# 12 The Haar wavelet

## 1. Definition

Now define desired wavelet $\psi(x)$

$$\equiv \begin{cases} 1 \quad \text{if } 0 \leq x \leq 1/2 & -1 \ \text{ if } 1/2 \leq x < 1 \\ & 0 \quad \text{otherwise} \end{cases}$$

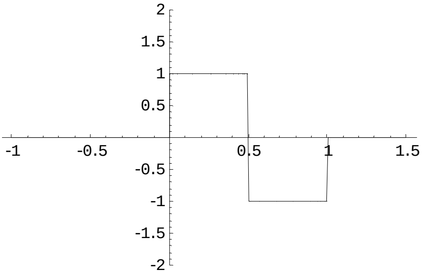



fig 16: $\psi(x)$

Now define family of Haar wavelets by translating:

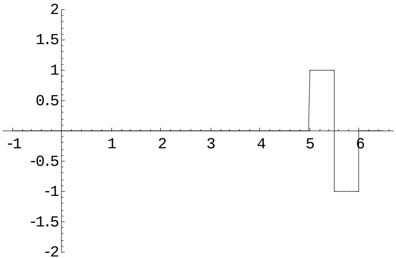



fig 17 : $\psi(x-5) = \psi_{0,5}$

and stretching:

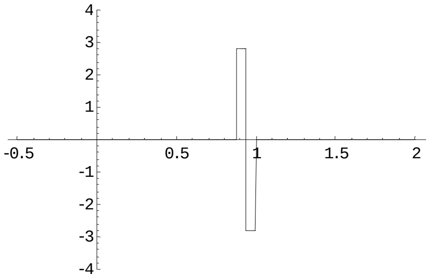



fig 18: $2^{3/2}\psi(2^3x-7)=\psi_{3,7}$

In general:

$$\psi_{jk} \equiv 2^{j/2}\,\psi(2^j x - k)$$

Show Haar wavelets are orthogonal, i.e.,

$$\left\langle \psi_{jk},\, \psi_{j'k'} \right\rangle \equiv \int_{-\infty}^{\infty} dx\, \psi_{jk}(x)\psi_{j'k'}(x) = 0$$

if $j \neq j'$ or $k \neq k'$ :

(i) if $j = j'$, $k \neq k'$:

$$\langle \psi_{jk}, \psi_{j'k'} \rangle = 0$$

because $\psi_{jk} = 0$ wherever $\psi_{jk'} \neq 0$ and vice-versa.

(ii) if $j \neq j'$:

$$\langle \psi_{jk}, \psi_{j'k'} \rangle = 0$$

because:

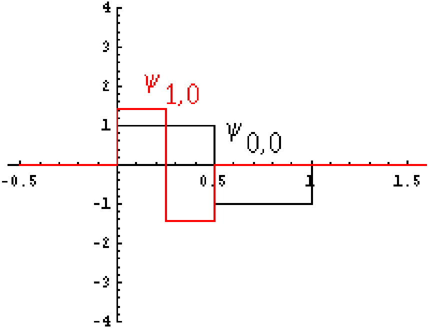



fig 19

$\Rightarrow$ integral $\langle \psi_{jk}, \psi_{j'k'} \rangle$ is $0$.

**2. Can any function be represented as a combination of Haar wavelets?**

[A general approach:]

**Recall:**

$$V_j = \text{square int. functions of form } \sum_k a_k \phi(2^j x - k)$$

$=$ square int. functions constant on dyadic intervals of length $2^{-j}$.

[note if j negative then intervals are of length > 1:]

$V_{-1}$ = functions constant on intervals of length 2

$V_{-2} =$ functions constant on intervals of length 4

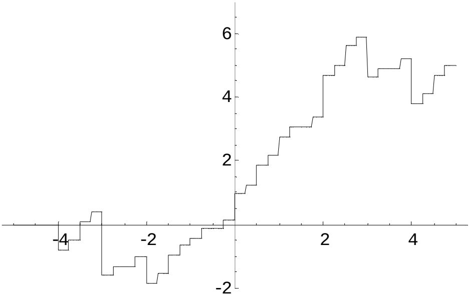



fig. 20: function in $V_j \ (j = 2)$

We have:

(a) $$\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset V_3 \ldots$$

[i.e., piecewise constant on integers $\Rightarrow$ piecewise constant on half-integers, etc.]

