11 Basics of Wavelets

References: I. Daubechies (Ten Lectures on Wavelets; Orthonormal Bases of Compactly Supported Wavelets)

Also: Y. Meyer, S. Mallat

Outline:

1. Need for time-frequency localization
2. Orthonormal wavelet bases: examples

3. Meyer wavelet
4. Orthonormal wavelets and multiresolution analysis

Signal:

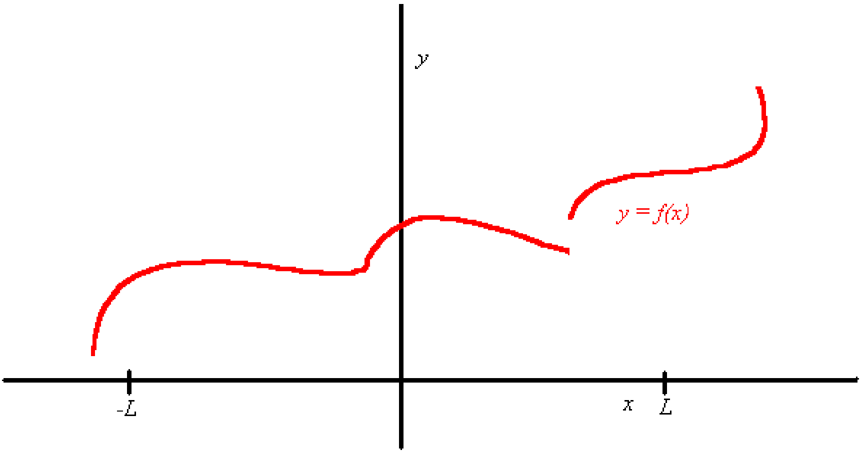


fig 1

Interested in “frequency content” of signal, locally in time. E.G., what is the frequency content in the interval [.5, .6]?

Standard techniques: write in Fourier series as sum of sines and cosines: given function defined on $[-L, L]$ as above:

$$f(x) =$$

$$\frac{1}{2}\,a_0 + \sum_{n=1}^{\infty} a_n \cos nx(\pi/L) + b_n \sin nx(\pi/L)$$

$(a_{n,}\, b_n$ constants)

$$a_n = \frac{1}{L}\int_{-L}^{L} dx\, f(x) \cos nx(\pi/L)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} dx\, f(x) \sin\, nx\,(\pi/L)$$

(generally f is complex-valued and a_n, b_n are complex numbers).

THEORY OF FOURIER SERIES

Consider function $f(x)$ defined on $[-L, L]$.

Let $L^2[-L, L] =$ square integrable functions

$$= \left\{ f : [-L, L] \to \mathbb{C} \,\middle|\, \int_{-L}^{L} dx \, |f^2(x)| < \infty \right\}$$

where $\mathbb{C} =$ complex numbers. Then L^2 forms a Hilbert space.

Basis for Hilbert space:

$$\left\{ \frac{1}{\sqrt{L}} \cos nx(\pi/L),\ \frac{1}{\sqrt{L}} \sin nx(\pi/L) \right\}_{N=1}^{\infty}$$

(together with the constant function $1/\sqrt{2L}$).

These vectors form an orthonormal basis for L^2 (constants $1/\sqrt{L}$ give length 1).

2. Complex form of Fourier series (see previous lecture):

Equivalent representation:

Can use Euler's formula $e^{ib} = \cos b + i \sin b$. Can show similarly that the family

$$\left\{ \frac{1}{\sqrt{2L}} e^{inx(\pi/L)} \right\}_{n=-\infty}^{n=\infty}$$

$$= \left\{ \frac{1}{\sqrt{2L}} \cos nx(\pi/L) + \frac{i}{\sqrt{2L}} \sin nx(\pi/L) \right\}_{n=}^{\infty}$$

is orthonormal basis for L^2 .

Function $f(x)$ can be written

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \phi_n(x)$$

where

$$c_n = \left\langle \phi_n, f \right\rangle = \int_{-L}^{L} dx \, \overline{\phi_n(x)} f(x),$$

and

$\phi_n(x) = n^{th}$ basis element $= \frac{1}{\sqrt{2L}} e^{inx(\pi/L)}$.

