13 GENERAL MULTIRESOLUTION ANALYSIS

1. Other constructions:

Suppose we use another “pixel” function $\phi(x)$:

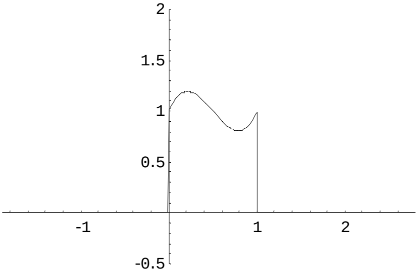


fig 31: another pixel function

Can we use this to build approximations to other functions? Consider linear combination:

$$2\phi(x) + 3\phi(x-1) - 2\phi(x-2) + \phi(x-3)$$

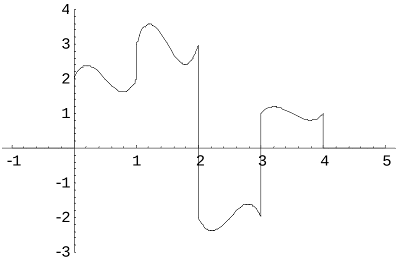


fig 32: graph of linear combination of translates of ϕ

Note we can try to approximate functions with other pixel functions.

Question: Can we repeat the above process with this pixel (scaling) function? What would be the corresponding wavelet?

Assumptions: $|\phi(x)|$ has finite integral and $\int \phi(x)dx \neq 0$.

More general construction:

As before define $V_0 =$ all L^2 linear combinations of ϕ and its translates:

$$= \{f(x) = \sum_k a_k \, \phi_{0k}(x) \Big| a_k \in \mathbb{R};\ f \in L^2\}. \qquad (2)$$

with

$$\phi_{0k}(x) = \phi(x-k).$$

and

$$V_1 = \{f(x) = \sum_k a_k\, \phi_{1k}(x) \,|\, a_k \in \mathbb{R};\ f \in L^2\}. \quad (3)$$

$$\phi_{1k}(x) = 2^{1/2}\, \phi(2x - k)$$

etc.

We want the same theory as earlier.

[Note V_0 no longer piecewise constant functions]

Recall condition

(d) $f(x) \in V_n \Rightarrow f(2x) \in V_{n+1}$

This is automatically true by definition of V_n, since if $f(x) \in V_0$, then f has the form of an element of (2). Then $f(2x)$ has form of an element of (3), and $f(2x) \in V_1$.

Similarly can be shown that (d) holds for any pair of spaces V_n and V_{n+1} of above form.

2. Some basic properties of F.T.:

Assume that $\widehat{f} = \mathcal{F}(f)$. Then

(a) $\mathcal{F}(f(x-c))(\omega) = e^{-i\omega c}\,\widehat{f}(\omega)$
(b) $\mathcal{F}(f(cx)) = \frac{1}{c}\widehat{f}(\omega/c)$

Proofs: Exercises.

3. Orthogonality of the ϕ's:

Another property of V_j :

(f) The basis $\{\phi(x-k)\}$ for V_0 is orthogonal, i.e. $\langle\phi(x-k),\phi(x-\ell)\rangle = 0$ for $k \neq \ell$.

Not automatic. Let $\mathcal{F}(f) \equiv$ F.T. of $f \equiv \widehat{f}(\omega)$.

Require a condition on ϕ of the following sort: if $k \neq \ell$, then (note use ω as Fourier variable) :

$$0 = \langle \phi(x-k), \phi(x-\ell) \rangle = \langle \mathcal{F}(\phi(x-k)), \mathcal{F}(\phi(x-\ell)) \rangle$$

$$\begin{aligned} &= \langle e^{-i\omega k}\,\widehat{\phi}\,(\omega), e^{-i\omega\ell}\,\widehat{\phi}(\omega) \rangle \\ &= \int_{-\infty}^{\infty} e^{i\omega(k-\ell)} |\widehat{\phi}(\omega)|^2 \, d\omega \end{aligned}$$