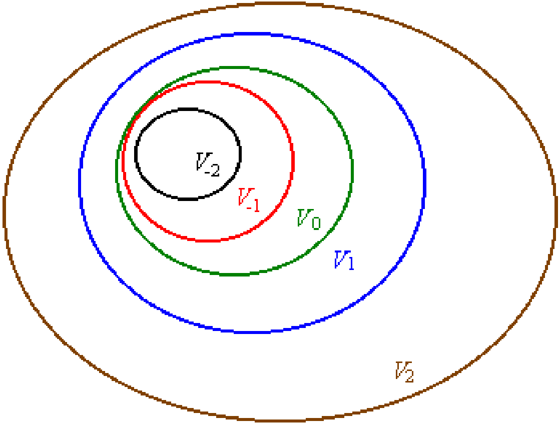



Fig. 21: Relationship of nested spaces $V_j$

(b) $\bigcap_n V_n = \{0\}$ (only $0$ function in all spaces)

[if a function is in all the spaces, then must be constant on arbitrarily large intervals $\Rightarrow$ must be everywhere constant; also must be square integrable; so must be 0].

(c) $\bigcup_n V_n$ is dense in $L^2(\mathbb{R})$

[i.e. the collection of all functions of this form can approximate any function $f(x)$]

Proof: First consider a function of the form $f(x) = \chi_{[a,b]}(x)$. Assume that $a = k/2^n - a_1$, and $b = \ell/2^n + b_1$, where $a_1, b_1 < 1/2^n$.

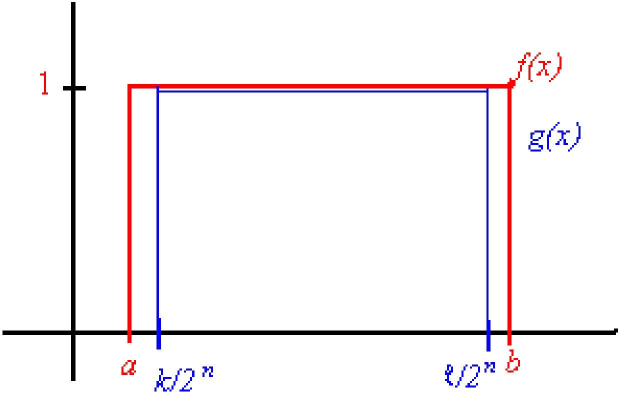



fig 22: Relationship of $a, b$ with $k/2^n$ and $\ell/2^n$.

Let

$$g(x) = \chi_{[k/2^n, \ell/2^n]}(x) \in \bigcup_j V_j.$$

Then

$$\|f - g\| = \int dx\,(f-g)^2 = \text{area under } f - g \leq 2/2^n$$

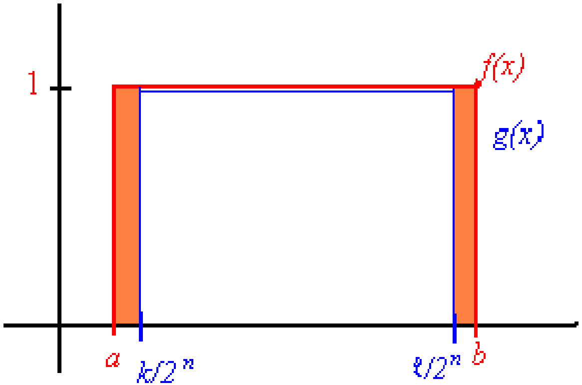



fig 23: area under $f - g$

Since $n$ is arbitrary, $\| f - g \|$ can be made arbitrarily small. Thus arbitrary char. functions $f$ can be well-approximated by functions $g(x) \in \bigcup_i V_i$.

Now if $f(x)$ is a step function:

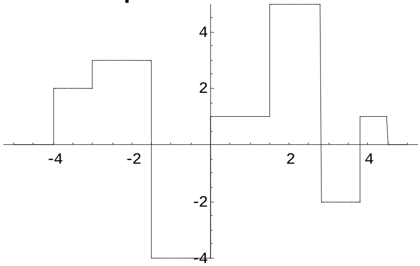



fig 24: step function

We can write

$f(x) = \sum_i c_i \chi_{[a_i,b_i]}(x) =$ linear combination of char. functions.

