Math 123, Solutions to Practice Questions for Exam #2, November 10, 2008

1. Find the following:

   (a) This is an application of L'Hopital's rule.

$$\lim_{y\to\infty}\left(1+\frac{2}{y}\right)^y = \lim_{y\to\infty}\exp\left(y\ln\left(1+\frac{2}{y}\right)\right) = \exp\left(\lim_{y\to\infty}\frac{\ln\left(1+\frac{2}{y}\right)}{\frac{1}{y}}\right) =$$

$$\exp\left(\lim_{y\to\infty}\frac{\frac{1}{1+\frac{2}{y}}\left(-\frac{2}{y^2}\right)}{-\frac{1}{y^2}}\right) = \exp(2) = e^2$$

   (b) This is an application of L'Hopital's rule, twice.

$$\lim_{x\to 0}\frac{\cos\left(\sqrt{5}\,x\right)-1}{x^2} = \lim_{x\to 0}\frac{-\sqrt{5}\sin\left(\sqrt{5}\,x\right)}{2x} = \lim_{x\to 0}\frac{-5\cos\left(\sqrt{5}\,x\right)}{2} = -\frac{5}{2}$$

   (c) This is an indeterminate form of type $\infty-\infty$.

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right) = \lim_{x\to 0}\frac{\sin x - x}{x\sin x} = \lim_{x\to 0}\frac{\cos x - 1}{x\cos x+\sin x} = \lim_{x\to 0}\frac{-\sin x}{\cos x - x\sin x+\cos x} = 0$$

   (d) The horizontal and vertical asymptotes of

$$y = \frac{4-3x}{\sqrt{16x^2+1}}$$

   Since $\lim_{x\to\infty}\frac{4-3x}{\sqrt{16x^2+1}} = -\frac{3}{4}$ and $\lim_{x\to-\infty}\frac{4-3x}{\sqrt{16x^2+1}} = \frac{3}{4}$, the horizontal asymptotes are $y=\frac{3}{4}$ and $y=-\frac{3}{4}$. There are no vertical asymptotes since $\frac{4-3x}{\sqrt{16x^2+1}}$ exists for all x values.

2. A man 6 feet tall is walking away from a light pole which is 30 feet high. If the tip of his shadow is moving at a rate equal to the distance between him and the light pole (in feet) then how fast is the man walking when he is 24 feet from the pole?

   Let l be the distance from the man to the lamp pole, x the distance from the lamp pole to the tip of the man's shadow, and t be time. We are told that $\frac{dx}{dt} = l$. We want to know what $\frac{dl}{dt}$ is when $l = 24$. By similar triangles, we obtain

$$\frac{6}{30} = \frac{x-l}{x}$$

   which can be solved to obtain $l = \frac{4}{5}x$. Therefore,

$$\frac{dl}{dt} = \frac{d}{dt}\left(\frac{4}{5}x\right) = \frac{4}{5}\frac{dx}{dt} = \frac{4}{5}l$$

   so when $l = 24$, $\frac{dl}{dt} = \frac{96}{5}$.

3. A spherical snowball is melting at a rate equal to its surface area. How fast is its radius shrinking when its volume is equal to its surface area?

   Let r be the radius of the spherical snowball, V the volume of the sphere, A be the surface area of the snowball, and t be time. We are told that $\frac{dV}{dt} = -A$. We want to know what $-\frac{dr}{dt}$ is when $V = A$. Recall that $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$ so
$$\frac{dV}{dt} = \frac{d}{dt}(\frac{4}{3}\pi r^3) = 4\pi r^2 \frac{dr}{dt} = A\frac{dr}{dt}.$$
   On the other hand, $\frac{dV}{dt} = -A$, therefore, combining with the previous equation, $\frac{dr}{dt} = -1$ at any moment in time.

4. Two nonnegative numbers are such that the sum of the first number and 3 times the second number equals 10. Find these numbers if the sum of their squares is as small as possible.

   Call the numbers x and y. We know that $x + 3y = 10$ or, in other words, $x = 10 - 3y$. Furthermore, let $f = x^2 + y^2 = (10 - 3y)^2 + y^2$. Notice that the fact that both x and y be nonnegative means that y must lie in the interval $[0, \frac{10}{3}]$. We want to find the absolute minimum of $f(y)$ on the interval $[0, \frac{10}{3}]$.