Thus conclude if $m \neq 0$,

$$0 = \int_{-\infty}^{\infty} e^{im\omega} |\widehat{\phi}(\omega)|^2 \, d\omega$$

$$= \left(\dots \int_{-4\pi}^{-2\pi} + \int_{-2\pi}^{0\pi} + \int_{0\pi}^{2\pi} + \right) e^{im\omega} |\widehat{\phi}(\omega)|^2 \, d\omega$$

$$= \sum_{n=-\infty}^{\infty} \int_{n \cdot 2\pi}^{(n+1) \cdot 2\pi} e^{im\omega} |\widehat{\phi}(\omega)|^2 \, d\omega$$

$$= \sum_{n=-\infty}^{\infty} \int_0^{2\pi} e^{im\omega} |\widehat{\phi}(\omega - 2n\pi)|^2 \, d\omega$$

$$= \int_0^{2\pi} e^{im\omega} \sum_{n=-\infty}^{\infty} |\widehat{\phi}(\omega - 2n\pi)|^2 \, d\omega$$

[since we can show that the integral of the absolute sum converges because $\sum_{n=-\infty}^{\infty} |\widehat{\phi}(\omega - 2n\pi)|^2 \, d\omega$ absolutely integrable; see exercises]

Conclude function $\sum_{n=-\infty}^{\infty} |\widehat{\phi}(\omega - 2n\pi)|^2$ on $[0, 2\pi]$ is in L^2 because it has square summable Fourier coefficients (in fact they are 0 if $m \neq 0$).

Further $\sum_{n=-\infty}^{\infty} |\widehat{\phi}(\omega - 2n\,\pi\,)|^2$ is 2π- periodic in ω, and has a Fourier series

$$\sum_{n=-\infty}^{\infty} |\widehat{\phi}(\omega - 2n\,\pi\,)|^2 = \sum_{m=-\infty}^{\infty} c_m\, e^{im\omega},$$

where

$$c_m = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\omega}\sum_{n=-\infty}^{\infty}|\widehat{\phi}(\omega-2n\pi)|^2\,d\omega = 0 \qquad \text{if } m \neq 0$$

And

$$c_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-im\omega}|\widehat{\phi}(\omega)|^2\,d\omega\,\Big|_{m=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty}|\widehat{\phi}(\omega)|^2\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}|\phi(x)|^2\,dx = \frac{1}{2\pi}.$$

Thus

$$\sum_{n=-\infty}^{\infty} |\widehat{\phi}(\omega - 2n\pi)|^2 = \sum_{m=-\infty}^{\infty} c_m e^{imx} = \frac{1}{2\pi}.$$

This condition equivalent to orthonormality of $\{\phi(x-k)\}$.

$V_0 \subset V_1$:

Recall the condition

(a) $V_0 \subset V_1$

What must be true of ϕ for this to hold in general? This says that every function in V_0 is in V_1. Thus since $\phi(x) \in V_0$, it follows $\phi(x) \in V_1$, i.e.

$\phi(x) =$ linear combination of translates of $\sqrt{2}\,\phi(2x)$

$$= \sum_k h_k \, \phi_{1k}(x) \qquad (4)$$

$$\phi_{1k}(x) = 2^{1/2}\phi(2x - k)$$

[recall normalization constant $\sqrt{2}$ is so we have unit L^2 norm].

Ex: If $\phi(x) =$ Haar wavelet, then

$$\phi(x) = \phi(2x) + \phi(2x - 1)$$

$$= \frac{1}{\sqrt{2}}\,\phi_{10}(x) + \frac{1}{\sqrt{2}}\,\phi_{11}(x)$$

$$= h_{10}\phi_{10}(x) + h_{11}\phi_{11}(x)$$

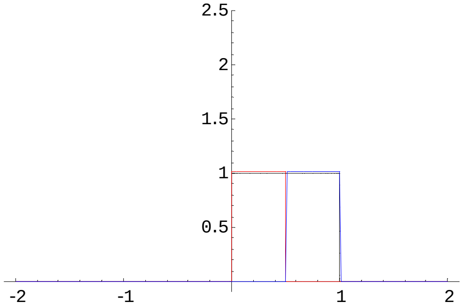


fig 33: $\phi(x) = \phi(2x) + \phi(2x-1)$

Thus in this case all h's are 0 except h_{10} and h_{11};

$$h_{10} = \frac{1}{\sqrt{2}}; \qquad h_{11} = \frac{1}{\sqrt{2}}.$$

Note in general that since this is an orthonormal expansion,

$$\sum_k h_k^2 = \|\phi(x)\|^2 < \infty.$$

4. What must be true of the scaling function for (4) above to hold?

Thus in general we have:

$$\phi(x) = \sum_{k=-\infty}^{\infty} h_k \phi_{1k}(x) = \lim_{N\to\infty} \sum_{k=-N}^{N} h_k \phi_{1k}(x) \quad (3)$$

in L^2 norm. Denote

$$\sum_{k=-N}^{N} h_k \phi_{1k}(x) \equiv F_N(x)$$

Specifically,

$$\|\phi(x) - \sum_{k=-N}^{N} h_k \phi_{1k}(x)\| \to 0.$$

[recall $\mathcal{F}$ is Fourier transform]

