MA 124 Test 1

The bonus problem is more difficult than the other problems, please do as much as you can on problems 1 - 5 before spending time on the bonus.

You may find the following equations useful.

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\sin 2x = 2 \sin x \cos x$$

Problem 1

Evaluate the following integrals

1.
$$\int e^{zp}\, dz$$

2.
$$\int \sin 2x \cos x\, dx$$

3.
$$\int_0^\infty x e^{-x}\, dx$$

4.
$$\int \frac{1}{(x^2 + 2x + 1)}\, dx$$

5.
$$\int_1^\infty \frac{1}{x^e}\, dx$$

Problem 2

For $x \geq 0$, the function $g(x)$ is given by,

$$g(x) = \int_0^x 2 + t - t^2 \, dt$$

1. Find the maximum value.
2. Interpret your result geometrically.

Problem 3

The Trapezoid, Midpoint and Simpson's rules for numerically approximating the integral of $f(x)$ on $[a, b]$ are given below. In each of these, $h = (b - a)/n$.

$$\begin{aligned}
T_n &= \frac{h}{2}(f_0 + 2f_1 + 2f_2 + \ldots + 2f_{n-1} + f_n) \\
M_n &= h(f(\bar{x}_1) + f(\bar{x}_2) + \ldots + f(\bar{x}_n)), \bar{x}_i = (x_{i-1} + x_i)/2 \\
S_n &= \frac{h}{3}(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \ldots + 2f_{n-2} + 4f_{n-1} + f_n)
\end{aligned}$$

note: Simpson's rule is defined for even n.

1. Show that $S_{2n} = (1/3)T_n + (2/3)M_n$.
2. Use part 1 to argue where S_{2n} falls relative to T_n, M_n.

Problem 4

By definition, odd functions f satisfy $f(-x) = -f(x)$. For example, $f(x) = x$ is an odd function. Prove that for odd functions the following is true

$$\int_{-a}^{+a} f(x) \, dx = 0$$

Problem 5

Two spheres with radius r have centers on the x-axis at $(0,0)$ and $(r,0)$.

1. Find the x-value where their surfaces intersect.
2. Find the volume common to the two spheres.

Bonus Problem

Mathematical induction is used to prove relationships true for all integers. Induction proofs involve the following two steps,

1. Show the relation is true for the smallest reasonable value of n.
2. Assume the relation is true for n and show that this implies the relation is then true for $n+1$.

Use induction to prove the following relationship,

$$\int_0^1 (1-x^2)^n \, dx = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

Hint. Derive a reduction formula for this integral.