3. FOURIER TRANSFORM

Fourier transform is "Fourier series" on entire line $(-\infty, \infty)$:

Start with function $f(x)$ on $(-L, L)$:

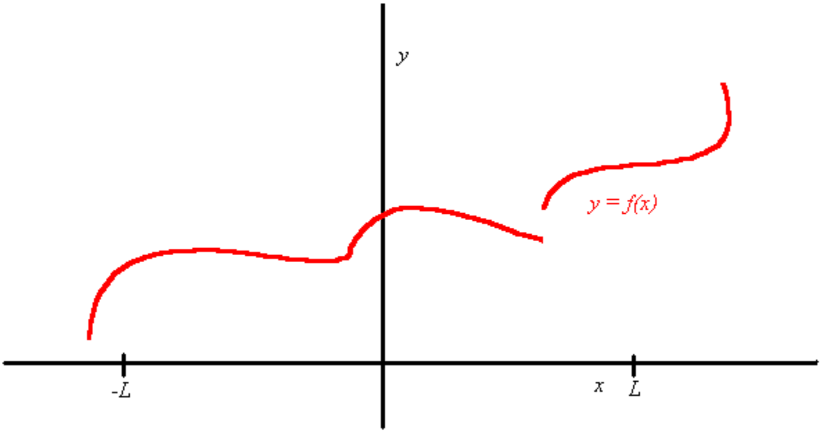


fig 2

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \phi_n(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx(\pi/L)} / \sqrt{2L}$$

Let $\xi_n = n\pi/L$; let $\Delta\xi = \pi/L$;

let $c(\xi_n) = c_n \sqrt{2\pi} / \left(\sqrt{2L}\, \Delta\xi \right)$.

Then:

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx(\pi/L)} / \sqrt{2L} \\
&= \sum_{n=-\infty}^{\infty} \left(c_n / \sqrt{2L}\right) e^{ix\xi_n} \\
&= \sum_{n=-\infty}^{\infty} c_n / \left(\sqrt{2L}\Delta\xi\right) e^{ix\xi_n} \Delta\xi \, . \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n \sqrt{2\pi} / \left(\sqrt{2L}\Delta\xi\right) e^{ix\xi_n} \Delta\xi. \\
&= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c(\xi_n) e^{ix\xi_n} \Delta\xi.
\end{aligned}
$$

Note as $L \to \infty$, we have $\Delta\xi \to 0$, and

$$
\begin{aligned}
c(\xi_n) &= c_n\sqrt{2\pi}/\Big(\sqrt{2L}\Delta\xi\Big) \\
&= \int_{-L}^{L} dx\, f(x)\, \overline{\phi_n(x)} \cdot \sqrt{2\pi}/\Big(\sqrt{2L}\,\Delta\xi\Big) \\
&= \int_{-L}^{L} dx\, f(x)\ \frac{\sqrt{2\pi}}{2L\Delta\xi}\, e^{-inx(\pi/L)} \\
&= \int_{-L}^{L} dx\, f(x)\, \frac{\sqrt{2\pi}}{2L(\pi/L)}\, e^{-inx(\pi/L)} \\
&= \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} dx\, f(x)\ \ e^{-ix\xi_n}\ .
\end{aligned}
$$

Now (informally) take the limit $L \to \infty$. The interval becomes

$$[-L, L] \to (-\infty, \infty).$$

We have

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c(\xi_n)\, e^{ix\xi_n} \Delta\xi$$

$$\xrightarrow[L \to \infty]{\text{fundamental thm. calculus}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c(\xi) e^{ix\xi}\, d\xi$$

[note this is like Riemann sum of calculus, which turns into integral].

