3

Infinite sequences and series

2.1 Ordered lists of numbers

A **sequence** is an **ordered** list[1] of real numbers $a_1, a_2, a_3, \ldots, a_n$ indexed by natural numbers $1, 2, 3, \ldots, n$. The general term a_n of the sequence corresponds—is a function—of natural index n. The term a_n of the sequence is the number found in the nth position counting from the left. For sequences the tidy **subscript** notation a_n is preferred to the cluttered function notation $a(n)$. We have seen in previous sections how perpetual or iterative algorithms such as the method of bisection generate such sequences. A sequence of N terms is denoted by $\{a_n\}_{n=1}^{N}$. An infinite sequence is denoted by $\{a_n\}_{n=1}^{\infty}$ or merely $\{a_n\}$.

Sequence a_n generated by the bisection method of section 1.6 is **bounded below** by 9/7. In fact, 9/7 is the greatest lower bound, the **glb**, of the sequence, namely, $a_n \geq 9/7$ for any n. Any term of sequence $\{b_n\}$ is an upper bound on $\{a_n\}$. For instance, $317/224 \geq a_n$ for any n. Irrational number $\sqrt{2}$ is the least upper bound, the **lub**, on $\{a_n\}$. That $\sqrt{2}$ is an upper bound on $\{a_n\}$ is obvious. That no number less than $\sqrt{2}$ is an upper bound on $\{a_n\}$ results from the fact that with a sufficiently large n, a_n can be brought as close to $\sqrt{2}$ as only desired. In the same way $\sqrt{2}$ is the greatest lower bound on sequence $\{b_n\}$. The least upper bound on $\{a_n\}$ is called its **supremum**. Here $\sup\{a_n\} = \sqrt{2}$. The greatest lower bound on $\{a_n\}$ is called its **infimum**. Here $\inf\{a_n\} = 9/7$. A sequence bounded below and above is said to be bounded. If the least upper bound on $\{a_n\}$ is a member of the sequence, then it is said to be a **maximum** of the sequence. Similarly, if the greatest lower bound of on $\{a_n\}$ is a member of the sequence, then it is said to be a **minimum** of the sequence. Formally;

Definition:

Real number u is the least upper bound, **lub**, *on $\{a_n\}$ if*

1. *$a_n \leq u$ for any n.*
2. *For any, willfully small, $\epsilon > 0$, natural number n exists such that $a_n > u - \epsilon$.*

Real number l is the greatest lower bound, **glb**, *on $\{a_n\}$ if*

1. *$a_n \geq l$ for any n.*
2. *For any, willfully small, $\epsilon > 0$, natural number n exists such that $a_n < l - \epsilon$.*

The infinite sequence $\{a_n = 1/n\}_{n=1}^{\infty}$, for example, is bounded below by any number less then zero, and it is bounded above by any number greater then one. In fact, $\inf\{a_n\} = 0$, and $\sup\{a_n\} = \max\{a_n\} = 1$.

Theorem (the completeness property of sequences): *A sequence bounded above has a least upper bound. A sequence bounded below has a greatest lower bound.*

Proof: Consider a sequence bounded above. Let b_1 be known to be an upper bound on the sequence, and a_1 known not to be an upper bound on the sequence. Starting a bisection algorithm with a_1 and b_1, under the assumption that for any number x in the interval

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[1] Related to *loose*, *loss* and *less*.

$a_1 \leq x \leq b_1$ a clear cut determination can be made as to wether it is an upper bound or not, we trap the lub of the sequence. Indeed, any number greater then the trapped number is an upper bound of the sequence, and any number less then the trapped number is not. In the same way we prove the second part of the theorem for sequences bounded below. End of proof

2.2 The limit of an infinite sequence

Definition 1: *real number l is said to be the limit of sequence $\{a_n\}$ if to any $0 < \epsilon < \epsilon_0$, and willfully small ϵ_0, there is a natural number N so that $|a_n - l| < \epsilon$ for all $n > N$.*

We formally write this as $a_n \rightarrow l$ as $n \rightarrow \infty$, pronounced a_n tends to l as n tends to infinity. Or we write this as $\lim_{n\rightarrow\infty} a_n = l$.

