Exact Smoothing Properties of Schrödinger Semigroups *

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Abstract

We study Schrödinger semigroups in the scale of Sobolev spaces, and show that, for Kato class potentials, the range of such semigroups in L^p has exactly two more derivatives than the potential; this proves a conjecture of B. Simon. We show that eigenfunctions of Schrödinger operators are generically smoother by exactly two derviatives (in given Sobolev spaces) than their potentials. We give applications to the relation between the potential's smoothness and particle kinetic energy in the context of quantum mechanics, and characterize kinetic energies in Coulomb systems. The techniques of proof invove Leibniz and chain rules for fractional derivatives which are of independent interest, as well as a new characterization of the Kato class.

1 Introduction

In this paper we attempt a precise study of the action of Schrödinger semigroups in the scale of Sobolev spaces. Work in this area was begun by B. Simon [Si4] in 1985. He proved a number of positive and negative results on such smoothing, and also presented an open question (in the form of a conjecture) regarding necessary and sufficient conditions on the potential V for smoothing of order s by the semigroup of functions in $L^2(\mathbb{R}^d)$.

Study of Schrödinger semigroups is of interest for a number of reasons (see [Si3]). An important one is that they are a key to the study of eigenfunctions of the Schrödinger operator $H = H_0 + V = -\frac{1}{2}\Delta + V$, or, equivalently, the stationary states of the corresponding Schrödinger equation $\frac{\partial\psi}{\partial t} = -iH\psi$. Here V is a potential, and Δ is the Laplacian; the operator $H_0 = -\frac{1}{2}\Delta$ is used instead of $-\Delta$ because this normalization is useful in path integral representations of Schrödinger semigroups (see [Si3] for details). We show here that some basic properties of Sobolev spaces under multiplication and composition imply precise results on how such semigroups act in Sobolev spaces, and from this also determine smoothness of eigenfunctions.

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As motivation, consider the heat diffusion semigroup e^{-tH_0} on $\mathbb{R}^d$. It is well known that this semigroup, being infinitely smoothing, maps any L^p space into the intersection $\bigcap_{s\geq 0} L^p_s$ of all Sobolev spaces. If $p=2$, this is easily seen since e^{-tH_0} acts in Fourier space as multiplication by $e^{-\frac{1}{2}t|\xi|^2}$. The situation is more complicated when one adds a potential to the Laplacian (yielding the Schrödinger semigroup), though mapping properties of the resulting operator can be summarized in a largely simple way, as is shown below.

Simon [Si4] studied Schrödinger semigroups in the scale of Sobolev spaces L^p_s and conjectured that the semigroup

$$S(t) = e^{-t(-\frac{1}{2}\Delta+V)}$$

maps L^p into $L^p_{s+2,\mathrm{loc}}$ if and only if $V \in L^p_{s,\mathrm{loc}}$. We prove this conjecture in rather full generality, showing that it holds for any V in the standard Kato class K_d (see below). Our techniques are largely from harmonic analysis related to multiplicative and compositional properties of Sobolev spaces.

Previous study of mapping properties of Schrödinger semigroups has focused on their action in scales of L^p spaces. The study of hyper- and supercontractivity, initially dealing with $L^p \to L^q$ mapping properties (with $p \leq q$) of the semigroup generated by the harmonic oscillator Hamiltonian, originated from the requirements of rigorous quantum field theory ([Seg], [GJ], [Ne], [Si2]). It has since been studied in a number of interesting settings for more general operators (see, e.g., [Gr], [Da1], [KS]).

Smoothing by semigroups in the scale of L^p spaces on $\mathbb{R}^d$ was studied for Schrödinger semigroups by Davies [Da1] Herbst and Sloan [HS], Carmona [Ca], and Simon [Si4]. For more background on this topic see [Si3], Sec. B1. The first systematic study for the scale of Sobolev spaces is in [Si4].

