curves with automorphisms can be stratified by these Weierstrass points. Less understood, however, is the generalization of this concept to the pluricanonical series $|nK|$, for $n \geq 2$, called higher-order Weierstrass points.

There are well-known connections between ordinary Weierstrass points and the fixed points of non-trivial automorphisms. Specifically, the automorphisms of $M$ fix as a set the Weierstrass points. There are also many connections to higher-order Weierstrass points [2]. We apply the methods from that book to the types of fixed points of non-trivial automorphisms in $\mathfrak{X}$, with the following results:

**Theorem 0.1.** *Let $T$ be an involution of a compact Riemann surface $M$ with canonical divisor $K$, and $s \geq 2$ the number of fixed points $A_l$ of $T$, $1 \leq l \leq s$, then*

$$w(nK, A_l) = \frac{1}{8}(-2+s)s.$$

*where $w(nK, A_l)$ is the weight of $A_l$ as an $n$-Weierstrass point.*

**Theorem 0.2.** *Let $T$ be an automorphism of prime order $t \geq 3$ on a compact Riemann surface $M$ of genus $g \geq 3$ with 2 fixed points. If $n \equiv 0$ or $1 \pmod{t}$ then the fixed points of $T$ are not $n$-Weierstrass points.*

Using these and previous results [1], we will show that, of the three types of fixed points of Klein's quartic curve, two types are higher-order Weierstrass points, in fact for infinitely many $n \geq 2$, while the third type is not. We will also determine the weights of the higher-order Weierstrass points.

We now consider the second, more transcendental aspect. The classical arithmetic-geometric mean (AGM) for elliptic curves has been generalized to genus 2 [12] and more recently to genus 3 [9]. It has also been shown that genus 3 is the last genus for which an AGM algorithm is feasible [5]. Some degree of explicitness was reached for genus 3 [9] using recent results [3] which associate the bitangents of a curve to its moduli. However, even though the 28 bitangents for the original curve $C$ will provide a quartic polynomial for the resulting curve $C'$, defined by a choice of Lagrangian subgroup of $Jac(C)[2]$, there is no algorithm to obtain $C'$ as a canonical quartic depending on the coefficients of $C$ because of the need for computer algebra systems in the calculations. The goal of this aspect is to develop a construction for the AGM image of $\mathfrak{X}$ using this curve's split Jacobian and the elliptic curve AGM. We will compare the resulting curve with the AGM for $\mathfrak{X}$ given by the previously defined genus 3 AGM.

We will begin to address our final aspect by considering some classical results. Gauss's AGM is defined for the real numbers $a$ and $b$ as the common limit of the sequences

$$\begin{array}{ll} a_0 = a & b_0 = b \\ a_{n+1} = \frac{a_n + b_n}{2} & b_{n+1} = (a_n b_n)^{1/2}. \end{array}$$

Generalizing this process to the complex numbers presents a new problem: the choice of the square root defining $b_{n+1}$ is no longer obvious. For the complex numbers, we need to make a choice for the square-root; a "right" choice can be defined [4] and, as long as only finitely many "wrong" choices are made, the sequences will have an interesting (i.e. non-zero) common limit. The collection of these limits is described using Jacobi's theta functions, $\theta_{00}(\tau) = 1 + 2\sum_{n=1}^{\infty} e^{2\pi i n^2}$ and $\theta_{01}(\tau) = 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{2\pi i n^2}$, which are modular forms of weight 1 for $\Gamma(2)_0$ and $\Gamma_2(4)$, respectively. This results from the relations

$$\theta_{00}(\tau)^2 + \theta_{01}(\tau)^2 = 2\theta_{00}(2\tau)^2$$

$$\theta_{00}(\tau)\theta_{01}(\tau) = \theta_{01}(2\tau)^2.$$

These relations also serve as the inspiration for the elliptic curve AGM. Let $E : y_0^2 = x_0(x_0 - a_0^2)(x_0 - b_0^2) \approx \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ be an elliptic curve. Then the classically known AGM of $E$ is the elliptic curve $E' : y_1^2 = x_1(x_1 - a_1^2)(x_1 - b_1^2) \approx \mathbb{C}/(\mathbb{Z} + 2\tau\mathbb{Z})$ [6], where $a_1$ and $b_1$ are the arithmetic and geometric means of $a_0$ and $b_0$, respectively. This process is easily seen to be moding out by the 2-torsion point, $(0, 0)$, on the analytic model, and $1/2$ on the algebraic model of $E$. It is also possible to make a similar construction moding out by the other 2-torsion points. In the case of complex values for $a_0$ and $b_0$, the complex case for the original AGM also extends to a complete understanding of the AGM for elliptic curves.

**Theorem 0.3.** *Let $E_0/\mathbb{C}$ be an elliptic curve $y_0^2 = x_0(x_0 + a_0^2)(x_0 + b_0^2)$ isomorphic to a torus $\mathbb{C}/(\tau\mathbb{Z} + \mathbb{Z})$ for $\tau \in F$, where $F$ is the fundamental domain for $\Gamma_2(4)$. Then for all $\gamma \in SL_2(\mathbb{Z})$ we get a different AGM depending as*

1. *$\gamma \in \{\pm 1\}\Gamma_2(4)$: all good sequences for $a_0$ and $b_0$, corresponding to $\{\gamma\tau, 1\} \mapsto \{2\gamma\tau, 1\}$*
2. *$\gamma \in \{\pm 1\}\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)\Gamma_2(4)$: all good sequences for $a_0$ and $-b_0$, corresponding to $\{\gamma\tau, 1\} \mapsto \{2\gamma\tau, 1\}$*
3. *$\gamma \in \{\pm 1\}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\Gamma_2(4)$: all good sequences for $\sqrt{a_0^2 - b_0^2}$ and $a_0$, corresponding to $\{[\gamma]\tau, 1\} \mapsto \{\frac{[\gamma]\tau}{2}, 1\}$*
4. *$\gamma \in \{\pm 1\}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)\Gamma_2(4)$: all good sequences for $-\sqrt{a_0^2 - b_0^2}$ and $a_0$, corresponding to $\{[\gamma]\tau, 1\} \mapsto \{\frac{[\gamma]\tau}{2}, 1\}$*
5. *$\gamma \in \{\pm 1\}\left(\begin{smallmatrix} 1 & 0 \\ 1 & -1 \end{smallmatrix}\right)\Gamma_2(4)$: all good sequences for $b_0$ and $\sqrt{b_0^2 - a_0^2}$, corresponding to $\{[\gamma]\tau, 1\} \mapsto \{\frac{1+[\gamma]\tau}{2}, 1\}$*
6. *$\gamma \in \{\pm 1\}\left(\begin{smallmatrix} 1 & 0 \\ 1 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)\Gamma_2(4)$: all good sequences for $b_0$ and $-\sqrt{b_0^2 - a_0^2}$, corresponding to $\{[\gamma]\tau, 1\} \mapsto \{\frac{1+[\gamma]\tau}{2}, 1\}$*

*where $[\gamma]$ is the $\Gamma_2(4)$ component of $\gamma$.*

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