Finally, from above

$$c(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} dx\, f(x)\; e^{-ix\xi}$$

$$\overrightarrow{L \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x)\; e^{-ix\xi}.$$

Thus, the informal arguments give that in the limit, we can write

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c(\xi)\, e^{ix\xi}\, d\xi,$$

where $c(\xi)$ (called *Fourier transform* of f) is

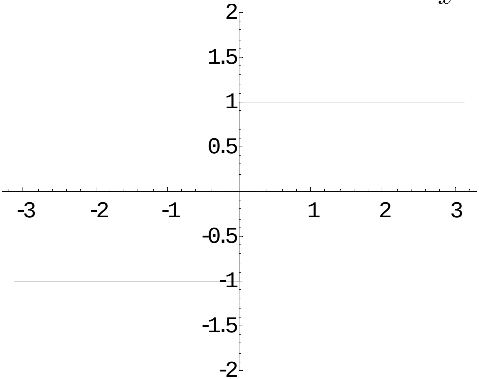
$$c(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x)\, e^{ix\xi}$$

(like Fourier series with sums replaced by integrals over the real line).

Note: can prove that writing $f(x)$ in the above integral form works for arbitrary $f \in L^2(-\infty, \infty)$.

4. FREQUENCY CONTENT AND GIBBS PHENOMENON

For now work with Fourier series on $\mathbb{R}$. If $f(x)$ discontinuous at $x = 0$, e.g. if $f(x) = \frac{|x|}{x}$:



first few partial sums of Fourier series are:

5 terms of FS:

$$\frac{4\sin x}{\pi} + \frac{4\sin 3x}{3\pi} + \frac{4\sin 5x}{5\pi}$$

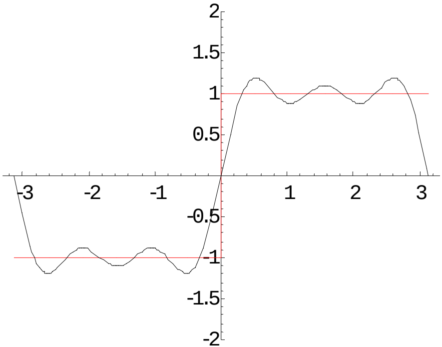
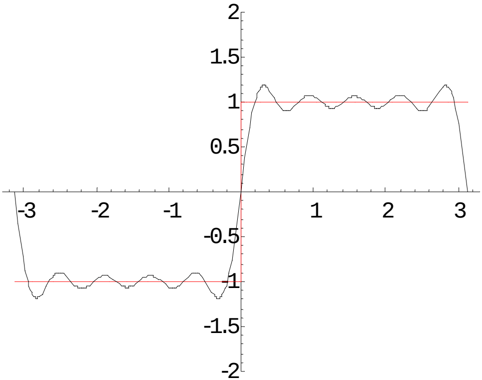


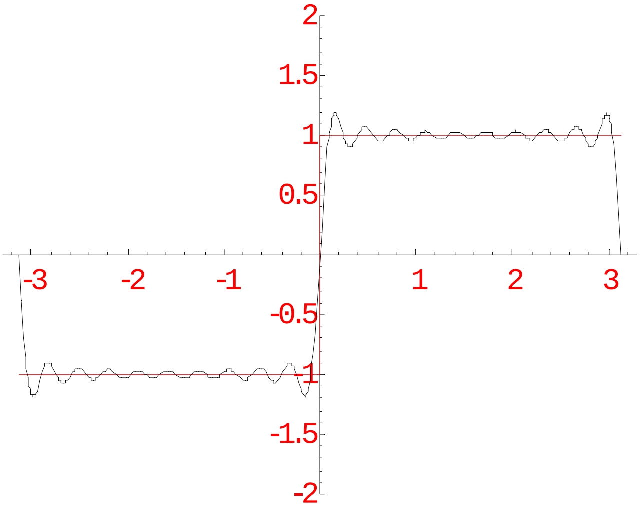
fig 3

10 terms:

$$\frac{4\sin x}{\pi} + \frac{4\sin 3x}{3\pi} + \frac{4\sin 5x}{5\pi} + \ldots + \frac{4\sin 10x}{10\pi}$$



20 terms:



40 terms:

