MA 124 Final Exam

Note that not all problems are worth the same number of points. Be sure to show all work. No credit will be given unless your work is shown. You may find the following equations useful.

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\sin 2x = 2 \sin x \cos x$$
$$\lim_{n \to \infty} \frac{x^n}{n!} = 0$$
$$\binom{k}{n} = \frac{k(k-1)\cdots(k-n+1)}{n!} \quad \text{for} \quad n \geq 1$$
$$\binom{k}{0} = 1$$

Problem 1 - 10 points

Find the volume of the solid obtained by rotating the region in the first quadrant bounded by the curves $y = x^{\frac{1}{3}}$, $x = 0$, $y = 1$ about the y-axis. Sketch the solid and the region bounded between the curves.

Problem 2 - 15 points

Decide if the following series are convergent or divergent. Make sure to justify your answer.

1.

$$\frac{1}{100} + \frac{1}{200} + \frac{1}{300} + \cdots$$

2.

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$$

3.

$$\sum_{n=1}^{\infty} \sin\left(\frac{2\pi n}{3}\right)$$

Problem 3 - 10 points

For the repeating decimal fraction $.121212\ldots$, answer the following questions:

1. Rewrite this number as a series. Remember, in positional notation, the first digit after the decimal point is the tenth's place, the second digit is the hundredth's place and so on.

2. Compute the sum of your series to find what fraction $.121212\ldots$ represents.

Problem 4 - 10 points

For the following alternating series,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

1. Is the series convergent?

2. Is the series absolutely convergent?

Problem 5 - 15 points

Let the function $f(x)$ be given by the formula

$$f(x) = x + x^3 + x^5 + \ldots$$

1. Find an explicit formula for $f(x)$.

2. What is the radius of convergence and the interval of convergence?

3. What happens to the function you found in part 1 at the endpoints of the interval of convergence?

Problem 6 - 20 points

Find the Taylor series expansion about 0 (Maclaurin series) for the following functions and find the radius of convergence.

1.

$$f(x) = e^{1-x}$$

2.

$$f(x) = \frac{1}{1 - 4x}$$

Problem 7 - 10 points

Evaluate the following integrals

1.

$$\int \sin 2x \cos x \, dx$$

2.

$$\int_0^\infty x e^{-x} \, dx$$

Problem 8 - 10 points

Find solutions to the following differential equations. Check that your answers are correct.

1.

$$x + y\frac{dy}{dx} = 0$$

2.

$$\frac{dy}{dx} = 1 - xy + x - y$$

Bonus

Each of these questions is worth 5 bonus points.

1. Xeno's Paradox: A guy named Xeno was shooting an arrow at a target that was 100 feet away. As he was thinking about the path the arrow would take on the way to the target, he came to the conclusion that the arrow could never really get there. He did this using the following reasoning: First, the arrow must move half the distance to the target (ie 50 feet), then it must move half the remaining distance (ie 25 feet). If it continues in this way, Xeno thought, it will always have half the remaining distance left to travel and can never actually reach the target.

   Represent the total distance the arrow travels using a series. Will the arrow ever get there? If Xeno had known calculus, he could have solved his paradox!

2. State the Fundamental Theorem of Calculus (both parts) and use it to differentiate

$$f(x) = \int_0^x x^2 \sin\left(t^2\right) dt$$