   Since $f'(y) = -60 + 20y$, the only stationary point of f (i.e. where $f'(y) = 0$) is $y = 3$. There are no singular points of f. Therefore, the only critical point of f is 3. The global minimum of f is then the smallest value of f evaluated on the endpoints (namely $f(0) = 100$, $f(\frac{10}{3}) = \frac{100}{9}$) or $f(3) = 10$. Therefore, the two numbers are $x = 10 - 3(3) = 1$ and $y = 3$.

5. If 12cm^3 of material is available to make a box with a square base and an open top, find the dimensions of the box which maximizes the the volume of the box. What is this maximum volume? Justify your answer.

   We are told that the surface area of the box $A = 12$. Since the base of the box is square, let l denote the length of a side of the base then the volume of the box
$$V = l^2 h.$$
   We wish to maximize V given $A = 12$.

   Let h denote the height of the box. The area of the bottom of the box, namely l^2, plus 4 times the area of each side, which is lh is equal to the surface area $A = 12$. In other words,
$$l^2 + 4lh = 12.$$
   Solving for h we obtain
$$h = \frac{12 - l^2}{4l} = \frac{3}{l} - \frac{l}{4}$$
   which, after plugging into V yields
$$V(l) = 3l - \frac{l^3}{4}.$$

Therefore, we need to find the absolute maximum of V on the interval $(0, \infty)$.

Since

$$V'(l) = 3 - \frac{3l^2}{4},$$

$V'(l) = 0$ occurs if and only if $l = \pm 2$. However, we are only interested in l in the interval $(0, \infty)$, hence, $l = 2$ is the only stationary point. There are no singular points since $V'(l)$ always exists. Hence, $l = 2$ is the only critical number of $V(l)$ on $(0, \infty)$. Since $V'(l) < 0$ if $l > 2$ and $V'(l) > 0$ if $0 < l < 2$, $l = 2$ must be a global maximum. In this case,

$$h = \frac{3}{2} - \frac{2}{4} = 1.$$

So the dimensions of the box are $2 \times 2 \times 1$ and the maximum volume is $V(2) = 4$.

6. Calculate the following:

   (a) Find the most general antiderivative $F(x)$ of $f(x) = \frac{x^3+4\sqrt{x}}{x}$.

   Since

   $$\frac{x^3 + 4\sqrt{x}}{x} = x^2 + 4x^{-\frac{1}{2}},$$

   we obtain

   $$F(x) = \int (x^2 + 4x^{-\frac{1}{2}})\, dx = \frac{x^3}{3} + 8x^{\frac{1}{2}} + C$$

   where C is any constant.

   (b) Find the antiderivative $F(x)$ from above satisfying $F(1) = 0$.

   Plugging in, $F(1) = 0$ becomes

   $$\frac{1}{3} + 8 + C = 0$$

   and, hence, $C = -\frac{25}{3}$. Therefore,

   $$F(x) = \frac{x^3}{3} + 8x^{\frac{1}{2}} - \frac{25}{3}.$$

7. Consider the function

   $$f(x) = 3x^4 - 4x^3 + 20000$$

   (a) On what interval(s) is f increasing? Since

   $$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$$

   $f'(x)$ is positive on the interval $(1, \infty)$.

   (b) On what interval(s) is f concave down? Since

   $$f''(x) = -24x + 36x^2 = 12x(-2 + 3x)$$

   $f''(x)$ is negative on the interval $(0, \frac{2}{3})$.

(c) Find the inflection point(s) of f.

The concavity changes across $x = \frac{2}{3}$ and $x = 0$.

(d) Find the critical points of f.

The stationary points are $x = 0, 1$. There are no singular points.

(e) Find the local maximum (maxima) of f.

There are no local maxima by the first derivative test.

(f) Find the global minimum of f on the interval $[-1, 2]$.

$x = 1$ is a local minimum by the first derivative test. Evaluating f on the endpoints ($f(-1) = 20007$, $f(2) = 20016$) and $f(1) = 19999$, the smallest is 19999 which occurs at $x = 1$.

