Math 123, Practice Exam #3 Solutions, December 5, 2008

(1) Find the following:

(a)

$$\begin{aligned}\int (5t^3 - \frac{7}{t^4} + 6\sqrt{t} - 4\sin t)\,dt &= \int (5t^3 - 7t^{-4} + 6t^{\frac{1}{2}} - 4\sin t)\,dt \\ &= \frac{5}{4}t^4 + \frac{7}{3}t^{-3} + 4t^{\frac{3}{2}} + 4\cos t + C\end{aligned}$$

(b)

$$\int (2e^x + \frac{3}{x} - \pi^x + 7\sec^2 x)\,dx = 2e^x + 3\ln|x| - \frac{\pi^x}{\ln \pi} + 7\tan x + C$$

(c)

$$\int_1^2 \frac{1 - x^3}{x^2}\,dx = \int_1^2 \left(\frac{1}{x^2} - x\right)\,dx = \left(-x^{-1} - \frac{x^2}{2}\right)\Big|_1^2 = -1$$

(d)

$$\frac{d}{dx}\int_3^x \sqrt{t^3+1}\,dt = \sqrt{x^3+1}$$

by using the fundamental theorem of calculus.

(e)

$$\int_0^{\sqrt{\pi}} \frac{d}{dt}\cos t^2\,dt = \cos t^2\big|_0^{\sqrt{\pi}} = \cos\pi - \cos 0 = -2$$

(f)

$$\begin{aligned}\int_{-1}^{10} f(x)\,dx &= \int_{-1}^{0} f(x)\,dx + \int_0^3 f(x)\,dx + \int_3^{10} f(x)\,dx \\ &= \int_{-1}^{0} (-x^2)\,dx + \int_0^3 (2x)\,dx + \int_3^{10} (-5)\,dx = -\frac{1}{3} + 9 - 35 = -\frac{79}{3}.\end{aligned}$$

(g) Let $u = x^3 - 8x^2 + 5x + 3$ then $du = \frac{du}{dx}dx = (3x^2 - 16x + 5)\,dx$ then

$$\begin{aligned}\int \frac{3x^2 - 16x + 5}{\sqrt{x^3 - 8x^2 + 5x + 3}}\,dx &= \int (3x^2 - 16x + 5)(x^3 - 8x^2 + 5x + 3)^{-\frac{1}{2}}\,dx \\ &= \int u^{-\frac{1}{2}}\,du = 2u^{\frac{1}{2}} + C = 2(x^3 - 8x^2 + 5x + 3)^{\frac{1}{2}} + C\end{aligned}$$

(h) Let $u = x^2 - 3x$ then $\frac{du}{dx} = (2x - 3)$ or $\frac{1}{2}\frac{du}{dx} = (x - \frac{3}{2})\,dx$ and

$$\int (x - \frac{3}{2})\sin(x^2 - 3x)\,dx = \int \sin(u)\,\frac{1}{2}\,du = \frac{1}{2}(-\cos u) + C = -\frac{1}{2}\cos(x^2 - 3x) + C$$

(2) Find the following:

(a) For any constant a, we have

$$\begin{aligned}\frac{d}{dx}\int_{\sin x}^{x^3} e^{t^2}\,dt &= \frac{d}{dx}\Big(\int_a^{x^3} e^{t^2}\,dt + \int_{\sin x}^{a} e^{t^2}\,dt\Big) \\ &= \frac{d}{dx}\Big(\int_a^{x^3} e^{t^2}\,dt - \int_a^{\sin x} e^{t^2}\,dt\Big) \\ &= e^{(x^3)^2}\frac{d}{dx}(x^3) - e^{(\sin x)^2}\frac{d}{dx}(\sin x) \\ &= e^{x^6}(3x^2) - e^{\sin^2 x}\cos x\end{aligned}$$

(b) $f(t)$ which satisfies the equation

$$\int_1^x \frac{f(t)}{t}\,dt = 3x^{\frac{1}{3}} - 3$$

Apply $\frac{d}{dx}$ to both sides of the equation to get

$$\frac{f(x)}{x} = 3(\frac{1}{3}x^{-\frac{2}{3}})$$

Solving for $f(x)$ and simplifying, we get

$$f(x) = x^{\frac{1}{3}}.$$

(c) the average speed an object moving along a line between $-2 \le t \le 6$ if its velocity at time t is given by $v(t) = t^2 - 3t - 4$

The speed at time t, $s(t) = |v(t)|$ by definition so the average speed we are after is

$$\overline{s} = \frac{1}{6-(-2)}\int_{-2}^{6} |t^2 - 3t - 4|\,dt = \frac{1}{8}\int_{-2}^{6} |t^2 - 3t - 4|\,dt.$$