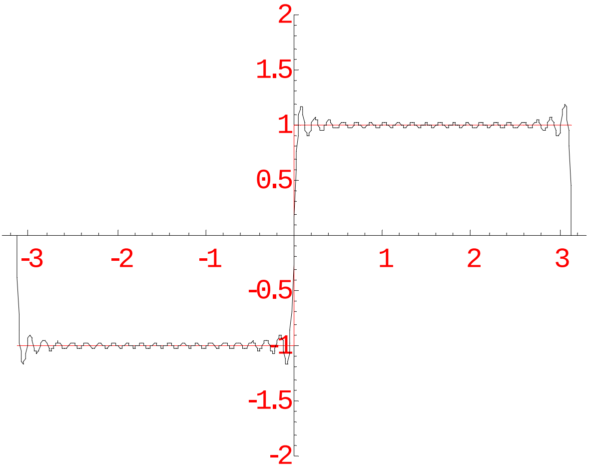


fig 4

Note there are larger errors appearing near “singularity" (discontinuity).

Specifically: “overshoot” of about 9% of the jump near singularity no matter how many terms we take!

In general, singularities (discontinuities in $f(x)$ or derivatives) cause high frequency components so that FS

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \, e^{inx} / \sqrt{2\pi}$$

has large c_n for n large (bad for convergence).

But notice that singularities are at only one point, but cause all c_n to be large.

Wavelets can deal with problem of localization of singularities, since they are localized.

Advantages of FS:

- “Frequency content" displayed in sizes of the coefficients a_k and b_k.

- Easy to write derivatives of f in terms of series (and use to solve differential equations)

Fourier series are a natural for differentiation.

Equivalently sines and cosines are eigenvectors of the derivative operator $\frac{d^2}{dx^2}$.

Disadvantages:

- Usual Fourier transform or series not well-adapted for time-frequency analysis (i.e., if high frequencies are there, we have large a_k and b_k for $k = 100$. But what part of the function has the high frequencies?

Where $x < 0$? Where $2 < x < 3$?

Possible solution:

Sliding Fourier transform -

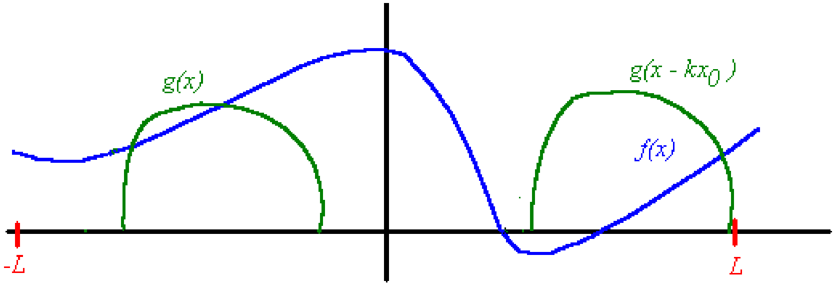


fig 5

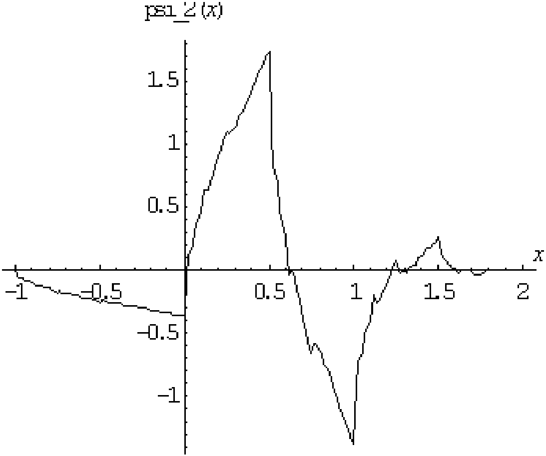
Thus first multiply $f(x)$ by "window" $g(x - kx_0)$, and look at Fourier series or take Fourier transform: look at

$$\int_{-L}^{L} dx\, f(x)\, g_{jk}(x) = \int_{-L}^{L} dx\, f(x)\; g(x - kx_0) e^{ij\frac{\pi}{L}x} \equiv c_{jk}$$

Note however: functions $g_{jk}(x) = g(x - kx_0) e^{ij\frac{\pi}{L}x}$ not orthonormal like sines and cosines; do not form a nice basis $-$ need something better.

