Math 721A1, Homework #5
Differential Topology I

1. Let $\Delta_1, \ldots, \Delta_h$ be integrable distributions on M, of dimensions $d_1, \ldots, d_h$, respectively. Suppose that for each p in M,
$$T_pM \;=\; (\Delta_1)_p \,\oplus\, \cdots \,\oplus\, (\Delta_h)_p.$$
Show that there is a coordinate system (x, U) around each point such that Δ_1 is spanned by $\frac{\partial}{\partial x^1} \ldots, \frac{\partial}{\partial x^{d_1}}$ and so forth.
2. Show that n functions $f_1, \ldots, f_n : M \to \mathbb{R}$ form a coordinate system in a neighborhood of p in M if and only if
$$df_1 \,\wedge\, df_2 \,\wedge\, df_n(p) \;\neq\; 0.$$
3. If $v_1, \ldots, v_n$ is a basis for V and $w_i \;=\; \sum_{j=1}^{n} \alpha_{ji}\, v_j$, show that
$$\det(\,\alpha_{ij}\,)\, w_1^* \wedge \cdots \wedge w_n^* \;=\; v_1^* \wedge \cdots \wedge v_n^*.$$
4. If ω is a $(k+1)$-form $(k \geq 0)$ then $\iota_X\, \omega$ is the k-form defined by
$$(\iota_X\, \omega)(X_1, \ldots, X_k) \;:=\; \omega(X, X_1, \ldots, X_k).$$
$\iota_X\, \omega$ is called the interior product (or contraction) of ω with X. (Define $\iota_X\, f \;:=\; 0$ for any smooth function f.) Show the following.
(a) Show that if ω is a k-form then so is $L_X\, \omega$.
(b) Show that
$$L_X\,(\omega_1 \,\wedge\, \omega_2) \;=\; (L_X\, \omega_1) \,\wedge\, \omega_2 \,+\, \omega_1 \,\wedge\, (L_X\, \omega_2)$$
(c) Show that
$$L_X\, \omega \;=\; d\,(\iota_X\, \omega) \,+\, \iota_X\,(d\,\omega).$$
(d) Using the previous, show that
$$d(\,L_X\, \omega) \;=\; L_X\,(d\,\omega).$$
5. Consider $\mathbb{R}^3$ with the standard Euclidean metric so that $T_p\mathbb{R}^3$ is identified with $T_p^*\mathbb{R}^3$. Let $X \;=\; \sum_{i=1}^{3} a^i \frac{\partial}{\partial x^i}$ be a vector field on $\mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ a function. Define the gradient of f by
$$\operatorname{grad} f \;:=\; \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\, \frac{\partial}{\partial x^i},$$
the divergence of X by
$$\operatorname{div} X \;:=\; \sum_{i=1}^{n} \frac{\partial a^i}{\partial x^i},$$
and the curl of X by
$$\operatorname{curl} X \;:=\; (\frac{\partial a^3}{\partial x^2} - \frac{\partial a^2}{\partial x^3})\, \frac{\partial}{\partial x^1} \,+\, (\frac{\partial a^1}{\partial x^3} - \frac{\partial a^3}{\partial x^1})\, \frac{\partial}{\partial x^2} \,+\, (\frac{\partial a^2}{\partial x^1} - \frac{\partial a^1}{\partial x^2})\, \frac{\partial}{\partial x^3}.$$
Define the differential forms on $\mathbb{R}^3$,
$$\omega_X \;=\; a^1\, dx^1 \,+\, a^2\, dx^2 \,+ a^3\, dx^3$$
and
$$\eta_X \;=\; a^1\, dx^2 \,\wedge\, dx^3 \,+\, a^2\, dx^3 \,\wedge\, dx^1 \,+ a^3\, dx^1 \,\wedge\, dx^2.$$

(a) Show that

$$df = \omega_{\mathrm{grad} f},$$

$$d(\omega_X) = \eta_{\mathrm{curl} X}$$

$$d(\eta_X) = (\mathrm{div} X)\, dx^1 \wedge dx^2 \wedge dx^3.$$

(b) Conclude that

$$\mathrm{curl}\,(\mathrm{grad}\, f) = 0$$

and

$$\mathrm{div}\,(\mathrm{curl}\, X) = 0.$$