Math 124, Practice Exam Solutions for Exam #1, February 23, 2001

1. Calculate the following:

(a) Let $u = \sin x$ then $du = \frac{du}{dx} dx = \cos x dx$ and

$$\int \sin^{100}(x)\cos(x)\,dx = \int u^{100}\,du = \frac{u^{101}}{101} + C = \frac{\sin^{101}(x)}{101} + C$$

(b) Let $u = x^3 - 8x^2 + 5x + 3$ then $du = \frac{du}{dx}dx = (3x^2 - 16x + 5)\,dx$ then

$$\begin{aligned}\int \frac{3x^2 - 16x + 5}{\sqrt{x^3 - 8x^2 + 5x + 3}}\,dx &= \int (3x^2 - 16x + 5)(x^3 - 8x^2 + 5x + 3)^{-\frac{1}{2}}\,dx \\ &= \int u^{-\frac{1}{2}}\,du = 2u^{\frac{1}{2}} + C = 2(x^3 - 8x^2 + 5x + 3)^{\frac{1}{2}} + C\end{aligned}$$

(c) Let $u = x^2 - 3x$ then $\frac{du}{dx} = (2x - 3)$ or $\frac{1}{2}\frac{du}{dx} = (x - \frac{3}{2})\,dx$ and

$$\int (x - \frac{3}{2})\sin(x^2 - 3x)\,dx = \int \sin(u)\,\frac{1}{2}\,du = \frac{1}{2}(-\cos u) + C = -\frac{1}{2}\cos(x^2 - 3x) + C$$

(d)

$$\int x\ln x^3\,dx = \int x(3\ln x)\,dx = 3\int x\ln x\,dx = 3\int u\,dv$$

where $u = \ln x$ and $v = \frac{x^2}{2}$. Using integration by parts, we obtain

$$\int x\ln x^3\,dx = 3\left(\frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{d}{dx}(\ln x)\,dx\right) = \frac{3x^2}{2}\ln x - \frac{3x^2}{4} + C$$

(e) Let's use the substitution method. Let $\hat{u} = \sqrt{x}$ then $du = \frac{1}{2}x^{-\frac{1}{2}}dx$ or, in other words,

$$dx = 2x^{\frac{1}{2}}d\hat{u} = 2\hat{u}d\hat{u}$$

then

$$\int e^{\sqrt{x}}dx = \int e^{\hat{u}}2\hat{u}d\hat{u} = 2\int e^{\hat{u}}\hat{u}d\hat{u}.$$

Now we use integration by parts by setting $u = \hat{u}$ and $v = e^{\hat{u}}$ then

$$2\int e^{\hat{u}}\hat{u}d\hat{u} = 2(\hat{u}e^{\hat{u}} - \int e^{\hat{u}}d\hat{u}) = 2(\hat{u}e^{\hat{u}} - e^{\hat{u}}) + C = 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C.$$

(f) Choose $u = \tan x$ then $du = \sec^2 x\,dx$. Also, when $x = 0$ then $u = \tan 0 = 0$ and when $x = \frac{\pi}{4}$ then $u = \tan\frac{\pi}{4} = 1$ so

$$\int_0^{\frac{\pi}{4}} \tan^5 x\sec^2 x\,dx = \int_0^1 u^5\,du = \left.\frac{u^6}{6}\right|_0^1 = \frac{1}{6}$$

(g) Let $u = f(x)$ then $du = \frac{du}{dx}dx = f'(x)\,dx$. Therefore,

$$\int_2^4 f'(x)\sin(f(x))dx = \int_{f(2)}^{f(4)} \sin(u)\,du = \int_1^7 \sin(u)\,du = -\cos(7) + \cos(1).$$

(h) Notice that

$$\int (2x-8)e^{-x}\,dx \;=\; 2\int\; x\,e^{-x}\,dx - 8\int\; e^{-x}\,dx \;=\; 2\int x\,e^{-x}\,dx \;+\; 8e^{-x}$$

but by integration by parts,

$$\int x\,e^{-x}\,dx \;=\; -e^{-x} - xe^{-x} + C$$

therefore,

$$\int (2x-8)e^{-x}\,dx \;=\; 6e^{-x} - 2xe^{-x} \;+\; C$$

and

$$\int_1^3 (2x-8)e^{-x}\,dx \;=\; -\frac{4}{e}$$

(i)

$$\int_{-\infty}^{3} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \int_t^3 \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} (\arctan(3) - \arctan(t)) = \arctan(3) - (-\frac{\pi}{2}) = \arctan(3) + \frac{\pi}{2}.$$