Examples.

1. Obviously, $\lim_{n\rightarrow\infty}(1/n) = 0$. Writing the sequence $1/1, 1/2, 1/3, \ldots$ we will never have occasion to write 0 no matter how long we keep writing. But we **know** that $1/n$ can be made willfully small with a sufficiently large n. In fact, $\lim_{n\rightarrow\infty}(c/n) = 0$ for any number c.

2. To prove that

$$\lim_{n\rightarrow\infty}\frac{2n+5n}{3n-7} = \lim_{n\rightarrow\infty}\frac{2+5/n}{3-7/n} = \frac{2}{3}.$$

we first establish that

$$\left|\frac{2n+5n}{3n-7} - \frac{2}{3}\right| = \left|\frac{29}{9n-21}\right|$$

and assume that $n > 2$, so that $9n - 21 > 0$ and $29/(9n-21) < 29/6$. We need to show now that a natural n exists so that $29/(9n-21) < \epsilon$ for any $0 < \epsilon < 29/6$. Simple algebra leads to $n > (29 + 21\epsilon)/(9\epsilon)$.

3. We verify that the sequence $\{a_n = (n^2+1)/(n+2)\}_{n=1}^{\infty}$ has no limit—the sequence does not **converge** but **diverges**. We also verify that the sequence $\{a_n = 1 + (-1)^n\}_{n=1}^{\infty}$ is without a limit.

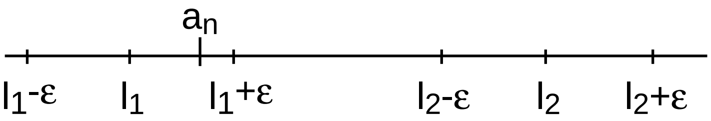
Definition 2: *Real number l is the limit of $\{a_n\}_{n=1}^{\infty}$ if for any, willfully small, $0 < \epsilon < \epsilon_0$ almost all, namely, all except for a finite number, terms of the sequence are located in the interval $l - \epsilon \leq x \leq l + \epsilon$.*

Theorem: *Definitions 1 and 2 for the limit of a sequence are equivalent.*

Proof: Assume convergence according to definition 1. Then for any $0 < \epsilon < \epsilon_0$ there is a natural N so that $l - \epsilon < a_n < l + \epsilon$ for all $n > N$ implying that all terms of the sequence except for possibly $a_1, a_2, a_3, \ldots, a_N$ are in this interval. The sequence converges also according to definition 2. Assume convergence according to definition 2. Then for any $0 < \epsilon < \epsilon_0$ most terms of the sequence are such that $l - \epsilon < a_n < l + \epsilon$. Namely, there is a natural number N so that $l - \epsilon < a_n < l + \epsilon$ for all $n > N$. The sequence converges also according to definition 1. End of proof.

Theorem: *A convergent sequence has only one limit.*

Proof. By contradiction. Suppose $\{a_n\}$ converges to the two distinct limits l_1 and l_2 such that $l_2 > l_1$. Chose $0 < \epsilon < (l_2 - l_1)/2$. By the assumption that $\{a_n\}$ converges natural number N exists such that, if $n > N$, then $|a_n - l_1| < \epsilon$ and $|a_n - l_2| < \epsilon$, which is impossible with this choice of ϵ. See the figure below. End of proof.



Theorem: *An increasing (non decreasing), bounded above, sequence has a limit equal to its least upper bound. A decreasing (non increasing), bounded below, sequence has a limit equal to its greatest lower bound.*

Proof: We shall prove the first part only; proof of the second part is analogous. An increasing, bounded above, sequence has a least upper bound u. There is at least one $n = N$ such that $u - a_N < \epsilon$ for any, willfully small $\epsilon > 0$. Since the sequence is non decreasing, $a_{n+1} \geq a_n$, it follows that $u - a_n < \epsilon$ for all $n \geq N$. The rest of the sequence is in this interval. End of proof

Theorem: *If the three sequences* $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ are such that $a_n \leq c_n \leq b_n$ for all $n > N$ and such that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = l$, then also $\lim_{n\to\infty} c_n = l$.