Our main result gives necessary and sufficient conditions on the potential V for the semigroup e^{-tH} to map L^p spaces into local Sobolev spaces. Let $V^{\pm}$ denote the positive and negative parts of V. We prove:

THEOREM 1.1 *Let $H = -\frac{1}{2}\Delta + V$ be a Schrödinger operator having a potential $V \in K_{d,\mathrm{loc}}$, with negative part $V^- \in K_d$, and let $1 < p < \infty$, $s \geq 0$. Then the semigroup e^{-tH} maps L^p into $L^p_{s+2,\mathrm{loc}}$ if and only if $V \in L^p_{s,\mathrm{loc}}$.*

The semigroup results here alternatively provide necessary and sufficient conditions for the solution of the perturbed heat equation $\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t) - V(x)u(x,t)$ in $\mathbb{R}^d$ to be in the local Sobolev spaces L^p_s, in terms of the smoothness of the perturbation coefficient $V(x)$ (which physically corresponds to external heat loss).

The above necessary and sufficient results are appropriate to local spaces $L^p_{s,\mathrm{loc}}$. These convert only to sufficient conditions for the global spaces L^p_s. The global smoothing result is given by:

THEOREM 1.2 *Let $V \in K_d$. Then $V \in L^p_s$ with $1 < p < \infty$, $s \geq 0$ implies $e^{-tH}: L^p \to L^p_{s+2}$.*

The converse is false, as follows from the example $V \equiv 1$. In this case the semigroup is infinitely smoothing but the potential V is not in L^p_s.

We note that the Kato class is essentially the largest class of potentials for which these results hold. For potentials V outside this class, essentially anything can happen with regard to smoothing by the semigroup. In particular there exist highly singular positive potentials for which infinite smoothing by the semigroup occurs. See [Ko] for more details.

We exploit the above results on Sobolev mapping of the semigroup to deduce analogous properties for eigenfunctions of a Schrödinger operator H. Smoothing results for semigroups provide information about smoothness of eigenfunctions, since any such eigenfunction is also an eigenfunction of the semigroup, and hence belongs to its range.

The following results show that the smoothness of an eigenfunction ϕ of the Schrödinger operator $H = -\frac{1}{2}\Delta + V$ is uniquely tied to that of the potential V. We will need a good supply of eigenfunctions of H, so we introduce the following notion: we say that the L^p eigenfunctions of the operator H are *non-trivial* at the point x if there exists an eigenfunction ϕ such that $\phi(x) \neq 0$. More generally, the system of eigenfunctions is non-trivial at all x (or simply non-trivial) if it is non-trivial at every $x \in \mathbb{R}^d$. The value $\phi(x)$ is defined because all eigenfunctions are in the range of the semigroup, which consists only of continuous functions (see [Si3], Corollary B.3.2). A special case of non-triviality is provided when there is a ground state, which, when it exists, is the L^2-eigenfunction corresponding to the eigenvalue $\lambda = \inf \sigma(H)$, where $\sigma(H)$ is the L^2 spectrum of H. The ground state is always strictly positive (its non-negativity follows from equation (A7) in [Si4], and strict positivity follows by reasoning involving the Feynman-Kac formula, as in the necessity proof in Theorem 1.1 below). The other extreme occurs in the case of a free particle ($H = -\frac{1}{2}\Delta$), for which there are no L^p eigenfunctions for $p < \infty$.

We have

THEOREM 1.3 *Let $V \in K_{d,\mathrm{loc}}$, with $V^- \in K_d$, $1 < p < \infty$, and let the L^p-eigenfunctions $\{\phi_\lambda\}_{\lambda \in \Lambda}$ of the corresponding Schrödinger operator H be nontrivial at all points. Then for $s \geq 0$, the eigenfunctions satisfy $\phi_\lambda \in L^p_{s+2,\mathrm{loc}}$ for all λ if and only if $V \in L^p_{s,\mathrm{loc}}$.*

Theorem 1.3 can be strengthened (by inspection of the proofs) to a local version which relates smoothness of individual eigenfunctions to that of the potential:

THEOREM 1.3$'$: *Assume that* $V \in K_{d,\text{loc}}$ *and* $V^- \in K_d$*, and that there exists an eigenfunction* ϕ *which is nonvanishing in a domain* Ω*. Then* $\phi \in L^p_{s+2,\text{loc}}(\Omega)$ *if and only if* $V \in L^p_{s,\text{loc}}(\Omega)$*.*

It is clear from the proofs given later that the hypothesis of non-triviality is needed only for the forward implication above, and not the reverse.

