14. GENERAL WAVELETS

1. What are ψ_{jk}?

[Recall norms and inner products of functions are preserved when we take Fourier transform. Let's take FT to see.]

Note if we find $W_0 = V_1 \ominus V_0$, then we will be done.

[Let's look at Fourier transforms of functions in these spaces:]

Note that if $f \in V_0$, then

$$f(x) = \sum_k a_k \phi(x-k) = \sum_k a_k \, \phi_{0k}(x) \tag{9}$$

gives by F.T.:

$$\widehat{f}(\omega) = \sum_k a_k \, \mathcal{F}\big(\phi_{0k}(x)\big) = \sum_k a_k e^{-ik\omega} \widehat{\phi}(\omega) \equiv m_f(\omega)\widehat{\phi}(\omega) \tag{10}$$

where

$$m_f(\omega) \equiv \sum_k a_k e^{-ik\omega}.$$

is a 2π periodic $L^2[0, 2\pi]$ function which depends on f. In fact reversing argument shows (9) and (10) are equivalent.

Similarly can show under Fourier transform that $g \in V_1$ equivalent to:

$$\widehat{g}(\omega) = m_g(\omega/2)\,\widehat{\phi}(\omega/2). \tag{11}$$

with $m_g(\,\cdot\,)$ some other 2π periodic function on $L^2[0, 2\pi]$.

Notice functions m_f and m_g both have period 2π (look at their Fourier series). Also note above steps are reversible, so equation (10) implies (9) by reverse argument.

Thus:

$$f \in V_1 \Leftrightarrow \widehat{f} = m_f(\omega/2)\,\widehat{\phi}(\omega/2)$$

Recall: we want to characterize $f \in W_0$; such an f has the property that $f \in V_1$ and $f \perp V_0$.

Now note:

$$f \perp V_0 \iff f \perp \phi_{0k} \ \forall k \iff \widehat{f} \perp \widehat{\phi}_{0k},$$

$$\iff \int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{i\omega k}\, \overline{\widehat{\phi}(\omega)}\, d\omega = 0$$

$$\iff 0 = \int_{-\infty}^{\infty} \widehat{f}(\omega)\, e^{i\omega k}\, \overline{\widehat{\phi}(\omega)}\, d\omega = \sum_m \int_{2\pi m}^{2\pi(m+1)} \widehat{f}(\omega)\, e^{i\omega k}\, \overline{\widehat{\phi}(\omega)}$$

$$= \sum_m \int_0^{2\pi} \widehat{f}(\omega + 2\pi m) e^{ik(\omega+2\pi m)} \overline{\widehat{\phi}(\omega + 2\pi m)} \, d\omega$$

$$= \int_0^{2\pi} e^{ik\omega} \sum_m \widehat{f}(\omega + 2\pi m) \, \overline{\widehat{\phi}(\omega + 2\pi m)} \, d\omega \, .$$

where above identities hold for all k.

Hence [viewing sum as some function of ω]

$$\sum_m \widehat{f}(\omega + 2\pi m) \, \overline{\widehat{\phi}(\omega + 2\pi m)} = 0.$$

Thus:

$$0 = \sum_m \widehat{f}(\omega + 2\pi m)\,\overline{\widehat{\phi}(\omega + 2\pi m)}$$

$$= \sum_m m_f((\omega + 2\pi m)/2)\widehat{\phi}((\omega + 2\pi m)/2)$$

$$\times\, \overline{m_0((\omega + 2\pi m)/2)\widehat{\phi}((\omega + 2\pi m)/2)}$$

$$= \sum_m m_f(\omega/2 + \pi m)\widehat{\phi}(\omega/2 + m)$$

$$\times\, \overline{m_0(\omega/2 + \pi m)\widehat{\phi}(\omega/2 + \pi m)}$$

$$= \sum_{m \text{ even}} + \sum_{m \text{ odd}} m_f(\omega/2 + \pi m)\,\widehat{\phi}(\omega/2 + \pi m)$$