Proof: For any $0 < \epsilon < \epsilon_0$ there exist a natural number N_1 so that $-\epsilon < a_n - l < \epsilon$ for all $n > N_1$. Also, for any $0 < \epsilon < \epsilon_0$ there exist a natural number N_2 so that $-\epsilon < b_n - l < \epsilon$ for all $n > N_2$. Let $N = \max\{N_1, N_2\}$. For any $n > N$ we have that

$$-\epsilon < a_n - l \leq c_n - l \leq b_n - l < \epsilon$$

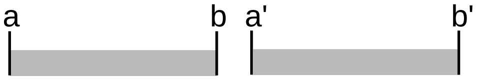
or $|c_n - l| < \epsilon$ and $\lim_{n\to\infty} c_n = l$. End of proof.

2.3 The Cauchy and the Bolzano-Weierstrass theorems

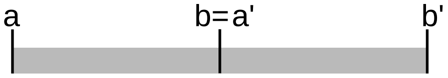
It stands to reason that sequence converges if and only if variation of its terms subside as their index n climbs. This is the essence of the following theorem, known as Cauchy's criterion.

Lemma: *The intersection of open intervals is an open interval.*

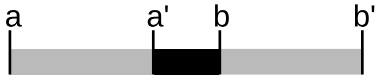
Proof. See the figure below.



(a,b) ∩ (a',b')=0

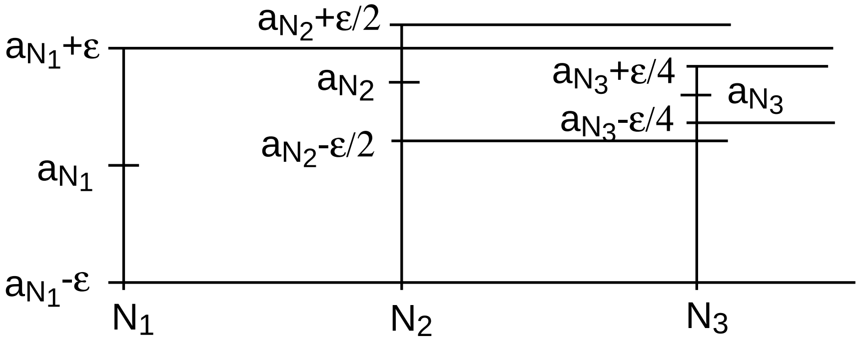


(a,b) ∩ (a',b')=0



(a,b) ∩ (a',b')=(a',b)

Theorem (Cauchy): *A necessary and sufficient condition for sequence* $\{a_n\}_{n=1}^{\infty}$ *to converge is that for any* $0 < \epsilon < \epsilon_0$ *there exist a natural number* N *so that* $|a_n - a_m| < \epsilon$ *for all* $n, m \geq N$.



Proof. Refer to the figure above.

Assume first that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to the limit l. By this assumption there exists a natural number N depending on the choice of $\epsilon > 0$, so that for every $n \geq N$ the inequality $|a_n - l| < \frac{\epsilon}{2}$ holds. Consequently for every $m, n \geq N$

$$|a_m - a_n| = |a_m - l + l - a_n| \leq |a_m - l| + |l - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which is the Cauchy criterion.

Assume now that the sequence satisfies the conditions of the theorem—that it be a Cauchy sequence. We will show that the sequence has a limit.