5. Wavelet transform

Try: Wavelet transform - fix appropriate function $h(x)$.



Then form all translations by integers, and all 'scalings' by powers of 2:

$$h_{jk}(x) = 2^{j/2} h(2^j x - k)$$

($2^{j/2}$ $=$ normalization constant)

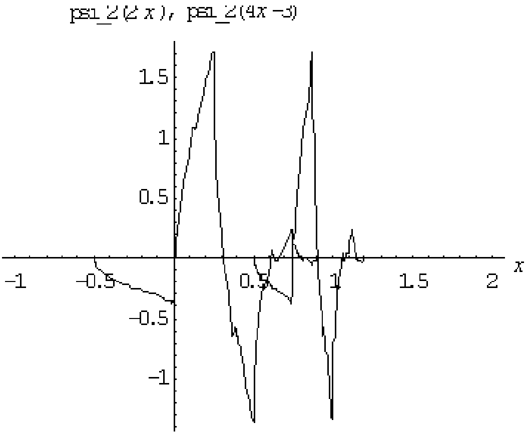


fig. 6: $h(2x)$ and $h(4x-3)$

Let

$$c_{jk} = \int dx f(x)\, h_{jk}(x).$$

If h chosen properly, then can get back f from the c_{jk}:

$$f(x) = \sum_{j,k} c_{jk}\, h_{jk}(x)$$

These new functions and coefficients are easier to manage. Sometimes much better –

Advantages over windowed Fourier transform:

- Coefficients c_{jk} are all real
- For high frequencies (j large), functions $h_{jk}(t)$ have good localization (get thinner as $j \to \infty$; above diagram). Thus short lived (i.e. of small duration in x) high frequency components can be seen from wavelet analysis, but not from windowed Fourier transform.

Note h_{jk} has width of order 2^{-j}, and is centered about $k2^{-j}$ (see diagram earlier).

DISCRETE WAVELET EXPANSIONS:

Take a basic function $h(x)$ (the basic wavelet);

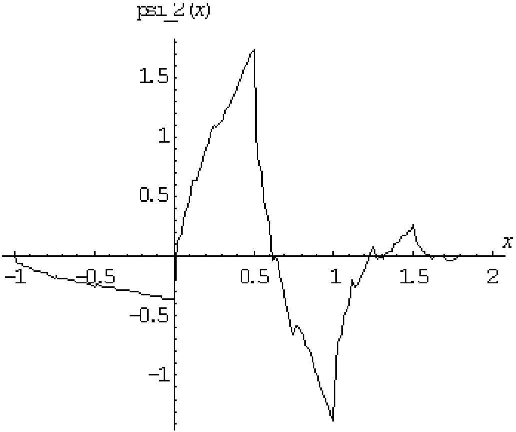


fig 7

let

$$h_{jk}(x) = 2^{j/2} h(2^j x - k).$$

Form discrete wavelet coefficients:

$$c_{jk} = \int dx\, f(x)\, h_{jk}(x) \equiv \langle f,\, h_{jk} \rangle.$$

Questions:

- Do c_{jk} characterize f?
- Can we expand f in an expansion of the h_{jk}?
- What properties must h have for this to happen?
- How can we reconstruct f in a numerically stable way from knowing c_{jk}?

We will show: It is possible to find a function h such that the functions h_{jk} form such a perfect basis for functions on $\mathbb{R}$.

That is, h_{jk} are orthonormal:

$$\langle h_{jk},\, h_{j'k'} \rangle \equiv \int h_{jk}(x) h_{j'k'}(x) dx = 0$$

unless $j = j'$ and $k = k'$.