To see that the nonvanishing assumption in Theorem 1.3 cannot be removed, it suffices to consider the hydrogen atom in three dimensions, with $H = -\frac{1}{2}\Delta - \frac{1}{|x|}$. In this case, eigenfunctions have the following form in spherical coordinates:

$$u_{nlm}(r,\theta,\phi) = e^{-r/n}(\frac{2}{n}r)^l P^{2l+1}_{n+l}(\frac{2}{n}r)Y_{lm}(\theta,\phi),$$

where Y_{lm} is a spherical harmonic, and $P^{2l+1}_{n+l}(r)$ is an associated Laguerre polynomial with leading order

$$P^{2l+1}_{n+l}(r) = -\frac{[(n+l)!]^2}{(n-l-1)!(2l+1)!} + O(r),$$

so that $P^{2l+1}_{n+l}(r)$ never vanishes at 0. Above $n \geq 1, \quad 0 \leq l \leq n-1, \quad -l \leq m \leq l$. It is easy to check by explicit differentiation that for any $s > 0$, the eigenfunction u_{nlm} is in L^2_s for l sufficiently large (note that since we let l become large, integer derivatives suffice to check this). Thus in this case eigenfunctions (which vanish at the origin) are arbitrarily smooth, whereas the potential $V = -1/r$ is in $L^2_{s,\text{loc}}$ only for $s < 1/2$ (here $r = |x|$ in $\mathbb{R}^3$). Therefore the nonvanishing assumption cannot be removed in Theorem 1.3$'$. On the other hand, the ground state has the form $u_{100}(x) = Ce^{-|x|}$. This function belongs to $L^2_{s,\text{loc}}$ only for $s < 5/2$, as required by the Theorem (see below).

The result on smoothness of eigenfunctions has a quantum mechanical interpretation, since smoothness of eigenfunctions gives direct information about the dispersion of momentum (and hence kinetic energy) in given eigenstates. Given an eigenfunction $\phi(x)$, its Fourier transform's modulus $|\hat{\phi}(\xi)|^2$ is the quantum mechanical probability distribution of momenta in ϕ, and ϕ is in a given L^2-Sobolev space L^2_s if and only if the momentum distribution $|\hat{\phi}(\xi)|^2$ has moments to order $2s$, i.e., if $\int_{\mathbb{R}^d} |\hat{\phi}(\xi)|^2(1+|\xi|^2)^s \, d\xi < \infty$. Thus our results state that the dispersion of kinetic energies is closely tied to the lack of smoothness of the potential, i.e., that high kinetic energies result from singular potentials, via lack of smoothness in eigenfunctions. That is, the Sobolev class of the potential determines this dispersion, measured in moments of the kinetic energy distribution.

It can be shown from these results that for a Coulomb system in $\mathbb{R}^3$ with

$$V(x_1,\ldots,x_n) = -\sum_{i>j} c_{ij}|x_i - x_j|^{-1},$$

a ground state if it exists is in $L^2_{s,\mathrm{loc}}$ if and only if $s < s/2$. Thus according to the above, the probability distribution of momenta in the ground state of any three dimensional Coulomb system (e.g., any atomic system) has finite moments of all orders less than 5, but not 5 or greater. The corresponding statement for kinetic energy distributions limits their moments to order strictly less than $\frac{5}{2}$ in any Coulomb system, since kinetic energy behaves like the square of momentum.

In order to prove the above results we develop a Leibniz-type rule of independent interest. Its new feature is that it allows for derivatives of negative order in the form of either Bessel or Riesz potentials. Our methods are essentially based on an approach originally developed by Coifman and Meyer [CM1],[CM2], who answered a question asked by Kato (information on their method is in an expository paper of Semmes [Sem]). The ideas of Coifman and Meyer were used and extended by Kato and Ponce [KP], Kenig, Ponce and Vega [KPV], Christ and Weinstein [CW], and Taylor [Ta]. In [KP], [Ta] negative derivatives are also used in special cases.

We denote by J^α, $-\infty < \alpha < \infty$, the Bessel potential, namely $J^\alpha = (I - \Delta)^{\alpha/2}$ and by D^α the Riesz potential $D^\alpha = (-\Delta)^{\alpha/2}$.