Corollary of Plancherel Theorem:

Corollary: *The Fourier transform is a bounded linear transformation. In particular, if the sequence of functions* $\{F_N(x)\}$ *converges in* L^2 *norm, then*

$$\mathcal{F}(\lim_{n\to\infty} F_N)(\omega) = \lim_{N\to\infty} \mathcal{F}(F_N)(\omega)$$

in L^2 *norm, i.e., Fourier transforms commute with limits.*

Thus since ∞ sums are limits and $\mathcal{F}$ is linear:

$$\mathcal{F}\left(\sum_{K=-\infty}^{\infty} h_k \phi_{1k}(x) \right) = \sum_{k=-\infty}^{\infty} h_k \ \mathcal{F}(\phi_{1k}(\omega))$$

[i.e., $\mathcal{F}$ commutes with ∞ sums]

Let $\mathcal{F}(\phi)(\omega) = \widehat{\phi}(\omega)$. Then generally:

$$\mathcal{F}(\phi_{jk})(\omega) = \mathcal{F}(2^{j/2}\phi(2^j x - k))(\omega)$$

$$= 2^{j/2}\,\mathcal{F}(\phi(2^j x - k))(\omega)$$

[recall dilation properties of Fourier transform earlier]

$$= 2^{j/2}\, \frac{1}{2^j}\, \mathcal{F}(\phi(x-k))(\omega/2^j)$$

[recall translation by k pulls out an $e^{-i\omega k}$]

$$= 2^{-j/2}\, e^{-i\omega k/2^j}\, \mathcal{F}(\phi(x))(\omega/2^j)$$

$$= 2^{-j/2}\, e^{-i\omega k/2^j}\, \widehat{\phi}(\omega/2^j)$$

Specifically for $j = 1$:

$$\mathcal{F}(\phi_{1k})(\omega) = \sqrt{2}\, e^{-i\omega k/2}\, \frac{1}{2}\, \widehat{\phi}(\omega/2)$$

Recall (3):

$$\phi(x) = \sum_{k=-\infty}^{\infty} h_k \phi_{1k}(x)$$

Fourier transforming both sides:

$$
\begin{aligned}
\widehat{\phi}(\omega) &= \mathcal{F}(\phi)(x) \\
&= \mathcal{F}\left(\sum_{k=-\infty}^{\infty} h_k \, \phi_{1k}(x) \right) \\
&= \sum_{k=-\infty}^{\infty} h_k \, \frac{1}{\sqrt{2}} \, e^{-ik(\omega/2)} \, \widehat{\phi}(\omega/2)
\end{aligned} \tag{5}
$$

Define

$$m(\omega/2) = \sum_{k=-\infty}^{\infty} h_k \frac{1}{\sqrt{2}} e^{-ik(\omega/2)} \qquad (6)$$

note m is 2π- periodic – Fourier series of $m(\omega/2)$ given above.

Note $m(\omega) \in L^2[0, 2\pi]$, since $\sum_k h_k^2 < \infty$.

Thus by (5):

$$\widehat{\phi}(\omega) = m(\omega/2)\,\widehat{\phi}(\omega/2).$$

with $m(\,\cdot\,)$ a 2π-periodic L^2 function.

[Note: This condition exactly summarizes our original demand that $V_0 \subset V_1$!]

Note if $V_0 \subset V_1$, then it follows (same arguments) that $V_1 \subset V_2$, and $V_j \subset V_{j+1}$ in general.

5. Some preliminaries:

Given a Hilbert space H and a closed subspace V, for $f \in H$ write

$$f = v + v^{\perp}$$

where $v \in V$ and $v^{\perp} \in V^{\perp}$.