So by above argument, step functions $f$ can be approximated arbitrarily well by $g \in \bigcup_j V_j$.

Now step functions are dense in $L^2(\mathbb{R})$ (see R&S, problem II.2), so that $\bigcup_j V_j$ must be dense in $L^2(\mathbb{R})$. $\square$

(d) $f(x) \in V_n \quad \Rightarrow \quad f(2x) \in V_{n+1}$

[because function constant on intervals of length $2^{-n}$ when is constant on intervals of length $2^{-n-1}$]

(e) $f(x) \in V_0 \quad \Rightarrow \quad f(x-k) \in V_0$

[i.e. translating function by integer does not change that it's constant on integer intervals]

(f) There is an orthogonal basis for the space $V_0$ in the family of functions

$$\phi_{0k} \equiv \phi(x-k)$$

where $k$ varies over the integers. This function $\phi$ is (in this case) $\phi = \chi_{[0,1]}(x)$.

$\phi$ is called a *scaling function*.

**Definition:** A sequence of spaces $\{V_j\}$ together with a scaling function $\phi$ which generates $V_0$ so that (a) - (f) above are satisfied, is called a *multiresolution analysis.*

## 3. Some more Hilbert space theory

**Recall:** Two subspaces $M_1$ and $M_2$ of an inner product vector space $V$ are *orthogonal* if every vector $w_1 \in M_1$ is perpendicular to every vector $w_2 \in M_2$.

**Ex:** Consider $V = L^2(-\pi, \pi)$. Then let

$$M_1 = \{f(x) : f(x) = \sum_{n=0}^{\infty} a_n \cos nx\}$$

be the set of Fourier cosine series. Let

$$M_2 = \{f(x) : f(x) = \sum_{n=1}^{\infty} b_n \sin nx\}$$

be the set of Fourier sine series.

Then if $f_1 = \sum_{n=1}^{\infty} a_n \cos \; nx \in M_1$ and if $f_2 = \sum_{k=1}^{\infty} b_n \sin \; kx \in M_2$, then using usual arguments:

$$\langle f_1, f_2 \rangle = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_n b_k \langle \cos nx, \sin kx \rangle = 0$$

Thus $M_1$ is orthogonal to $M_2$.

**Recall:** A vector space $V$ is a *direct sum* $M_1 \oplus M_2$ of subspaces $M_1, M_2$ if
every vector $v \in V$ can be written uniquely as a sum of vectors $w_1 \in M_1$ and $w_2 \in M_2$.

$V$ is an *orthogonal* direct sum $M_1 \oplus M_2$ if the above holds and in addition $M_1$ and $M_2$ are orthogonal.

**Ex:** If $V = \mathbb{R}^3$, and

$$M_1 = x - y\,\text{plane} \;= \{(x, y, 0) : x, y \in \mathbb{R}\}$$

$$M_2 = z - \text{axis} = \{(0, 0, z) : z \in \mathbb{R}\},$$

then every $(x, y, z) \in V$ can be written uniquely as sum of $(x, y, 0) \in M_1$ and $(0, 0, z) \in M_2$, so $V$ is orthogonal direct sum $M_1 \oplus M_2$.

**Ex:** $V = L^2[-\pi, \pi]$. Then every function $f(x) \in V$ can be written uniquely as

$$f(x) = \sum_{n=0}^{\infty} a_n \cos nx + \sum_{k=1}^{\infty} b_n \sin kx$$

[note first sum in $M_1$ and second in $M_2$]

Thus $L^2 = M_1 \oplus M_2$ is orthogonal direct sum. Note: not hard to show

$M_1$ = even functions in $L^2$

$M_2$ = odd functions in $L^2$

[thus L$^2$ $=$ orthogonal direct sum of even functions and odd functions]

**Theorem 1:** *If $V$ is a Hilbert space and if $M_1 \perp M_2$ and $V = M_1 + M_2$, i.e., $\forall \quad v \in V \quad \exists \quad m_i \in M_i$ s.t. $v = m_1 + m_2$, then $V = M_1 \oplus M_2$ is an orthogonal direct sum of $M_1$ and $M_2$*

Pf: In exercises.