(j) The average value by definition is

$$\overline{f} \;=\; \frac{1}{8-3}\int_3^8 x^2\,dx = \frac{97}{3}.$$

2. First, note that that two graphs intersect when $x^3 - x^5 = 0$ or, equivalently, when $x^3(1-x^2) = 0$ which occurs when $x = 0, \pm 1$. Therefore, since $x \geq 0$, R lies between $x = 0$ and $x = 1$.

(a) The area, A, of R is

$$A \;=\; \int_0^1 (x^3 - x^5)dx \;=\; \frac{1}{12}.$$

(b) The centroid $(\overline{x}, \overline{y})$ is defined by

$$\overline{x} \;=\; \frac{1}{A}\int_0^1 x(x^3 - x^5)dx \;=\; \frac{24}{35}$$

and

$$\overline{y} \;=\; \frac{1}{A}\int_0^1 \frac{1}{2}(x^3 - x^5)^2 dx = \frac{16}{231}$$

where the last integral has been evaluated using the fact that $(x^3 - x^5)^2 = x^6 - 2\,x^8 + x^{10}$.

(c) The area of an equilateral triangle with a side of length s is

$$\frac{1}{2}\left(\frac{\sqrt{3}}{2}s\right)(s) = \frac{s^2\sqrt{3}}{4}.$$

The length of an edge of the triangle located at the position x (where $0 \leq x \leq 1$) is $x^3 - x^5$. Therefore, the area of such a triangle is thus

$$A(x) = \frac{\sqrt{3}}{4}(x^3 - x^5)^2.$$

The volume of S is

$$V = \int_0^1 A(x)dx = \int_0^1 \frac{\sqrt{3}}{4}(x^3 - x^5)^2 dx = \frac{\sqrt{3}}{4}\int_0^1 (x^6 - 2x^8 + x^{10})dx.$$

Performing the latter integral, one obtains

$$V = \frac{\sqrt{3}}{4}(\frac{1}{7} - 2\left(\frac{1}{9}\right) + \frac{1}{11}) = \frac{2\sqrt{3}}{693}.$$

3. The arc length L is given by

$$L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

but

$$\frac{dy}{dx} = 3x^{\frac{1}{2}}.$$

Plugging this into the arc length formula, we need to calculate the integral

$$L = \int_1^4 \sqrt{1 + 9x}\, dx$$

but by using substitution, we can see that

$$\int \sqrt{1 + 9x} dx = \frac{2}{27}(1 + 9x)^{\frac{3}{2}}$$

Plugging in, we obtain

$$L = \frac{-20\sqrt{10}}{27} + \frac{74\sqrt{37}}{27}.$$

4. The graphs intersect when $0 = (x-2)^2 - x = x^2 - 5x + 4 = (x-4)(x-1)$; in otherwords, at $x = 1, 4$.

The volume V of the region is

$$V = \int_1^4 (\pi x^2 - \pi((x-2)^2)^2)\, dx$$

but

$$\int (\pi x^2 - \pi((x-2)^2)^2)\, dx = -16\,\pi\, x + 16\,\pi\, x^2 - \frac{23\,\pi\, x^3}{3} + 2\,\pi\, x^4 - \frac{\pi\, x^5}{5}.$$

Plugging in the limits, we obtain

$$V = \frac{72}{5}\pi.$$

5. (a) Just calculate the following:

$$\frac{d}{dx}\left(Ce^{-2x} + \frac{1}{3}e^x\right) + 2\left(Ce^{-2x} + \frac{1}{3}e^x\right) = -2Ce^{-2x} + \frac{1}{3}e^x + 2Ce^{-2x} + \frac{2}{3}e^x = e^x$$

(b) Just plug into the general solution

$$8 = y(0) = C + \frac{1}{3}$$

then

$$C = 8 - \frac{1}{3} = \frac{23}{3}.$$