**Math 124, Exam #2 Solutions, May 2, 2001**
**Prof. Takashi Kimura**

1. (45 points) Answer the following questions.

(a) Does the following series converge? Justify your answer.

$$\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$$

The series converges since it is a p-series with p $= \pi > 1$.

(b) Does the following series converge? Justify your answer.

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sin^2 n}{n}$$

This problem turned out to be (unintentionally) surprisingly difficult. The alternating series test cannot be used as is since it's not clear that for $n$ large enough, $\frac{\sin^2(n+1)}{n+1} < \frac{\sin^2 n}{n}$. Since the solution of this problem is beyond the scope of the course, **everyone was given full credit for this problem**.

(c) Does the following series converge? Justify your answer.

$$\sum_{n=1}^{\infty} \frac{8n^5 + 7n^2 + 1600}{3n^6 + 8n + 1}$$

Since

$$\begin{aligned}\lim_{n\to\infty} \frac{8n^5 + 7n^2 + 1600}{3n^6 + 8n + 1} / (\tfrac{1}{n}) &= \lim_{n\to\infty} \frac{(8n^5 + 7n^2 + 1600)n}{3n^6 + 8n + 1} \\ &= \lim_{n\to\infty} \frac{8n^6 + 7n^3 + 1600n}{3n^6 + 8n + 1} \\ &= 8/3 > 0,\end{aligned}$$

the series diverges according to the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n}$ (which diverges by the p-test).

(d) Does the following series converge? Justify your answer.

$$\sum_{n=2}^{\infty} (\ln n)^{-1} \frac{1}{n}$$

Let $f(x) = (\ln x)^{-1}/x$. Note that for $x \geq 2$, $f > 0$, $f$ is continuous, and decreasing. Hence the integral test can be used. Substitute $u = \ln x$, to get

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \int_{\ln(2)}^{\infty} \frac{1}{u}\, du = \ln(u)\Big|_{\ln(2)}^{\infty} = \infty.$$

Therefore, the series diverges.

(e) Does the following series converge? Justify your answer.

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

Since $\lim_{n\to\infty} (-1)^n \neq 0$, the series diverges by the divergence test.

2. (20 points) Consider the function

$$f(x) = \frac{7x^3}{3x-8}.$$

(a) Write $f(x)$ as a power series.

$$\begin{aligned} f(x) &= \frac{7x^3}{3x-8} \\ &= \frac{-7x^3}{8}\left(\frac{1}{1-\frac{3x}{8}}\right) \\ &= \frac{-7x^3}{8}\left(\sum_{n=0}^{\infty}(\tfrac{3x}{8})^n\right) \quad \text{(geometric series)} \\ &= -\sum_{n=0}^{\infty}(\tfrac{7}{8})(\tfrac{3}{8})^n x^{n+3} \\ &= -\sum_{j=3}^{\infty}(\tfrac{7}{8})(\tfrac{3}{8})^{j-3} x^j \end{aligned}$$

(b) Find its radius of convergence.
The radius of convergence is determined by the geometric series (see above) which will converge when $|3x/8| < 1$. Hence $R = 8/3$. This can also be found by using the ratio test on the final power series for $f$.

(c) Find its interval of convergence.
Substute $x = \pm 8/3$ to determine that the series diverges in both cases. Or observe that the convergence of the power series for $f$ depends upon the convergence of the geometric series (line 3 above) which must diverge when $|3x/8| \geq 1$. Either way, $I = (-8/3, 8/3)$.

3. (20 points) Consider the series

$$\sum_{n=0}^{\infty} \frac{(x-4)^n}{7^n(n+2)}$$

(a) Find its radius of convergence.
Use the ratio test:

$$\lim_{n\to\infty}\left|\frac{(x-4)^{n+1}7^n(n+2)}{7^{n+1}(n+3)(x-4)^n}\right| = \lim_{n\to\infty}\left|\frac{(x-4)(n+2)}{7(n+3)}\right| = \frac{|x-4|}{7}.$$

Now set $|x-4|/7 < 1$ to get $|x-4| < 7$ and conclude $R = 7$ and interval is at least $(-3, 11)$.

(b) Find its inverval of convergence.
Test endpoints to determine interval. When $x = -3$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-7)^n}{7^n(n+2)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+2)}$$

which converges by the alternating series test. When $x = 11$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(7)^n}{7^n(n+2)} = \sum_{n=0}^{\infty} \frac{1}{(n+2)}$$

which diverges by using comparison to p-series (p = 1). Hence $I = [-3, 11)$.

4. (15 points) Find the Taylor series about 2 of

$$f(x) = \frac{1}{x}.$$

First observe that by the power rule,

$$f^{(n)}(x) = (-1)(-1-1)(-1-2)\cdots(-1-(n-1))x^{-1-n}.$$

So

$$f^{(n)}(2) = (-1)(-1-1)(-1-2)\cdots(-1-(n-1))2^{-1-n} = (-1)^n n! 2^{-(n+1)}$$

Using Taylor's formula, we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum_{n=0}^{\infty}(-1)^n 2^{-(n+1)}(x-2)^n.$$