Note: no assumption of uniqueness of $v_i$ necessary above.

**Def:** If $V = W_1 \oplus W_2$ is an orthogonal direct sum, we also write

$$W_1 = V \ominus W_2; \qquad W_2 = V \ominus W_1.$$

**Recall:** Given a subspace $M \subset V$,

$M^\perp =$ vectors which are perpendicular to everything in $M$

$$= \{v \in V : v \perp w \forall w \in W\}$$

**Ex:** If $V = \mathbb{R}^3$, and $W = x$-$y$ plane, then $W^\perp = z$-axis

**Ex:** If $V = L^2$, then if $W =$ even functions, $W^\perp =$ odd functions.

**Pf.** exercise

**Recall** (R&S, Theorem II.3)**:** Given a complete inner product space $V$ and a complete subspace $M$, then $V$ is an orthogonal direct sum of $M$ and $M^\perp$

## 4. Back to wavelets:

**Recall:**

- $V_j =$ functions constant on dyadic intervals$[k2^{-j}, (k+1)2^{-j}]$.
- $\ldots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \ldots$

Since $V_0 \subset V_1$, there is a subspace $W_0 = V_1 \ominus V_0$ such that $V_0 \oplus W_0 = V_1$. Indeed, we can make $W_0 =$ perp space of $V_0$ as a subset of $V_1$.

That is we can define $W_0 = \{v \in V_1 | v \perp V_0\}$; follows easily that $W_0$ is closed.

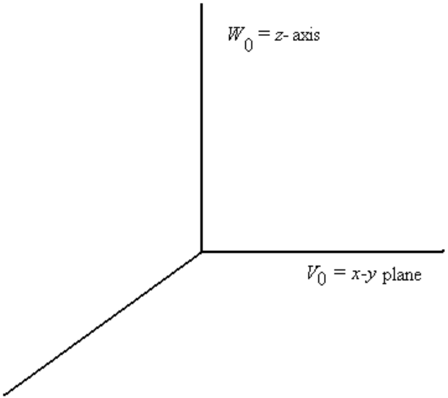



fig 25: Schematic relationship of $V_0$ and $W_0$ : $V_0$ as the $x$-$y$ plane and $W_0$ as the $z$ axis ... $V_0 \oplus W_0 = V_1 = \mathbb{R}^3$.

Similarly define

$$W_1 = V_2 \ominus V_1.$$

Generally:

$$W_{j-1} = V_j \ominus V_{j-1}$$

Then relationships are:

$$\begin{array}{ccccccccc} \ldots V_{-2} & \subset & V_{-1} & \subset & V_0 & \subset & V_1 & \subset & V_2 \ldots \\ & W_{-2} & & W_{-1} & & W_0 & & W_1 & \end{array}$$

Also note, say, for $V_3$:

$$
\begin{aligned}
V_3 &= V_2 \oplus W_2 \\
&= V_1 \oplus W_1 \oplus W_2 \\
&= V_0 \oplus W_0 \oplus W_1 \oplus W_2 \\
&= V_{-1} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2
\end{aligned}
$$

So if $v_3 \in V_3$ we have:

$$
\begin{aligned}
v_3 &= v_2 + w_2 \\
&= v_1 + w_1 + w_2 \\
&= v_0 + w_0 + w_1 + w_2 \\
&= v_{-1} + w_{-1} + w_0 + w_1 + w_2,
\end{aligned}
$$

with $v_i \in V_i$ and $w_i \in W_i$.

[successively decomposing the $v$ into another $v$ and a $w$].

In general :

$$v_3 = v_{-n} + \sum_{k=-n}^{2} w_k. \qquad (1)$$

Now let $n \to \infty$ . Since all vectors in above sum orthogonal, we have (see exercises):

$$\|v_3\|^2 = \|v_{-n}\|^2 + \sum_{k=-n}^{2} \|w_k\|^2.$$

Thus

$$\sum_{k=-n}^{2} \|w_k\|^2 \leq \|v_3\|^2$$

$\forall\, n$, so

$$\sum_{k=-\infty}^{2} \|w_k\|^2 < \infty.$$

**Lemma:** In a Hilbert space $H$, if $w_k$ are orthogonal vectors and the sum

$\sum_k \|w_k\|^2 < \infty$, then the sum $\sum_k w_k$ converges.