$$\times \; \overline{m_0(\omega/2 + \pi m)\widehat{\phi}(\omega/2 + \pi m)}$$

$$= \sum_m \; m_f(\omega/2 + 2\pi m)\widehat{\phi}(\omega/2 + 2\pi m)$$

$$\times \overline{m_0(\omega/2 + 2\pi m)\widehat{\phi}(\omega/2 + 2\pi m)}$$

$$+ \sum_m m_f(\omega/2 + \pi + 2\pi m)\widehat{\phi}(\omega/2 + \pi + 2\pi m)$$

$$\times \overline{m_0(\omega/2 + \pi + 2\pi m)\widehat{\phi}(\omega/2 + \pi + 2\pi m)}$$

$$= m_f(\omega/2)\overline{m_0(\omega/2)} \sum_m \widehat{\phi}(\omega/2 + 2\pi m)\, \overline{\widehat{\phi}(\omega/2 + 2\pi m)}$$

$$+ m_f(\omega/2 + \pi)\overline{m_0(\omega/2 + \pi)}$$

$$\times \sum_m \widehat{\phi}(\omega/2+\pi+2\pi m)\overline{\widehat{\phi}(\omega/2+\pi+2\pi m)}$$

$$= m_f(\omega/2)\overline{m_0(\omega/2)}\sum_m |\widehat{\phi}(\omega/2+2\pi m)|^2$$

$$+ m_f(\omega/2+\pi)\overline{m_0(\omega/2+\pi)}\sum_m |\widehat{\phi}(\omega/2+\pi+2\pi m)|^2$$

$$= (m_f(\omega/2)\overline{m_0(\omega/2)} \cdot \tfrac{1}{2\pi} + m_f(\omega/2+\pi)\overline{m_0(\omega/2+\pi)}) \cdot \tfrac{1}{2\pi}$$

(3) $\Rightarrow m_f(\omega')\overline{m_0(\omega')} + m_f(\omega'+\pi)\overline{m_0(\omega'+\pi)} = 0$

Thus (note $m_0(\omega')$ and $m_0(\omega'+\pi)$ cannot vanish together); let $\omega' \to \omega$:

$$m_f(\omega) = -\frac{m_f(\omega+\pi)}{\overline{m_0(\omega)}}\,\overline{m_0(\omega+\pi)} \equiv \lambda(\omega)\,\overline{m_0(\omega+\pi)}\,, \quad (12)$$

where

$$\lambda(\omega) \equiv -\frac{m_f(\omega+\pi)}{\overline{m_0(\omega)}}$$

and so $\lambda(\omega)$ is 2π periodic. Also,

$$\lambda(\omega) + \lambda(\omega+\pi) = -\frac{m_f(\omega+\pi)}{\overline{m_0(\omega)}} - \frac{m_f(\omega+2\pi)}{\overline{m_0(\omega+\pi)}} \quad (13)$$

combining fractions and using (3)

$$= 0.$$

Define $\nu(2\omega) \; = \; \lambda(\omega)\, e^{-i\omega}$.

Then

$$\nu(2\omega + 2\pi) = \ \lambda(\omega + \pi)\, e^{-i(\omega+\pi)}$$

$$= -\lambda(\omega)e^{-i\omega}e^{-i\pi} = \lambda(\omega)e^{-i\omega} = \nu(2\omega)$$

so ν has period 2π.

Thus

$$\widehat{f}(\omega) = m_f(\omega/2)\widehat{\phi}(\omega/2) = \lambda(\omega/2)\overline{m_0(\omega/2 + \pi)}\,\widehat{\phi}(\omega/2)$$

$$= \nu(\omega)\, e^{i\omega/2}\, \overline{m_0(\omega/2 + \pi)}\,\widehat{\phi}(\omega/2).$$

Thus we define the wavelet $\psi(x)$ by its Fourier transform:

$$\widehat{\psi}(\omega) = e^{i\omega/2}\,\overline{m_0(\omega/2+\pi)}\,\widehat{\phi}(\omega/2) \tag{14}$$

Thus

$$\widehat{f}(\omega) = \nu(\omega)\widehat{\psi}(\omega).$$

Going back in Fourier transform, we would get (compare with how we got $\widehat{f}(\omega) = m_f(\omega/2)\widehat{\phi}(\omega/2)$)

$$f(x) = \sum_k a_k \psi(x-k). \qquad (15)$$

where a_k are coefficients of the Fourier series of $\nu(\omega)$, i.e.,

$$\nu(\omega) = \sum_k a_k e^{ik\omega}.$$

To justify process of Fourier transformation as above, need to also show that the coefficients a_k are square summable (i.e. $\sum_k |a_k|^2 < \infty$), since we do not know whether Fourier transform properties which we have used in getting (15) are valid otherwise.