Definition: The operator P defined by

$$Pf = P(v + v^{\perp}) = v$$

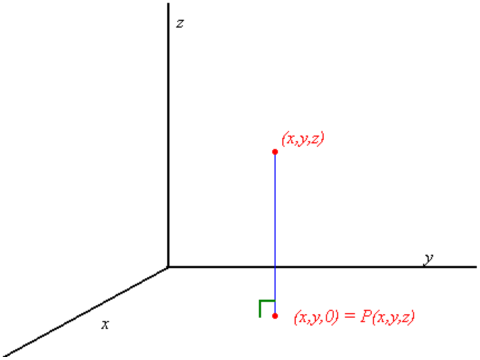
is the *orthogonal projection* onto V.

Note P is a bounded linear operator (see exercises).

Easy to check that $\|P\| = 1$ if $P \neq 0$ (see exercises).

Ex: $V = \mathbb{R}^3$. $P(x, y, z) = (x, y, 0) =$ is the orthogonal projection onto the $x-y$ plane.

$P(x, y, z) = (0, 0, z) =$ orthogonal projection onto z axis.



Ex: $V \subset L^2[-\pi, \pi]$ is the even functions. Then for $f \in L^2$

$$Pf(x) = f_{\text{even}}(x) = \frac{f(x) + f(-x)}{2}$$

(see exercises).

6. How to construct the wavelet?

Recall we have now given conditions on the scaling function:

Condition

(a) $$\dots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset V_3 \dots$$

is equivalent to:

(i) $$\widehat{\phi}(\omega) = m_0(\omega/2)\widehat{\phi}(\omega/2),$$

where m_0 is a function of period 2π.

Condition

(f) There is an orthogonal basis for the space V_0 in the family of functions

$$\phi_{0k} \equiv \phi(x-k)$$

is equivalent to:

(ii)
$$\sum_k |\widehat{\phi}(\omega + 2\pi k)|^2 = \frac{1}{2\pi}$$

Condition

(b) $$\bigcap_n V_n = \{0\}$$

can also be shown to follow from (ii) as follows:

Proposition: If $\phi \in L^2(\mathbb{R})$ and satisfies (ii), then $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

Proof: Denote C_c to be compactly supported continuous functions. Let $f \in \bigcap_{j\in\mathbb{Z}} V_j$. Let $\epsilon > 0$ be arbitrarily small.

By arguments as in problem II.2 in R&S, C_c is dense in $L^2(\mathbb{R})$, so that there exists an $\tilde{f} \in C_c$ with

$$\|f - \tilde{f}\| < \epsilon,$$

with $\| \cdot \|$ denoting L^2 norm. Let

$$P_j = \text{orthogonal projection onto } V_j.$$

Then since $f \in V_j$:

$$\|f - P_j\tilde{f}\| = \|P_j f - P_j\tilde{f}\| = \|P_j(f - \tilde{f})\| \leq \|f - \tilde{f}\| \leq \epsilon.$$

Thus by triangle inequality

$$\|f\| \leq \|f - P_j\tilde{f}\| + \|P_j\tilde{f}\| \leq \epsilon + \|P_j\tilde{f}\|. \qquad (7)$$

Since $P_j\tilde{f} \in V_j,$ we have

$$P_j\tilde{f} = \sum_k c_{jk}\phi_{jk}(x).$$

where $c_{jk} = \langle \phi_{jk}, f \rangle$ (recall $\{\phi_{jk}(x)\}_{k=-\infty}^{\infty}$ is an orthonormal basis for V_j).

Thus if $\|f\|_\infty = \sup_x |f(x)|$,

$$\|P_j \tilde{f}\|^2 = \sum_k |c_{jk}|^2 = \sum_k |\langle \phi_{jk}, \tilde{f} \rangle|^2$$

$$= \sum_k \left| \int \overline{\phi_{jk}(x)}\, \tilde{f}(x)\, dx \right|^2$$

[assuming $\tilde{f}$ is supported in $[-R, R]$]

$$\leq 2^j \|\tilde{f}\|_\infty^2 \sum_k \left(\int_{[-R,R]} 1 \cdot |\phi(2^j x - k)| dx \right)^2$$

[using Schwartz inequality $\langle a(x) b(x) \rangle \leq \|a(x)\| \|b(x)\|$]

$$\leq 2^j \|\tilde{f}\|_\infty^2 \sum_k \int_{[-R,R]} 1^2 \, dx \int_{[-R,R]} |\phi(2^j x - k)|^2 dx$$

$$= 2^j \|\tilde{f}\|_\infty^2 2R \sum_k \int_{[-R,R]} |\phi(2^j x - k)|^2 dx$$

$$\overset{y=2^j x-k}{=} \|\tilde{f}\|_\infty^2 \, 2R \int_{S_{R,j}} |\phi(y)|^2 dy$$

