MA 124 Test 2

This exam has 4 problems and a bonus. Note that not all problems are worth the same number of points.

You may find the following equations useful.

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\sin 2x = 2 \sin x \cos x$$

Problem 1 - 15 points

Find solutions to the following differential equations. Check that your answers are correct.

1. $$\frac{dy}{dx} = xe^{-x}$$

2. $$x + y\frac{dy}{dx} = 0$$

3. $$\frac{dy}{dx} = 1 - xy + x - y$$

Problem 2 - 30 points

Your great aunt invested some money for you when you were born 20 years ago in an account that compounded interest continuously with annual rate r. Sadly, your aunt died and you've now inherited the account.

1. If you inherit P dollars, how much did your aunt initially set aside for you? That is, find P_0 in terms of P.
2. Your younger sister is 10 years old and also inherited an account that was funded at her birth. That account is now worth Q dollars. If $P = 2Q$ and $r = 10\%$, who did your great aunt love the most, you or your sister? (Look at the ratio P_0/Q_0 and determine whose account started with the most money).
3. What do you conclude if $P = 2Q$ and $r = 5\%$?

Problem 3 - 25 points

Water is pumped into a cylindrical tank with height H and constant cross-sectional area A at rate r. The tank has a small hole in the bottom and water drains out according to Torricelli's law. Torricelli's law gives the rate at which water flows through a hole with area a as,

$$ka\sqrt{2gy} \tag{1}$$

In this equation g is the constant acceleration due to gravity, y is the water depth and k is a positive proportionality constant. Note that $ka\sqrt{2gH}$ is the maximum rate that water can flow out of the tank. For the following questions, assume that $0 < r < ka\sqrt{2gH}$.

1. Let $y(t)$ be the water depth. Derive a differential equation for y.
2. Find any equilibrium solutions.
3. Describe what happens if $y(0) = 0$. Give a qualitative answer, do not solve the differential equation.
4. Describe what happens if $y(0) = H$. Give a qualitative answer, do not solve the differential equation.

Problem 4 - 30 points

A cup of coffee cools according to Newton's law of cooling. That is, the rate of cooling is proportional to the temperature difference between the coffee and the surrounding room. Let R be the constant temperature of the surrounding room.

1. Let $T(t)$ be the temperature of the coffee as a function of time. Write a differential equation for $T(t)$.
2. Define the half-life of a cup of coffee as the time it takes to cool half way to R. That is, if T is initially T_0, find t such that $T(t) - R = (T_0 - R)/2$. You will need to first solve for $T(t)$.
3. Is the half-life you found in (2) above independent of T_0? Would a half-life that depends on initial conditions be useful?

Bonus Problem - 10 points

If fish are harvested with constant effort then the catch will be proportional to the population size and the following Logistic model applies.

$$\frac{dP}{dt} = rP(1 - \frac{P}{K}) - aP$$

In this equation aP is the rate at which fish are harvested and $a \geq 0$. Constant effort means that the same number of boats fish the same number of days each year.

1. What condition on the parameter a results in 2 equilibrium solutions?
2. Assume that a is such that there are two equilibrium solutions. Draw a diagram showing qualitatively how solutions $P(t)$ behave.
3. A sustainable yield is the rate at which fish can be caught indefinitely. Find an expression for the sustainable yield (as a function of a) and determine the value of a which maximizes the sustainable yield.