And any function $f(x)$ can be represented by the h_{jk}:

$$f(x) = \sum_{j,k} c_{jk}\, h_{jk}(x).$$

So: like Fourier series, but h_{jk} are better (e.g., non-zero only on a small sub-interval, i.e., compactly supported)

6. A SIMPLE EXAMPLE: HAAR WAVELETS

Motivation: suppose have basic function

$$\phi(x) = \begin{cases} 1 \text{ if } 0 \leq x \leq 1 \\ 0 \text{ otherwise} \end{cases} = \text{basic "pixel".}$$

We wish to build all other functions out of pixel and translates $\phi(x-k)$

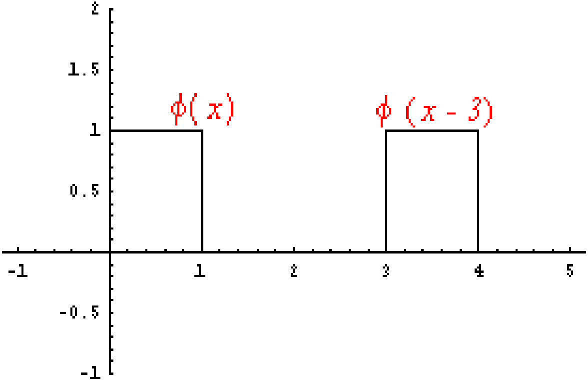


fig 8: ϕ and its translates

Linear combinations of the $\phi(x-k)$:

$$f(x) = 2\phi(x) + 3\phi(x-1) - 2\phi(x-2) + 4\phi(x-3)$$

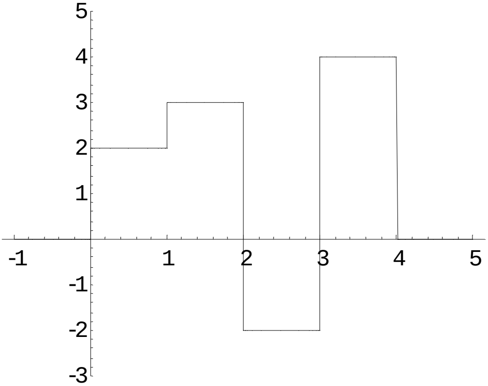


fig 9: linear combination of $\phi(x-k)$

[Note that any function which is constant on the integers can be written in such a form:]

Given function $f(x)$, approximate $f(x)$ by a linear combination of $\phi(x-k)$:

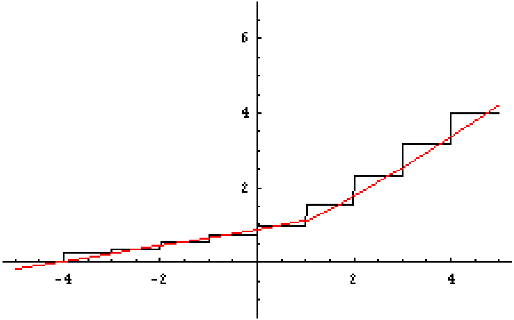


fig 10: approximation of $f(x)$ using the pixel $\phi(x)$ and its translates.

Define V_0 = all square integrable functions of the form

$$g(x) = \sum_k a_k \phi(x-k)$$

= all square integrable functions which are constant on integer intervals

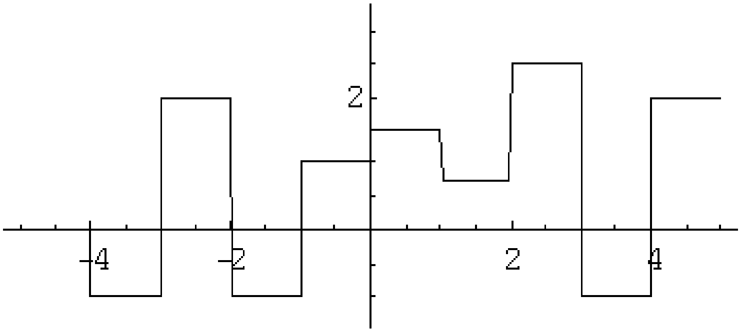


fig 11: a function in V_0

To get better approximations, shrink the pixel :

