Math 563A1, Midterm Exam, March 15, 1999
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This exam is due at the beginning of class on Monday March 22. The exam is open book and you may work with others although you must write up the solutions yourself. Good luck!

1. (10 points) Let $\alpha : I \to \mathbf{R}$ be a regular curve parametrized by arc length whose curvature is nowhere vanishing. Let

$$x(s,v) = \alpha(s) + r\,(n(s)\cos(v) + b(s)\sin(v))$$

(where r is a nonzero constant and s belongs to I) be a parametrized surface (the *tube* of radius r around α), where n is the normal vector and b is the binormal vector of α. Show that if x is regular then its unit normal vector is

$$N(s,v) = -(n(s)\cos(v) + b(s)\sin(v)).$$

2. (10 points) Let S be a regular surface. The *gradient* of a differentiable function $f : S \to \mathbf{R}$ is a differentiable map $\operatorname{grad} f : S \to \mathbf{R^3}$ which assigns to each point p in S a vector $\operatorname{grad} f(p)$ in T_pS such that

$$\operatorname{grad} f(p) \cdot v := f_*(v)$$

for all v in T_pS. Let x be a parametrization about p. Show the following:

(a) If $E = x_u \cdot x_u$, $F = x_u \cdot x_v$, $G = x_v \cdot x_v$ then

$$\operatorname{grad} f = \frac{f_uG - f_vF}{EG - F^2}\,x_u + \frac{f_vE - f_uF}{EG - F^2}\,x_v$$

In particular, if $S = \mathbf{R^2}$ with coordinates x and y then

$$\operatorname{grad} f = (f_x, f_y)$$

thereby reducing to the usual definition of gradient in the plane.

(b) Choose a fixed p in S. Let v vary in the unit circle with $|v| = 1$ in T_pS. $f_*(v)$ is a maximum if and only if

$$v = \frac{\operatorname{grad} f}{|\operatorname{grad} f|}.$$

In other words, $\operatorname{grad} f(p)$ gives the direction of maximum variation of f at p.

3. (10 points) Determine the umbilical points of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

4. When two functions $f, g : U \subset \mathbf{R^2} \to \mathbf{R}$ satisfy the Cauchy-Riemann equations

$$\frac{\partial f}{\partial u} = \frac{\partial g}{\partial v} \qquad\qquad \frac{\partial f}{\partial v} = -\frac{\partial g}{\partial u},$$

the pair (f, g) are said to be *harmonic conjugate*. Let x and y be isothermal parametrizations of minimal surfaces such that their component functions are pairwise harmonic conjugate then x and y are said to be *conjugate minimal surfaces*. Prove that

(a) The helicoid and the catenoid are conjugate minimal surfaces.

(b) Given two conjugate minimal surfaces, x and y, the surface

$$z(u,v) := \cos(t)\,x(u,v) + \sin(t)\,y(u,v)$$

is again minimal for all t in $\mathbf{R}$.

(c) All surfaces of the one-parameter family above satisfy $E = x_u \cdot x_u = y_u \cdot y_u$, $F = x_u \cdot x_v = y_u \cdot y_v = 0$, and $G = x_v \cdot x_v = y_v \cdot y_v$.

5. (10 points) Exercise 2.2 in Chapter 4 (page 131) of our text.