Note since a_k are coefficients of Fourier series of ν, we just need to show ν is square integrable on $[0, 2\pi]$ (recall this is equivalent to the a_k being square summable).

To show that ν is square integrable, note that with m_f as in (0):

$$\infty \overset{\text{use } m_f \in L^2[0,2\pi]}{>} \int_0^{2\pi} d\omega\, |m_f(\omega)|^2$$

$$\overset{\text{by (12)}}{=} \int_0^{2\pi} d\omega\, |\lambda(\omega)|^2\, |m_0(\omega+\pi)|^2$$

$$= \left(\int_0^{\pi} + \int_{\pi}^{2\pi}\right) d\omega\, |\lambda(\omega)|^2$$

$$|m_0(\omega+\pi)|^2$$

[substitute $\omega' = \omega - \pi$ in second integral; then rename $\omega' = \omega$ again]

$$= \int_0^{\pi} d\omega \, |\lambda(\omega)|^2 \, |m_0(\omega + \pi)|^2$$

$$+ \int_0^{\pi} d\omega \, |\lambda(\omega + \pi)|^2 \, |m_0(\omega + 2\pi)|^2$$

[recall that by periodicity $|m_0(\omega + 2\pi)|^2 = |m_0(\omega)|^2$ and use (13)]

$$= \int_0^{\pi} d\omega \, |\lambda(\omega)|^2 \, \left(|m_0(\omega + \pi)|^2 + |m_0(\omega)|^2\right)$$

$$\overset{\text{use (8)}}{=} \int_0^{\pi} d\omega \, |\lambda(\omega)|^2$$

$$= \int_0^{\pi} d\omega \, |\nu(2\omega)|^2$$

$$\overset{\omega' = 2\omega}{=} \frac{1}{2} \int_0^{2\pi} d\omega \, |\nu(\omega)|^2$$

Thus we have that $\infty > \int_0^{2\pi} d\omega \, |\nu(\omega)|^2$, so that ν is square integrable, as desired.

This was only thing left to show $\psi(x-k)$ span W_0. Wish to show also orthonormal. Use almost exactly the same argument as was used to show the same for $\phi(x-k)$:

$$\sum_k |\widehat{\psi}(\omega + 2\pi k)|^2 \overset{\text{use (14)}}{=} \sum_k |m_0(\omega/2 + \pi k + \pi)|^2 |\widehat{\phi}(\omega/2 + \pi k)|^2$$

[now break up the sum into even and odd k again and use the same method as before]

$$= \left(\sum_{k \text{ even}} + \sum_{k \text{ odd}} \right) |m_0(\omega/2 + \pi k + \pi)|^2 |\widehat{\phi}(\omega/2 + \pi k)|^2$$

$$
\begin{aligned}
&= \sum_k |m_0(\omega/2 + \pi \cdot 2k + \pi)|^2 |\widehat{\phi}(\omega/2 + \pi \cdot 2k)|^2 \\
&\quad + \sum_k |m_0(\omega/2 + \pi \cdot (2k+1) + \pi)|^2 \\
&\qquad \times |\widehat{\phi}(\omega/2 + \pi \cdot (2k+1))|^2 \qquad (16) \\
&= |m_0(\omega/2 + \pi)|^2 \sum_k |\widehat{\phi}(\omega/2 + \pi \cdot 2k)|^2 \\
&\quad + |m_0(\omega/2)|^2 \sum_k |\widehat{\phi}(\omega/2 + \pi \cdot (2k+1))|^2
\end{aligned}
$$

$$\overset{\text{using (ii) above again}}{=} \left(|m_0(\omega/2+\pi)|^2 + |m_0(\omega/2)|^2\right) \cdot \frac{1}{2\pi}$$

$$= \frac{1}{2\pi}$$

By same arguments as used for $\phi(x-k)$, it follows by (16) $\psi(x-k)$ orthonormal.