Since $v(t) = (t-4)(t+1)$, $v(t) \ge 0$ when either $t \ge 4$ or $t \le -1$ while $v(t) < 0$ when $-1 \le t \le 4$. Thus,

$$|v(t)| = \begin{cases} t^2 - 3t - 4, & \text{if} \quad t \ge 4 \quad \text{or} \quad t \le -1 \\ -(t^2 - 3t - 4) & \text{if} \quad -1 \le t \le 4 \end{cases}$$

and we have

$$\begin{aligned}\int_{-2}^{6} |t^2 - 3t - 4|\,dt &= \int_{-2}^{-1} |t^2 - 3t - 4|\,dt + \int_{-1}^{4} |t^2 - 3t - 4|\,dt + \int_{4}^{6} |t^2 - 3t - 4|\,dt \\ &= \int_{-2}^{-1} (t^2 - 3t - 4)\,dt + \int_{-1}^{4} -(t^2 - 3t - 4)\,dt + \int_{4}^{6} (t^2 - 3t - 4)\,dt. \\ &= \frac{17}{6} + \frac{125}{6} + \frac{38}{3} = \frac{109}{3}.\end{aligned}$$

Therefore, the average speed is

$$\overline{s} = \frac{1}{8}(\frac{109}{3}) = \frac{109}{24}.$$

(3) Michelle begins walking along a line at time $t = 0$. Her acceleration at time $t \ge 0$ is $a(t) = 6t - 7$. Suppose that her initial velocity is 1 and her initial position is 3. If $s(t)$ denotes her position at time t and $v(t)$ denotes her velocity at time t then answer the following:

(a) Find her velocity at $t = 4$. Since $v(t)$ is an antiderivative of $a(t)$,

$$v(t) = \int (6t - 7)\,dt = 3t^2 - 7t + v_0$$

where v_0 is a constant. Furthermore, $1 = v(0) = v_0$ so

$$v(t) = \int (6t - 7)\,dt = 3t^2 - 7t + 1.$$

Therefore, $v(4) = 3(4)^2 - 7(4) + 1 = 21$.

(b) Find her position at $t = 2$. Since $s(t)$ is an antiderivative of $v(t)$,

$$s(t) = \int (3t^2 - 7t + 1)\,dt = t^3 - \frac{7}{2}t^2 + t + s_0$$

where s_0 is a constant. Furthermore, $3 = s(0) = s_0$ so

$$s(t) = t^3 - \frac{7}{2}t^2 + t + 3.$$

Therefore, $s(2) = 2^3 - \frac{7}{2}(2)^2 + 2 + 3 = -1$.

(c) When does she return to her starting position? This occurs when $s(t) = s(0)$ where $t > 0$. But $s(0) = 3$ so we are really solving the equation:

$$t^3 - \frac{7}{2}t^2 + t = 0$$

The left hand side is

$$t(t^2 - \frac{7}{2}t + 1) = 0.$$

Now, since we are interested in $t > 0$, we need to solve:

$$t^2 - \frac{7}{2}t + 1 = 0$$

By the quadratic formula, Michelle returns to the starting point when

$$t = \frac{7 \pm \sqrt{33}}{4}$$

both of which are positive.

(4) Consider the following Riemann sum:

$$I := \lim_{n \to \infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)^8 \left(\frac{2}{n}\right).$$

(a) Write I as a definite integral.

$$\int_1^3 x^8\, dx$$

(b) Calculate I (using any method you like).

$$\int_1^3 x^8\, dx = \frac{3^9}{9} - \frac{1^9}{9} = \frac{19682}{9}$$

(5) Consider the graph below. Find the following:

(a)

$$\int_{-5}^{0} f(x)\, dx = \int_{-5}^{-4} f(x)\, dx + \int_{-4}^{-3} f(x)\, dx + \int_{-3}^{0} f(x)\, dx = (3)(1) + (-4)(1) - \frac{1}{2}(4)(3) = -7.$$

(b)

$$F'(x) = \frac{d}{dx}\int_{-6}^{x} f(t)\, dt = f(x)$$

Therefore, $F'(-3) = f(-3) = -4$.

(c)

$$\int_{-3}^{1} |\, f(x)\,|\, dx = \int_{-3}^{0} |\, f(x)\,|\, dx + \int_{0}^{1} |\, f(x)\,|\, dx = \frac{1}{2}(3)(4) + \frac{1}{2}(1)(2) = 7$$

(d)

$$\int_{-3}^{4} f'(x)\, dx = f(4) - f(-3) = -1 - (-4) = 3.$$

(e) Let $u = x^2$ then $du = \frac{du}{dx}\, dx = 2x dx$. Furthermore, $(-1)^2 = 1$ and $2^2 = 4$ thus,

$$\int_{-1}^{2} f'(x^2)\, x\, dx = \int_{1}^{4} f'(u)\, \frac{1}{2}\, du = \frac{1}{2}(f(4) - f(1)) = \frac{1}{2}(-1 - 2) = -\frac{3}{2}.$$

(f)

$$\int_{5}^{7} (9(f(x))^2 - 8)\, dx = \int_{5}^{7} (9(2)^2 - 8)\, dx = 56$$

(g) $g(x)$ is increasing whenever $g'(x) > 0$ but $g'(x) = f(x)$. Therefore, $g(x)$ is increasing on the interval $(0, 1)$ since we are restricting to x values $-4 \leq x \leq 1$.
Similarly, $g''(x) = f'(x)$ but $f'(x) > 0$ which occurs on the interval $(-3, 0) \cup (0, 1)$.