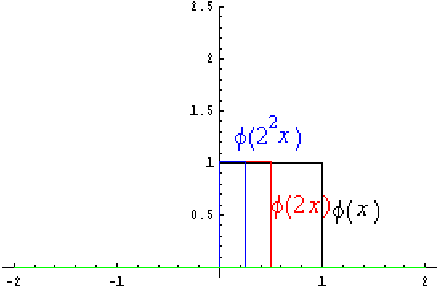


fig 12: $\phi(x)$, $\phi(2x)$, and $\phi(2^2x)$

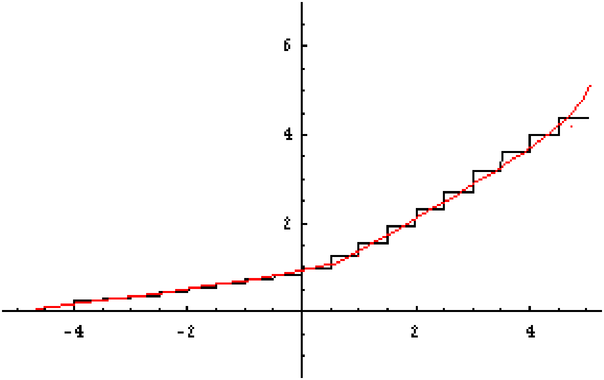


fig 13: approximation of $f(x)$ by translates of $\phi(2x)$.

Define

V_1 = all square integrable functions of the form

$$g(x) = \sum_k a_k \phi(2x - k)$$

$=$ all square integrable functions which are constant on all half-integers

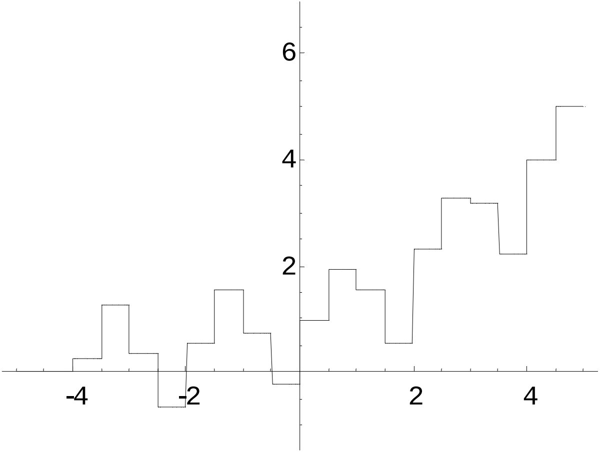


fig 14: Function in V_1

Define V_2 = sq. int. functions

$$g(x) = \sum_k a_k \phi(2^2 x - k)$$

= sq. int. fns which are constant on quarter integer intervals

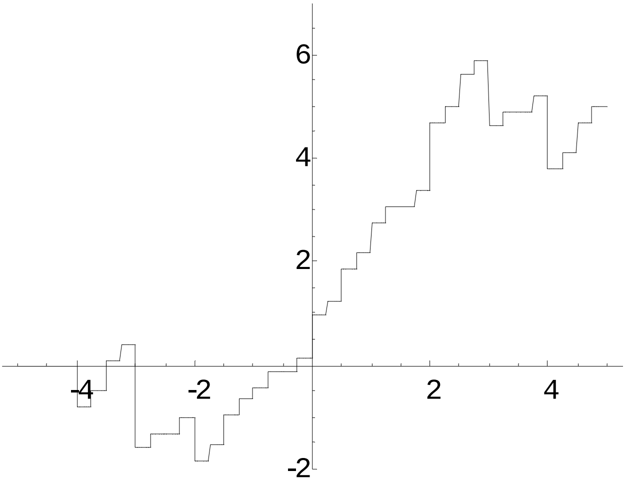


fig 15: function in V_2

Generally define V_j = all square integrable functions of the form

$$g(x) = \sum_k a_k \phi(2^j x - k)$$

= all square integrable functions which are constant on 2^{-j} length intervals

[note if j is negative the intervals are of length greater than 1].