**Pf:** We can show that the sum $\sum_{k=1}^{N} w_k$ forms a Cauchy sequence by noting if $N > M$:

$$\|\sum_{k=1}^{N} w_k - \sum_{k=1}^{M} w_k\|^2 = \|\sum_{k=M+1}^{N} w_k\|^2 = \sum_{k=M+1}^{N} \|w_k\|^2 \xrightarrow[N,\,M \to \infty]{} 0.$$

Thus we have a Cauchy sequence. The sequence must converge ($H$ is complete), and so $\sum_{k=1}^{\infty} w_k$ exists. $\square$

From above:

$$v_{-n} = v_3 - \sum_{k=-n}^{2} w_k \,.$$

Letting $n \to \infty$, get $v_{-n} \xrightarrow[n \to \infty]{} v_3 - \sum_{k=-\infty}^{2} w_k$. Thus vectors $v_{-n}$ have limit as $n \to \infty$ : $v_{-n} \to v_{-\infty}$.

But notice

$$v_{-n} \in V_{-n} \subset V_{-n+1} \subset V_{-n+2} \cdots$$

$\Rightarrow$

$$v_{-n} \in V_{-n} \cap V_{-n+1} \cap V_{-n+2} \ldots = \bigcap_{k=-n}^{\infty} V_k$$

Thus $v_{-\infty} \in$ intersection of all $V_n$'s $=$

$\Rightarrow v_{-\infty} = 0$ (by condition (b) on spaces $V_j$).

Thus taking the limit as $n \to \infty$ in (1) :

$$v_3 \ = \ v_{-n} \ + \sum_{k=-n}^{2} w_k \tag{1}$$

get

$$v_3 \ = \sum_{k=-\infty}^{2} w_k \ .$$

So by definition of direct sum:

$$V_3 = \ldots W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 = \bigoplus_{k=-\infty}^{2} W_k$$

i.e., every vector in $V_3$ can be uniquely expressed as a sum of vectors in the $W_j$. Further this is an orthogonal direct sum since $W_j$'s orthogonal.

Generally:

$$V_n = \ldots W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \ldots \oplus W_{n-1}$$

$$= \bigoplus_{k=-\infty}^{n-1} W_k$$

Now note

$$L^2 = V_3 \oplus V_3^{\perp}$$

$$= V_4 \oplus V_4^{\perp}$$

$$= V_3 \oplus W_3 \oplus V_4^{\perp}.$$

Thus comparing above get

$$V_3^{\perp} = W_3 \oplus V_4^{\perp}.$$

Similarly,

$$V_4^{\perp} = W_4 \oplus V_5^{\perp}.$$

So

$$V_3^{\perp} = W_3 \oplus W_4 \oplus V_5^{\perp}.$$

Generally:

$$V_3^{\perp} = W_3 \oplus W_4 \oplus W_5 \oplus \ldots \oplus W_n \oplus V_{n+1}^{\perp}.$$

Letting $n \to \infty$ and using same arguments, we see that the $V_{n+1}^{\perp}$ components "go to $0$" as $n \to \infty$, so that

$$V_3^{\perp} = W_3 \oplus W_4 \oplus W_5 \oplus \ldots$$

Thus:

$$L^2 = V_3 \oplus V_3^{\perp} = \ldots W_{-2} \oplus W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \ldots$$

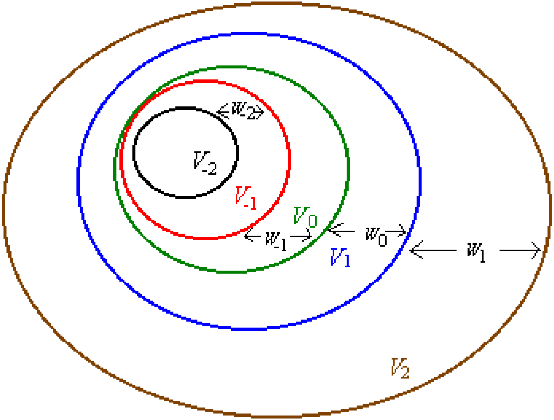
[Thus every function in $L^2$ can be uniquely written as a sum of functions in the $W_j$'s].