The sequence is such that for every, willfully small, $\epsilon > 0$ there is a natural number N such that if $m, n \geq N$, then $|a_m - a_n| < \epsilon$ or specifically $|a_N - a_n| < \epsilon$. Hence, for all $n \geq N$ $a_N - \epsilon < a_n < a_N + \epsilon$. Designate $N = N(\epsilon)$ by N_1. For the choice of $\epsilon/2$ we have that all $a_n, n \geq N_2$ are in the intersection interval

$$(a_{N_1} - \epsilon, a_{N_1} + \epsilon) \cap (a_{N_2} - \epsilon/2, a_{N_2} + \epsilon/2).$$

In the figure above it is $(a_{N_2} - \epsilon/2, a_{N_1} + \epsilon)$. Continuing in this way we generate an endless sequence of, ever diminishing, nested intervals, which close upon the limit of the sequence. End of proof.

Theorem (Bolzano-Weierstrass): *Every bounded, above and below, sequence has a convergent subsequence.*

Proof: Let a_1 and b_1 be the glb and lub, respectively, of sequence $\{x_n\}_{n=1}^{\infty}$ so that $a_1 \leq x_n \leq b_1$ for any natural n. We propose to bisect interval $[a_1, b_1]$ into the two intervals $[a_1, (a_1 + b_1)/2]$ and $[(a_1 + b_1)/2, b_1]$. At least one of these intervals contains an endless number of terms of the sequence for otherwise the sequence would have been finite. Denote the interval containing the infinite number of terms by $[a_2, b_2]$. Continuing with this bisection procedure we obtain after k steps section $[a_k, b_k]$ that still contains an infinity of terms and that is included in all previous sections. As k tends to ∞, $b_k - a_k$ tends to zero. According to Cantor's lemma these intervals trap a number l that is the common limit of sequences $\{a_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = l.$$

We shall now progressively construct a convergent subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}$. We start by choosing $n_1 = 1$. Obviously $a_1 \le x_{n_1} \le b_1$. Interval $[a_2, b_2]$ contains an endless number of terms of $\{x_n\}$ and there exists $n_2 > n_1$ so that $a_2 \le x_{n_2} \le b_2$. Continuing this way we obtain the subsequence $\{x_{n_k}\}$ so that $a_k \le x_{n_k} \le b_k$ for any k. Since $a_k \to l$ and $b_k \to l$ also $x_{nk} \to l$ as $k \to \infty$. End of proof

2.4 More on irrational exponents

The computation of a^p for a rational $a > 0$ and an irrational p is carried out with a good rational approximation to p. We shall now substantiate this practice and define a^p as a limit.

Let $\{p_n\}_{n=1}^{\infty}$ be an increasing sequence such that $p_n \to p$. If $a < 1$, then a^{p_n} is decreasing with n but is bounded below by zero. If $a \ge 1$, then a^{p_n} is increasing but is bounded above by a^r, r being any rational number greater than p. In any event, a^{p_n} tends to a limit as $n \to 0$. Before defining this limit as a^p we need to show that this limit is independent on the particular choice of $\{p_n\}$. First we prove the next two lemmas.

Lemma 1: *If real $a > 0$, then $\lim_{n\to\infty} a^{\frac{1}{n}} = \lim_{n\to\infty} a^{-\frac{1}{n}} = 1$.*

Proof. To prove that $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$ we need to show that for any $0 < \epsilon < 1$, natural number N exists such that $|a^{\frac{1}{n}} - 1| < \epsilon$ for all $n > N$. The inequality may be written as $1 - \epsilon < a^{\frac{1}{n}} < 1 + \epsilon$ or $(1-\epsilon)^n < a < (1+\epsilon)^n$. Since $0 < 1 - \epsilon < 1$ the sequence $(1-\epsilon)^n$ decreases and tends to zero, and since $1 + \epsilon > 1$ the sequence $(1+\epsilon)^n$ increases without bound. Hence natural number N certainly exists such that $(1-\epsilon)^n < a < (1+\epsilon)^n$ for all $n > N$. The second part of the theorem is proved likewise. End of proof.