8. Consider the function

$$f(x) = |x^2 - 5|$$

(a) On what interval(s) is f increasing? We need to calculate $f'(x)$. We recall that

$$|u| = \begin{cases} u, & \text{if } u \geq 0; \\ -u, & \text{if } u < 0 \end{cases}.$$

In our case, $u = x^2 - 5$ and $x^2 - 5 \geq 0$ is equivalent to saying that either $x \geq \sqrt{5}$ or $x \leq -\sqrt{5}$. Similarly, $x^2 - 5 < 0$ is equivalent to $-\sqrt{5} < x < \sqrt{5}$. Therefore, have

$$f(x) = \begin{cases} x^2 - 5, & \text{if } x \geq \sqrt{5} \text{ or } x \leq -\sqrt{5}; \\ -(x^2 - 5), & \text{if } -\sqrt{5} < x < \sqrt{5} \end{cases}.$$

Calculating $f'(x)$, we obtain

$$f'(x) = \begin{cases} 2x, & \text{if } x > \sqrt{5} \text{ or } x < -\sqrt{5}; \\ -2x, & \text{if } -\sqrt{5} < x < \sqrt{5}; \\ \textbf{DNE}, & \text{if } x = \pm\sqrt{5} \end{cases};$$

The stationary points of f occur when $f'(x) = 0$. This is precisely when $x = 0$. The singular points of f are $x = \pm\sqrt{5}$.

Therefore, f is increasing whenever $f'(x) > 0$ which is on the intervals $(-\sqrt{5}, 0)$ or $(\sqrt{5}, \infty)$.

(b) On what interval(s) is f concave down?

$$f''(x) = \begin{cases} 2, & \text{if } x > \sqrt{5} \text{ or } x < -\sqrt{5}; \\ -2, & \text{if } -\sqrt{5} < x < \sqrt{5}; \\ \textbf{DNE}, & \text{if } x = \pm\sqrt{5} \end{cases};$$

f is concave down whenever $f''(x) < 0$ which is on the interval $(-\sqrt{5}, \sqrt{5})$.

(c) Find the inflection point(s) of f.

The inflection points are those points where f changes concavity. This occurs when $x = \pm\sqrt{5}$.

(d) Find the critical points of f.

We did this above: $x = \pm\sqrt{5}, 0$.

(e) Find the local maximum (maxima) of f.

The only local maximum occurs when $x = 0$ and the local maximum is $f(0) = 5$.

(f) Find the global minimum of f on the interval $[-2, 3]$.

$f(-2) = 1$, $f(3) = 4$, $f(0) = 5$, $f(\pm\sqrt{5}) = 0$. Therefore, the global minimum occurs when $x = \pm\sqrt{5}$ and the global minimum is $f(\pm\sqrt{5}) = 0$.

9. Suppose the graph on the following page is of $y = f(x)$

(a) Find the critical numbers of f.

$x = -2, 0, 2$ or $x \geq 6$ are the critical numbers.

(b) On what interval(s) is f increasing?

$(-\infty, -2)$ or $(0, 2)$.

(c) On what interval(s) is f concave down?

$(-2, 0)$ or $(0, 4)$.

(d) Find the values of x on the interval $(-\infty, \infty)$ where f has a local minimum.

$x = 0$ or $x \geq 6$.

(e) Find the values of x on the interval $[0, 4]$ where f has a global minimum.

$x = 0$ or $x = 4$.

(f) Find the x values of all inflection points of f.

$x = 4$.

10. Suppose the graph on the following page is of $y = f'(x)$ (**NOT** $f(x)$).

(a) Find the critical numbers of f.

The stationary points are where $f'(x) = 0$ which occurs when $x = -4, 0, 4$. The singular points are where $f'(x)$ does not exist. There are no such points in this case. Therefore, $x = -4, 0, 4$ are the critical numbers.

(b) On what interval(s) is f increasing?

$f(x)$ is increasing on the intervals $(-4, 0)$ and $(0, 4)$.

(c) On what interval(s) is f concave down?

$f(x)$ is concave down on $(-2, 0)$ and $(2, 6)$.

(d) Find the values of x on the interval $(-\infty, \infty)$ where f has a local minimum.

$f(x)$ has a local minimum when $x = -4$ only.

(e) Find the values of x on the interval $[0, 4]$ where f has a global minimum.

$x = 0$ is the global minimum since $f(x)$ is increasing on $(0, 4)$.

(f) Find the x values of all inflection points of f.

$x = -2, 0, 2$