Thus:

**Theorem:** Every vector $v \in L^2(-\infty, \infty)$ can be uniquely expressed as a sum

$$\sum_{j=-\infty}^{\infty} w_j \text{ where } w_j \in W_j \,.$$

## Conclusion - relationship of $V_j$ and $W_j$:



## 5. What are the $W_j$ spaces?

Consider $W_0$.

**Claim:** $W_0 = A \equiv$ functions which are constant on half-integers and take equal and opposite values on half of each integer interval.

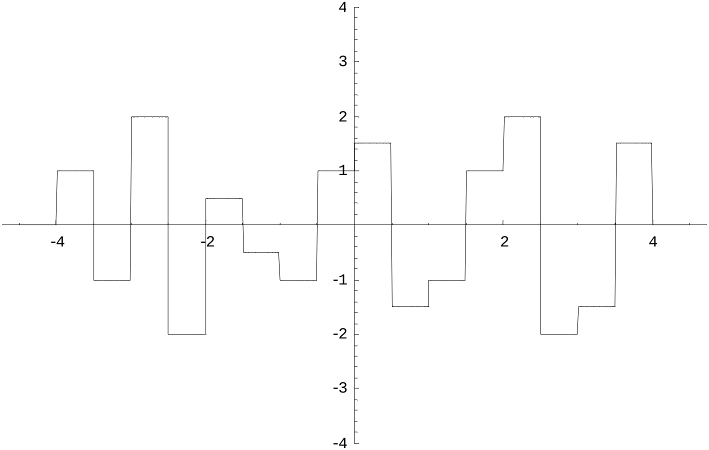



fig 25: Typical function in $A$

**Proof:** Will show that with above definition of $A$,

$$V_0 \oplus A = V_1,$$

and that $V_0$ and $A$ are orthogonal. Then it will follow that

$$A = V_1 \ominus V_0 \equiv W_0,$$

First to show $V_0$ and $A$ are orthogonal: let $f \in V_0$ and $g \in A$. Then $f$ looks like:

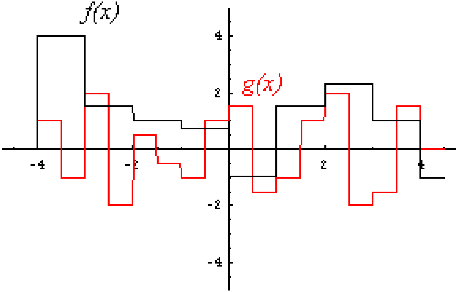



fig 25: $f(x) \in V_0$; $g(x) \in W_0$

Thus

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)dx$$

$$= \left( \int_{-2}^{-1} + \int_{-1}^{0} + \int_{0}^{1} + \int_{1}^{2} + \dots \right) f(x)g(x)dx \ = 0$$

since $f(x)g(x)$ takes on equal and opposite values on each half of every integer interval above, and so integrates to $0$ on each interval.

Thus $f$ and $g$ orthogonal, and so $V_0$ and $A$ are orthogonal.

Next will show that if $f \in V_1$, then $f = f_0 + g_0$, where $f_0 \in V_0$ and $g_0 \in A$ (which is all that's left to show).

Let $f \in V_1$. Then $f$ is constant on half integer intervals:

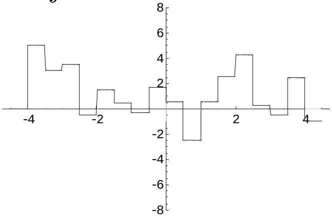



fig 26: $f(x) \in V_1$

Define $f_0$ to be the function which is constant on each integer interval, and whose value is the *average* of the two values of $f(x)$ on that interval:

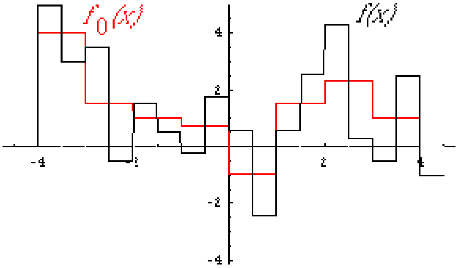



fig 27: $f_0(x)$ as related to $f(x)$.