Lemma 2: *If $\{t_n\}$ is a sequence of real numbers tending to zero, $t_n \to 0$ as $n \to \infty$, and $a > 0$ is real , then $\lim_{n\to\infty} a^{t_n} = 1$.*

Proof. We assume first that $a > 1$. According to lemma 1, to $\epsilon > 0$ there corresponds a natural number k such that $1 - \epsilon < a^{-\frac{1}{k}} < a^{\frac{1}{k}} < 1 + \epsilon$. Since $t_n \to 0$ natural N exists such that for all $n > N$ certainly $-1/k < t_n < 1/k$. Or $|a^{t_n} - 1| < \epsilon$ and $a^{t_n} \to 1$. The proof for the case $a \le 1$ is similar. End of proof.

Theorem: *Let $\{p_n\}_{n=1}^{\infty}$ be an increasing sequence such that $p_n \to p$ and let $\{q_n\}_{n=1}^{\infty}$ be another sequence also tending to p. Then $\lim_{n\to\infty} a^{p_n} = \lim_{n\to\infty} a^{q_n} = a^p$.*

Proof. $\lim_{n\to\infty}(p_n - q_n) = \lim_{n\to\infty} p_n - \lim_{n\to\infty} q_n = p - p = 0$. According to lemma 2, $\lim_{n\to\infty} a^{q_n} = \lim_{n\to\infty} a^{q_n - p_n} \lim_{n\to\infty} a^{p_n} = 1 \cdot a^p = a^p$. End of proof.

Moreover, $\lim_{n\to\infty} a^{p_n} = a^{\lim_{n\to\infty} p_n} = a^p$, and it may happen that p is rational.

2.5 Infinite series

The recursive relationship $S_{n+1} = S_n + a_{n+1},\ S_1 = a_1$, defines the sequence $\{S_n\}_{n=1}^{\infty}$ in terms of the given sequence $\{a_n\}_{n=1}^{\infty}$. Repeated recursions produces the nth **partial sum**

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

of an **infinite series**, and if $S_n \to S$ as $n \to \infty$, then the series is said to be convergent. If $a_n > 0$, then the series

$$S_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1} a_n$$

is **alternating**. For example

$$S_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + (-1)^{n+1} \frac{1}{n}.$$

Theorem (Leibnitz): *If the sequence $\{a_n\}_{n=1}^{\infty}$ is positive, $a_n > 0$, and strictly decreasing, $a_n > a_{n+1}$, then the infinite alternating series $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots$ converges to the sum S and $0 < S < a_1$. The remainder $R_n = S - S_n$ of the series is such that $(-1)^n R_n > 0$, and $|R_n| < a_{n+1}$.*

Proof. Since each term in the parentheses in positive, the even partial sum

$$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + (a_7 - a_8) + \cdots + (a_{2n-1} - a_{2n})$$

is increasing. The odd partial sum

$$S_{2n-1} = a_1 - (a_2 - a_3) - (a_4 - a_5) - (a_6 - a_7) - \cdots - (a_{2n-2} - a_{2n-1})$$

is decreasing. But

$$S_{2n-1} - S_{2n} = a_{2n} \geq 0, n = 1, 2, 3 \ldots$$

implying that S_{2n} is bounded above, and that S_{2n-1} is bounded below, and the two series are convergent. Since $a_n \to 0$ as $n \to \infty$, the two series converge to the same limit, which we denote by S. It results that $S_n \to S$ such that $S_1 > S, S_2 < S, S_3 > S, S_4 < S \ldots$, and the remainder of the series

$$R_m = S - S_m = (-1)^{m+2} a_{m+1} + (-1)^{m+3} a_{m+2} + \cdots$$

which is by itself an alternating series, alternates in sign. If index m is even, then$(-1)^{m+2} = 1$, and $0 < R_m < a_{m+1}$. If m is odd, then $(-1)^{m+2} = -1$, and $-a_{m+1} < R_m < 0$. In any event, $(-1)^m R_m > 0$, and $|R_m| < a_{m+1}$. End of proof.