This proves our choice of ψ gives a basis for W_0 as desired.

Specifically,

$$\psi_{0k}(x) = \psi(x - k)$$

form an orthogonal basis for W_0 (in fact can show their length is 1 so they are orthonormal).

In same way as for ϕ, can show immediately that since functions in W_j are functions in W_0 stretched by factor 2^j, the functions

$$\psi_{jk}(x) = 2^{j/2}\,\psi(2^j x - k)$$

form a basis for W_j (j fixed, k varies).

Since L^2 $=$ direct sum of the W_j spaces, conclude functions $\{\psi_{jk}(x)\}_{j,k=-\infty}^{\infty}$ over all integers j and k form orthonormal basis for L^2.

Conclusion:

If we start with a pixel function $\phi(x)$, which satisfies

(i) $\widehat{\phi}(\omega) = m_0(\omega/2)\widehat{\phi}(\omega/2)$ (with m_0 some 2π-periodic function)

(ii) $\sum_k |\phi(\omega + 2\pi k|^2 = \frac{1}{2\pi}$

then the set of spaces V_j form a multiresolution analysis, i.e., satisfy properties (a) - (f) from earlier.

Further, if define function $\psi(x)$ with Fourier transform:

$$\widehat{\psi}(\omega) = e^{i\omega/2}\, \overline{m_0(\omega/2 + \pi)}\, \widehat{\phi}(\omega/2)$$

(17)

(here m_0 is from (i) above), then

$$\psi_{jk}(x) = 2^{j/2}\psi\left(2^j x - k\right).$$

form orthonormal basis for L^2

[Next we'll construct some wavelets]

2. Additional remarks:

Note further that (17) has another interpretation without Fourier transform :

Recall the two scale equation:

$$\phi(x) = \sum_k h_k \phi_{1k}(x).$$

Also then we have (see eq. (5)) that if

$$m_0(\omega) = \sum_k \frac{h_k}{\sqrt{2}} e^{-ik\omega},$$

then:

$$\widehat{\phi}(\omega) = m_0(\omega/2)\widehat{\phi}(\omega/2).$$

Then we have from (17):

$$
\begin{aligned}
\widehat{\psi}(\omega) &= e^{i\omega/2}\sum_{k=-\infty}^{\infty}\frac{1}{\sqrt{2}}\overline{h_k}e^{ik(\omega/2+\pi)}\,\widehat{\phi}(\omega/2) \\
&= e^{i\omega/2}\sum_{k=-\infty}^{\infty}\frac{1}{\sqrt{2}}\overline{h_k}e^{ik\pi}e^{ik\omega/2}\,\widehat{\phi}(\omega/2) \\
&= \sum_{k=-\infty}^{\infty}\frac{1}{\sqrt{2}}\overline{h_k}(-1)^k e^{i(k+1)\omega/2}\,\widehat{\phi}(\omega/2)
\end{aligned}
$$

Inverse Fourier transforming:

$$
\begin{aligned}
\psi(x) &= \mathcal{F}^{-1}(\widehat{\psi}(\omega)) \\
&= \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2}} \overline{h}_k (-1)^k \mathcal{F}^{-1}(e^{i(k+1)\omega/2}\ \widehat{\phi}(\omega/2)) \\
&= \sum_{k=-\infty}^{\infty} \frac{\overline{h}_k}{\sqrt{2}} (-1)^k 2\phi(2x + (k+1)) \\
&= \sum_{k=-\infty}^{\infty} \frac{\overline{h}_{k-1}}{\sqrt{2}} (-1)^{k-1} \sqrt{2}\sqrt{2}\phi(2x + k)) \\
&= \sum_{k=-\infty}^{\infty} \overline{h}_{-k-1} (-1)^{-k-1} \phi_{1k}(x) \\
&= \sum_{k=-\infty}^{\infty} g_k \phi_{1k}(x)
\end{aligned}
$$

where

$$g_k = \overline{h}_{-1-k}(-1)^{-k-1} = \overline{h}_{-1-k}(-1)^{k+1}$$

$$\stackrel{\text{standard form}}{=} \overline{h}_{-1-k}(-1)^{k-1},$$

and (recall) h_k defined by

$$\phi(x) = \sum_k h_k \phi_{1k}(x).$$

3. Some comments on the scaling function:

Recall

$$\widehat{\phi}(\omega) = m_0(\omega/2)\,\widehat{\phi}(\omega/2)$$

from earlier. This stated that the Fourier transform of ϕ and its stretched version are related by some function $m_0(\omega/2)$, where m_0 is a periodic function of period 2π.