Then clearly $f_0(x)$ is constant on integer intervals, and so is in $V_0$.

Now define $g_0(x) = f(x) - f_0(x)$ :

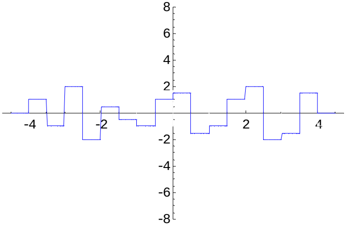



fig 28: $g_0(x) = f(x) - f_0(x)$

Then clearly $g_0$ takes on equal and opposite values on each half of every integer interval, and so is in $A$. Thus we have: for $f(x) \in V_1$,

$$f(x) = f_0(x) + g_0(x),$$

where $f_0 \in V_0$ and $g_0 \in A$. Thus $V_1 = V_0 \oplus A$ by Theorem 1 above.

Thus $A = V_1 \ominus V_0 = W_0$, so $A = W_0$.

Thus $W_0 =$ functions which take on equal and opposite values on each half of an integer interval, as desired. $\square$

Similarly, can show:

$W_j$ = functions which take on equal and opposite values on each half of the dyadic interval of length $2^{-j-1}$ and are square integrable:

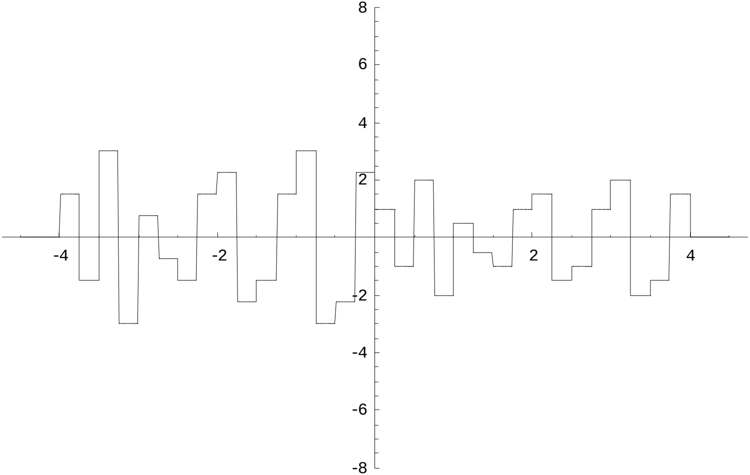



fig 29: typical function in $W_j$ $(j = 1)$

## 6 What is a basis for the space $W_j$?

Consider $W_0$ = functions which take equal and opposite values on each integer interval

What is a basis for this space? Let

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1/2 \\ -1 & \text{if } 1/2 < x \leq 1 \end{cases}.$$

Claim a basis for $W_0$ is $\{\psi(x-k)\}_{k=-\infty}^{\infty}$.

Note linear combinations of $\psi(x-k)$ look like:

$$g(x) = 2\psi(x) + 3\psi(x-1) - 2\psi(x-2) + 2\psi(x-3).$$

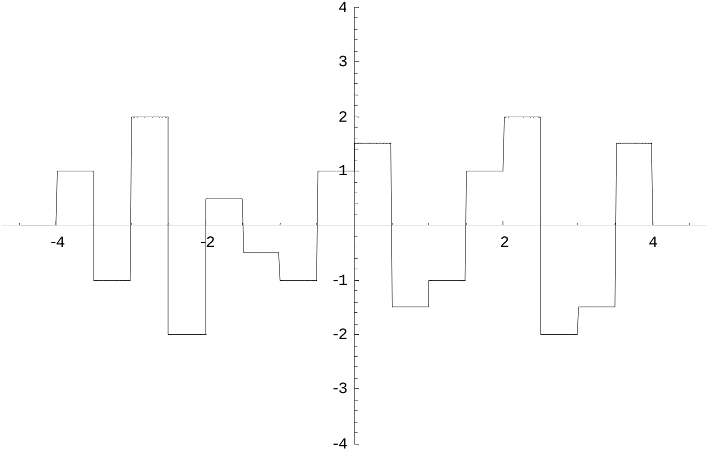



fig 30: graph of $g(x)$.