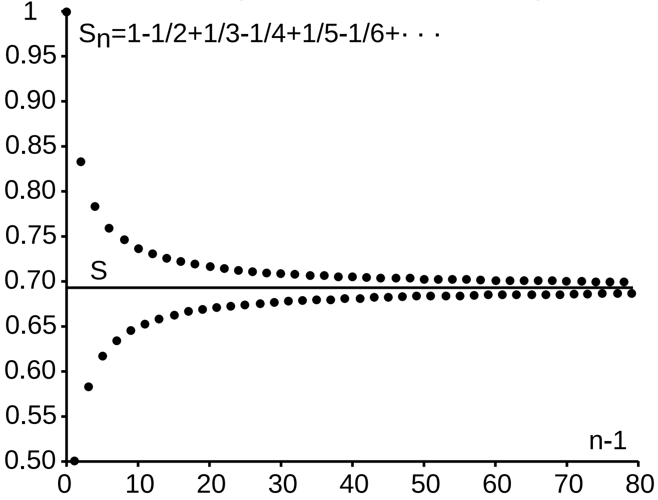
The series

$$S_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \cdots$$

satisfies the conditions of the theorem, and

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} < S < 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7}$$

or $-1/7 < S - S_7 < 0$. The convergence of this series is graphically depicted below.



Theorem: *Let S_n be the partial sum, S the total sum, and $R_n = S - S_n$ the remainder of the convergent alternating series $\Sigma_{n=1}^{\infty}(-1)^{n+1}a_n$. If $a_n - a_{n+1}$ decreases with n, then $a_{n+1}/2 < |R_n| < a_n/2$.*

Proof. We define $\Delta_n = a_n - a_{n+1}$, write the series in terms of these differences as

$$S = S_n + (-1)^n(\Delta_{n+1} + \Delta_{n+2} + \cdots)$$

and continue under the assumption that n is odd, so that

$$S = S_n - (\Delta_{n+1} + \Delta_{n+2} + \cdots), \text{ and } S = S_{n-1} + (\Delta_n + \Delta_{n+1} + \cdots).$$

By the assumption that $\Delta_n > \Delta_{n+1}$, $\Delta_{n+1} > \Delta_{n+2} \ldots$ it results that $S - S_{n-1} > S_n - S$. Using the relationship $S_n = S_{n-1} + a_n$ we obtain $0 < S_n - S = R_n < \frac{1}{2}a_n$. Repeating this argument under the assumption that n is even completes the first part of the proof. To prove the second part of the theorem, still under the assumption that n is odd, we write

$$S = S_n - (\Delta_{n+1} + \Delta_{n+2} + \cdots), \text{ and } S = S_{n+1} + (\Delta_{n+2} + \Delta_{n+3} + \cdots)$$

and use the fact that $S_{n+1} = S_n - a_{n+1}$. End of proof.

Whereas the alternating series converges, the **harmonic** series

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

diverges—S_n relentlessly increases without bound. We see this by noticing that

$S_2 = 1 + \frac{1}{2}$

$S_4 = S_2 + \frac{1}{3} + \frac{1}{4} > S_2 + \frac{1}{4} + \frac{1}{4} = 1 + \frac{2}{2}$

$S_8 = S_4 + \frac{1}{5} + \cdots + \frac{1}{8} > S_4 + \frac{1}{8} + \cdots \frac{1}{8} = 1 + \frac{3}{2}.$

In general

$$S_{2^m} > 1 + \frac{m}{2}$$

and the series diverges.

2.6 variable terms

The terms of an infinite sequence may depend on variable x. Consider the series

$$S = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

that is alternating for any value of variable x. Also,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_n(x) = \frac{1}{(2n-2)!}x^{2n-2} \to 0$$

for any x. To be a Leibnitz series it is required that $a_n(x) > a_{n+1}(x)$, which happens if $(2n-2)(2n-3) > x^2$. Thus, if $x < \sqrt{2}$, then the entire series is a Leibnitz series, but if $x > \sqrt{2}$, then the series becomes a Leibnitz series only after some $n > N$. This is sufficient for convergence, and the series converges for any x.