Lemma: The Fourier transform of an integrable function is continuous.

Proof: exercise

Assumption: $\phi(x)$ (the scaling function) is integrable (i.e., its absolute value has a finite integral).

Fact: Under our assumptions, it can be shown that

$$\int_{-\infty}^{\infty} dx\, \phi(x) = 1$$

[proof is an exercise]

Consequence: A consequence of the above assumption is that the Fourier transform $\widehat{\phi}\,(\omega)$ satisfies:

$$\widehat{\phi}(0) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, \phi(x)\, e^{-i\cdot 0x}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, \phi(x) = \frac{1}{\sqrt{2\pi}}.$$

Now recall we had

$$\widehat{\phi}(\omega) = m_0(\omega/2)\widehat{\phi}(\omega/2) \qquad (18)$$

for some periodic function m_0.

Replacing ω by $\omega/2$ above:

$$\widehat{\phi}(\omega/2) = m_0(\omega/4)\widehat{\phi}(\omega/4);$$

Plugging into (18):

$$\widehat{\phi}(\omega) = m_0(\omega/2)m_0(\omega/4)\widehat{\phi}(\omega/4). \qquad (19)$$

Now taking (18) and replacing ω by $\omega/4$, and then plugging into (19):

$$\widehat{\phi}(\omega) = m_0(\omega/2)m_0(\omega/4)m_0(\omega/8)\widehat{\phi}(\omega/8).$$

Continuing this way n times, we get:

$$\widehat{\phi}(\omega) = m_0(\omega/2)m_0(\omega/4)m_0(\omega/8)\ldots m_0(\omega/2^n)\widehat{\phi}(\omega/2^n).$$

or:

$$\widehat{\phi}(\omega) = \left(\prod_{j=1}^{n} m_0(\omega/2^j) \right) \widehat{\phi}(\omega/2^n)$$

$\Rightarrow$

$$\frac{\widehat{\phi}(\omega)}{\widehat{\phi}(\omega/2^n)} = \prod_{j=1}^{n} m_0(\omega/2^j). \qquad (20)$$

Now let $n \to \infty$ on both sides of equation. Since $\widehat{\phi}$ is continuous (above assumption), we get

$$\widehat{\phi}(\omega/2^n) \xrightarrow[n \to \infty]{} \widehat{\phi}(0) = \frac{1}{\sqrt{2\pi}}.$$

Since the left side of (20) converges as $n \to \infty$, the right side also converges. After letting $n \to \infty$ on both sides of (20):

$$\frac{\widehat{\phi}(\omega)}{\widehat{\phi}(0)} = \prod_{j=1}^{\infty} m_0(\omega/2^j),$$

$\Rightarrow$

$$\widehat{\phi}(\omega) = \sqrt{2\pi}\prod_{j=1}^{\infty} m_0(\omega/2^j).$$

Conclusion: If we can find $m_0(\omega)$, we can find the scaling function ϕ.

4. Examples of wavelet constructions using this technique:

Haar wavelets: Recall that we chose the scaling function

$$\phi(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases},$$

and then we defined spaces V_j.

From ϕ we constructed the wavelet ψ whose translates and dilates form a basis for L^2.

Such constructions can be made automatic if we use above observations.

Note first in Haar case:

$$\widehat{\phi}(\omega) = \frac{1}{\sqrt{2\pi}}\int_0^1 e^{-i\omega x}\,dx = -\frac{1}{\sqrt{2\pi}}\frac{e^{-i\omega x}}{i\omega}\bigg|_0^1 = \frac{1}{\sqrt{2\pi}}\left[-\frac{e^{-i\omega}}{i\omega}+\frac{1}{i\omega}\right]$$

$$= -\frac{2}{\sqrt{2\pi}\,\omega}\,e^{-i\omega/2}\left(\frac{e^{-i\omega/2}}{2i}-\frac{e^{i\omega/2}}{2i}\right)$$