[where $S_{R,j} = \cup_{k\in\mathbb{Z}}[k - 2^j R, \, k + 2^j R]$ (note we replaced $k \to -k$ in the union) assuming j large and negative, so $2^{-j}R < \frac{1}{2}$. Note that then the k sum becomes a sum over disjoint intervals after the change of variables above, and we therefore replace a sum over k by a union over these intervals, as above]

$$= \|f\|_\infty^2 \, 2R \int \chi_{S_{R,j}}(y)|\phi(y)|^2 dy \underset{j \to -\infty}{\longrightarrow} 0$$

by the dominated convergence theorem, since if $y \notin \mathbb{Z}$, $\chi_{S_{R,j}}(y) \underset{j \to \infty}{\longrightarrow} 0$.

Thus by (7), we have for j large and negative and all $\epsilon > 0$:

$$\|f\| \leq \|f - P_j\tilde{f}\| + \|P_j\tilde{f}\| \leq \epsilon + \|P_j\tilde{f}\| \leq 2\epsilon.$$

Thus $\|f\| = 0$ and $f = 0$. □

Condition

(c) $\underset{n}{\cup} V_n$ is dense in $L^2(\mathbb{R})$

also follows from (ii):

Proposition: If $\phi \in L^2(\mathbb{R})$ and satisfies (ii), then $\overline{\underset{j\in\mathbb{Z}}{\cup} V_j} = L^2(\mathbb{R})$.

Proof: Similarly technical proof.

Condition

(d) $$f(x) \in V_n \quad \Rightarrow \quad f(2x) \in V_{n+1}$$

is automatic from the definition of the V_n.

Condition

(e) $f(x) \in V_0 \quad \Rightarrow \quad f(x-k) \in V_0$

is also automatic from definition.

Thus we conclude:

Theorem: Conditions (i) and (ii) above are necessary and sufficient for the spaces $\{V_j\}$ and scaling function ϕ to form a multiresolution analysis.

Thus if (i), (ii) are satisfied for ϕ and we define the spaces V_j as usual, the spaces will satisfy properties (a) - (f) of a multiresolution analysis.

Recall: orthonormality of translates $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ is equivalent to:

(ii) $$\sum_k |\widehat{\phi}(\omega + 2\pi k)|^2 = \frac{1}{2\pi}$$

Rewrite (ii):

$$\sum_k |m_0(\omega/2 + \pi k)|^2 \, |\widehat{\phi}(\omega/2 + \pi k)|^2 = \tfrac{1}{2\pi}$$

$$\Rightarrow \ \tfrac{1}{2\pi} = \ \sum_k |m_0(\omega' + \pi k)|^2 \, |\widehat{\phi}(\omega' + \pi k)|^2$$

$$[\omega' = \omega/2]$$

$$= \sum_{k \text{ even}} |m_0(\omega' + \pi k)|^2 |\widehat{\phi}(\omega' + \pi k)|^2$$

$$+ \sum_{k \text{ odd}} |m_0(\omega' + \pi k)|^2 |\widehat{\phi}(\omega' + \pi k)|^2$$

$$= \sum_k |m_0(\omega' + \pi \cdot 2k)|^2 \, |\widehat{\phi}(\omega' + \pi \cdot 2k)|^2$$

$$+ \sum_k |m_0(\omega' + \pi(2k+1))|^2$$

$$|\widehat{\phi}(\omega' + \pi(2k+1)|^2$$

$$\overset{m_0 \text{ periodic}}{=} |m_0(\omega')|^2 \sum_k |\widehat{\phi}(\omega' + 2\pi k)|^2 + |m_0(\omega' + \pi)|^2 \sum_k |\widehat{\phi}(\omega' + \pi +$$

$$\overset{\text{by (ii)}}{=} |m_0(\omega')|^2 \cdot \tfrac{1}{2\pi} + |m_0(\omega' + \pi)|^2 \cdot \tfrac{1}{2\pi}.$$

This implies that

$$|m_0(\omega')|^2 + |m_0(\omega' + \pi)|^2 = 1. \tag{8}$$

What about wavelets? Recall we define $W_j = V_{j+1} \ominus V_j$. We now know that $\{\phi_{jk}(x)\}$ form basis for V_j. The wavelets ψ_{jk} will form basis for W_j.