Suggestions - PS 4

1*. (RS II.1). Suggestion: $(x_n,y_n) - (x_m,y_m) = (x_n,y_n) - (x_n,y_m) + (x_n,y_m) - (x_m,y_m)$. Why is (x_n,y_n) Cauchy? To show two different sequences x'_n and y'_n converging to x and y will give the same limit for $\langle x'_n, y'_n \rangle$, form larger (still Cauchy) sequences $x_1, x'_1, x_2, x'_2, x_3, x'_3$, etc., which should give the same limits (why?). For (b) first show for fixed $x \in V$, since $y \to (x,y)$ is a BLT on y, it can be extended to all $y \in \tilde{V}$, and (x,y) can be defined for all $x \in V$ and $y \in \tilde{V}$. Does this have the right properties? How can you use the BLT theorem to extend the inner product to all $x \in \tilde{V}$?

2* (RS II.2). Recall a step function is a finite sum of characteristic functions of intervals see - p. 10. Some of this can be done similarly to problem 18 in Chapter 1.

4. (RS II.6). For the second statement the harder part is showing $(\mathcal{M}^{\perp})^{\perp} \subset \overline{\mathcal{M}}$. Show $(\overline{\mathcal{M}})^{\perp} = \mathcal{M}^{\perp}$. Thus it suffices to assume $\mathcal{M}$ is closed (take its closure). To show that $(\mathcal{M}^{\perp})^{\perp} \subset \mathcal{M}$, note for $x \in (\mathcal{M}^{\perp})^{\perp}$ we can uniquely write x = z + w, where $z \in \mathcal{M}$ and $w \in \mathcal{M}^{\perp}$. Show $z \in (\mathcal{M}^{\perp})^{\perp}$. What does this say about w by Theorem II.3? What do we conclude about x by the same theorem?

5. (RS II.7). To show uniqueness in the theorem assume $x = z + w = z' + w'$, with $z, z' \in \mathcal{M}$ and $w, w' \in \mathcal{M}^{\perp}$. Then $z - z' = w - w'$. To show uniqueness in the Lemma note $\{y_n\}$ is Cauchy and has a unique limit. What would happen if two z and z′ minimized the inequality? Note the lemma proof shows if

(1) $$\| x - y_n \| \xrightarrow[n\to\infty]{} d$$

then y_n are Cauchy. Suppose you had a sequence y_n made up of just z and z′, alternating. Then (1) would hold ($\|x - y_n\|$ would be identically equal to d) and the sequence would be Cauchy. What would this say about z and z′?

8*. (RS II.15). (a) To show $\sum_0^{\infty} |b_n|^2 < \infty$, try Bessel's inequality. Then use integration by parts to relate c_n to b_n, via

$$b_n = \int_0^{2\pi} dx\, f'(x)\, e^{inx}/\sqrt{2\pi}\ .$$

Why do boundary terms cancel (recall f is periodic)?

(b) How can you use the Schwarz inequality? (Try writing $\sum|c_n| = \sum|nc_n| \cdot 1/n$. Note since the sequences of numbers form a Hilbert space ℓ^2, by the Schwarz inequality

$$|\langle a,b \rangle| = \leq \sqrt{\sum|a_n|^2 \ \sum|b_n|^2}$$

for sequences a and b.)

(c) Try the Weierstrass M-test.

(d) You may assume the conclusion of problem 14, namely that if $S_N(x)$ is the partial sum of the Fourier series of f(x), then if f(x) continuous, $S_N(x) \underset{N\to\infty}{\longrightarrow} f(x)$ in L^2, i.e., $\int |S_N(x) - f(x)|^2 dx \underset{n\to\infty}{\longrightarrow} 0$. Thus we can assume $S_N(x)$ converge to f in L^2, and need to show they also converge uniformly. We know from (c) $S_N(x)$ is uniformly convergent to some function, call it g(x). Show it also follows $S_N(x)$ converges to g(x) in L^2 (use the definition), so g(x) = f(x) (why?).

10. (II.16) Can you show there are 4 disjoint translates of a ball of this radius in the 2 dimensional unit ball? 6 in 3 dimensions? Use the same argument to show there must be an infinite number in infinite dimension. Show if you center the n^{th} of the balls at the point with $x_n = 1/2$ and all other $x_i = 0$, they will be disjoint.