$$= \frac{2}{\sqrt{2\pi}\,\omega}\,e^{-i\omega/2}\sin\omega/2.$$

For Haar wavelets we can find $m_0(\omega)$ from:

$$\widehat{\phi}(\omega) = m_0(\omega/2)\widehat{\phi}(\omega/2),$$

so

$$m_0(\omega/2) = \frac{\widehat{\phi}(\omega)}{\widehat{\phi}(\omega/2)} = \frac{1}{2}\, e^{-i\omega/4}\, \frac{\sin\omega/2}{\sin\omega/4}$$

$$= \frac{1}{2}\, e^{-i\omega/4}\, \frac{\sin\,(2\cdot\omega/4)}{\sin\omega/4}$$

$$= \frac{1}{2} e^{-i\omega/4} \frac{2 \sin \omega/4 \cos \omega/4}{\sin \omega/4}$$

$$= \frac{1}{2} e^{-i\omega/4} 2 \cos \omega/4$$

$$= e^{-i\omega/4} \cos \omega/4.$$

Recall wavelet Fourier transform is:

(4) $$\widehat{\psi}(\omega) = e^{i\omega/2}\,\overline{m_0(\omega/2+\pi)}\,\widehat{\phi}(\omega/2)$$

In this case

$\widehat{\psi}(\omega) =$

$$e^{i\omega/2}\, e^{i(\omega/4+\pi/2)} \cos(\omega/4+\pi/2)\, \frac{4}{\sqrt{2\pi\,\omega}}\, e^{-i\omega/4} \sin\omega/4.$$

[using

$\cos(\omega/4 + \pi/2) = \cos\omega/4 \cos\pi/2 - \sin\omega/4 \sin\pi/2$
$= -\sin\omega/4$]

$$= -\frac{4i}{\sqrt{2\pi}\,\omega}\, e^{i\omega/2} \sin^2(\omega/4)$$

Can check (below) this indeed is Fourier transform of usual Haar wavelet ψ, except the complex conjugate (which means the original wavelet is reflected about 0, i.e., translated and negated, which still yields a basis for W_0).

To check this, recall Haar wavelet:

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1/2 \\ -1 & \text{if } 1/2 \le x < 1 \end{cases}$$

Thus:

$$\widehat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_0^{1/2} + \int_{1/2}^{1} \right) \psi(x)\, e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{1/2} e^{-i\omega x}\, dx - \frac{1}{\sqrt{2\pi}} \int_{1/2}^{1} e^{-i\omega x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(- \frac{e^{-i\omega/2}}{i\omega} + \frac{1}{i\omega} \right) - \frac{1}{\sqrt{2\pi}} \left(- \frac{e^{-i\omega}}{i\omega} + \frac{e^{-i\omega/2}}{i\omega} \right)$$

$$= - \frac{2e^{-i\omega/2}}{\sqrt{2\pi}\, i\omega} + \frac{e^{-i\omega} + 1}{\sqrt{2\pi}\, i\omega}$$

$$= \frac{2}{\sqrt{2\pi}\, i\omega}\left(- e^{-i\omega/2} + e^{-i\omega/2}\, \frac{(e^{-i\omega/2} + e^{i\omega/2})}{2} \right)$$

$$= \frac{2}{\sqrt{2\pi}\, i\omega}\left(- e^{-i\omega/2} + e^{-i\omega/2} \cos \omega/2 \right)$$

$$= \frac{2}{\sqrt{2\pi}\, i\omega}\left(- e^{-i\omega/2} + e^{-i\omega/2} \cos 2 \cdot \omega/4 \right)$$

[using $\cos 2x = 1 - 2\sin^2 x$]

$$= \frac{2}{\sqrt{2\pi}\, i\omega} \Big(- e^{-i\omega/2} + e^{-i\omega/2}(1 - 2\sin^2\omega/4) \Big)$$

$$= \frac{-4}{\sqrt{2\pi}\, i\omega} \Big(e^{-i\omega/2} \sin^2\omega/4 \Big)$$

$$= \frac{4i}{\sqrt{2\pi}\, \omega} \Big(e^{-i\omega/2} \sin^2\omega/4 \Big)$$