Exercises

1. Define the (Fibonacci) sequence $\{a_n\}$ recursively as $a_1 = 0, a_2 = 1, a_{n+2} = a_{n+1} + a_n$. Is the sequence bounded above?

2. Prove that if the limit of $\{a_n\}$ is a and the limit of $\{b_n\}$ is b, then the limit of $\{a_n + b_n\}$ is $a + b$.

3. Prove that if the limit of $\{a_n\}$ is a and the limit of $\{b_n\}$ is b, then the limit of $\{a_n b_n\}$ is ab.

4. Prove that if the limit of $\{a_n\}$ is a and the limit of $\{b_n\}$ is b, then the limit of $\{a_n b_n\}$ is ab.

5. Prove that if the limit of $\{a_n \neq 0\}$ is $a \neq 0$, then the limit of $\{1/a_n\}$ is $1/a$.

6. Prove that a convergent sequence (of finite terms) is bounded.

7. Prove that if a_n is bounded and $b_n \to 0$, then $a_n b_n \to 0$.

8. Prove that if $a_n \to a$ and $b_n \to b > a$, then natural N exists such that $b_n > a_n$ for all $n > N$.

8. Prove that if $a_n \to a$ and $b_n \to b$ and $b_n \geq a_n$ for all $n > N$, then $b \geq a$.

9. Prove that if $a_n \to a$, then $|a_n| \to |a|$. Hint[1]: use the inequality $\big||a| - |b|\big| \leq |a - b|$.

10. Prove that if $a_n \geq 0$, and $a_n \to a$, then $\sqrt[k]{a_n} \to \sqrt[k]{a}$

11. Show that $\lim\limits_{n\to\infty} \dfrac{1+2n}{1+3n} = \dfrac{2}{3}$. For a given $\epsilon > 0$ find $N = N(\epsilon)$ such that for every $n > N$, $|(1+2n)/(1+3n) - 2/3| < \epsilon$.

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[1] 'hint' is related to 'hand' and 'hunt.' A hint may be a hand gesture gotten, as though captured by hand, by the intended recipient of the allusion

Chapter 3

12. Prove that the sequence $\{a_n = \frac{\sqrt{n}+1}{n+1}\}$ converges and find its limit.

13. Prove that for any two real numbers $a_1 > 0$ and $a_2 > 0$ the inequality $\sqrt{a_1 a_2} \le \frac{a_1 + a_2}{2}$ holds. This inequality is a special case of

$$(a_1 a_2 a_3 \cdots a_n)^{\frac{1}{n}} \le \frac{a_1 + a_2 + a_3 + \cdots + a_n}{n}$$

14. Prove that if $\{a_n > 0\} \to 1$ as $n \to \infty$, then $a_n^k \to 1$ as $n \to \infty$ for any natural k.

15. What is wrong with the following paradoxical 'proof?'

$S = 1 + 2 + 4 + 8 + 16 + \cdots$

$S = 1 + 2(1 + 2 + 4 + 8 + 16 + \cdots)$

$S - 1 = 2S$

$S = -1.$

16. Show that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = S$$

but

$$1 + (-\frac{1}{2} - \frac{1}{4}) + \frac{1}{3} + (-\frac{1}{6} - \frac{1}{8}) + \frac{1}{5} + (-\frac{1}{10} - \frac{1}{12}) + \frac{1}{7} + \cdots = \frac{1}{2}S$$

and

$$(1 + \frac{1}{3}) - \frac{1}{2} + (\frac{1}{5} + \frac{1}{7}) - \frac{1}{4} + (\frac{1}{9} + \frac{1}{11}) - \frac{1}{6} + \cdots = \frac{3}{2}S.$$

17. Show that

$$S_n = \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \cdots + \frac{1}{(2n-1) \cdot 2n}$$

converges.