I.E., linear combinations of translates $\psi(x-k) =$ functions equal and opposite on each half of every integer interval.

Can easily conclude:

$W_0 =$ functions in $L^2$ equal and opposite on integer intervals

$=$ functions in $L^2$ which are linear combinations of translates $\psi(x-k)$.

Also easily seen translates $\psi(x-k)$ are orthonormal.

Conclude: $\{\psi(x-k)\}$ form orthonormal basis for $W_0$.

Similarly can show $\{2^{1/2}\psi(2x-k)\}_k$ form orthonormal basis for $W_1$.

$\{2^{2/2}\psi(2^2x-k)\}_k$ form orthonormal basis for $W_2$.

Generally,

$$\{2^{j/2}\psi(2^j x - k)\}_{k=-\infty}^{\infty} \text{ form orthonormal basis for } W_j.$$

Define $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$.

Recall every function $f \in L^2$ can be written

$$f = \sum_j w_j$$

where $w_j \in W_j$. But each $w_j$ can be written

$$w_j = \sum_k a_k \psi_{jk}(x)$$

[note $j$ fixed above replace $a_k$ by $a_{jk}$ since need to keep track of $j$].

so:

$$f = \sum_j \sum_k a_{jk} \psi_{jk}(x).$$

Furthermore we have shown the $\psi_{jk}$ orthonormal. Conclude they form orthonormal basis for $L^2$.

## 7. Example of a wavelet expansion:

Let $f(x) = \begin{cases} x^2 & \text{if } 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$. Find wavelet expansion.

## 8. Some more Fourier analysis:

Recall Fourier transform (use $\omega$ instead of $\xi$ for Fourier variable):

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}(\omega) e^{ix\omega} \, d\omega$$

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \, f(x) \; e^{-ix\omega}.$$

[earlier had $\widehat{f}(\omega) = c(\omega)$ ]

Write $\widehat{f}(\omega) =$ Fourier transform of $\widehat{f}(\omega) = \mathcal{F}(f)$.

## 9. Plancherel theorem:

**Plancharel Theorem:**

*(i) The Fourier transform is a one to one correspondence from $L^2$ to itself.*

*That is, for every function $f(x) \in L^2$ there is a unique $L^2$ function which is its Fourier transform, and for every function $\widehat{g}(\omega) \in L^2$ there is a unique $L^2$ function which it is the Fourier transform of.*

*(ii) The Fourier transform preserves inner products, i.e., if $\widehat{f}$ is the FT of $f$ and $\widehat{g}$ is the FT of $g$, then $\langle \widehat{f}(\omega), \widehat{g}(\omega)\rangle = \langle f(x), g(x)\rangle$.*

*(iii) Thus*

$$\|f(x)\|^2 = \|\widehat{f}(\omega)\|^2 \,.$$

Now for a function $f \in L^2[-\pi, \pi]$, consider the Fourier series of $f$, given by

$$\sum_{k=-\infty}^{\infty} c_k \, e^{ikx}.$$

The above theorem has analog on $[-\pi, \pi]$. Theorem below follows immediately from fact that $\{e^{inx}/\sqrt{2\pi}\}_{n=-\infty}^{\infty}$ form orthonormal basis for $L^2[-\pi, \pi]$.

**Plancharel Theorem for Fourier series:**

*(i) The correspondence between functions* $f \in L^2[-\pi, \pi]$ *and the coefficients* $\{c_k\}$ *of their Fourier series is a one to one correspondence, if we restrict* $\sum_k c_k^2 < \infty$*. That is, for every* $f \in L^2[-\pi, \pi]$ *there is a unique series of square summable Fourier coefficients* $\{c_k\}$ *of* $f$ *such that* $\sum_k |c_k|^2 < \infty$*. Conversely for every square summable sequence* $\{c_k\}$ *there is a unique function* $f \in L^2[-\pi, \pi]$ *such that* $\{c_k\}$ *are the coefficients of the Fourier series of* $f$*.*

*(ii) Furthermore,* $\sum_k \|c_k\|^2 = \frac{1}{2\pi}\|f(x)\|^2$