We have

THEOREM 1.4 (Generalized Leibniz rule). *Suppose* $1 < p < \infty$, $s \geq 0$, $\alpha \geq 0$, $\beta \geq 0$, *and* $p^{-1} = p_i^{-1} + q_i^{-1}$ *with* $i = 1, 2$, $1 < q_1 \leq \infty$, $1 < p_2 \leq \infty$. *Then*

(1) $\|J^s(fg)\|_p \leq c(\|J^{s+\alpha}(f)\|_{p_1}\|J^{-\alpha}(g)\|_{q_1} + \|J^{-\beta}(f)\|_{p_2}\|J^{s+\beta}(g)\|_{q_2})$,

(2) $\|D^s(fg)\|_p \leq c(\|D^{s+\alpha}(f)\|_{p_1}\|D^{-\alpha}(g)\|_{q_1} + \|D^{-\beta}(f)\|_{p_2}\|D^{s+\beta}(g)\|_{q_2})$,

where the constants c *depend on all of the parameters above but not on* f *and* g.

Inequality (2) is a homogeneous version of (1), and is an immediate corollary of (1) by known properties of Bessel and Riesz potentials.

In addition, we prove and utilize a generalized chain rule. Specific forms analogous to the present chain rule for the case $0 \leq s \leq 1$ have appeared in [MS], [KPV] and [CW].

Note that the generalized Leibniz rule in Theorem 1.4 implies the known result that for $1 < p < \infty$, $L^p_s \cap L^\infty$ is an algebra (see [KP]). The same result is also implied by the fact that for any $f \in L^p_s \cap L^\infty$, the function $\phi(f) = f^2 \in L^p_s \cap L^\infty$. The same statement extends also to local spaces. The local version of this theorem is then a special case of the observation that for any $f \in L^p_{s,\mathrm{loc}} \cap L^\infty$ and function ϕ which has sufficiently many derivatives, we have $\phi(f) \in L^p_{s,\mathrm{loc}} \cap L^\infty$. In fact for $s < 1$ it is true that $\phi(f) \in L^p_{s,\mathrm{loc}}$ if we assume only $f \in L^p_{s,\mathrm{loc}}$, as follows from the proof of Lemma 3.1 below.

Let $[\cdot]$ denote the greatest integer function. We have

THEOREM 1.5 (Generalized chain rule) *Let $s \geq 0$ and $\phi: \mathbb{C} \to \mathbb{C}$ be a function with $([s]+1)$-th derivative which is locally bounded. Then for $1 < p < \infty$, ϕ preserves $L^p_{s,\mathrm{loc}} \cap L^\infty$ under composition, i.e., if $f \in L^p_{s,\mathrm{loc}} \cap L^\infty$, then so is $\phi(f)$.*

Investigation of the proof of Theorem below shows that for any $s > 0$ and for sufficiently smooth ϕ, the derivative $J^s(\phi(f))$ can be bounded in L^p by a sum of products of L^q_r norms of f, of a form to be expected based on the standard chain rule for integer s. This version of the chain rule for fractional derivatives is new in that it is stated as well for derivatives of order s greater than 1 (the method of proof is quite different for $s > 1$ from that for $s \leq 1$). The above statement of Theorem 1 is in its most simple form (and also that needed here), in terms which characterize Sobolev spaces preserved by composition with ϕ.

We now summarize the contents of this paper. In Section 2 we gather known definitions and preliminary results. We derive a new characterization of the global Kato class as second Bessel potentials of bounded uniformly continuous functions (see Lemma 2.1 below).

In Section 3 we prove the above-mentioned Leibniz and chain rules.

Section 4 is devoted to proofs of the main results of this paper concerning Sobolev smoothing of Schrödinger semigroups and smoothness of eigenfunctions of Schrödinger operators.

We remark that it would be interesting to extend the present results to the case of time-dependent potentials which are applicable to the study of nonlinear Schrödinger-type equations. It would also be interesting to consider the case of perturbations of more general elliptic operators.

2 Definitions and Notation

Let L^p be the space of all Lebesgue measurable functions f on $\mathbb{R}^d$ with norms

$$\|f\|_p \stackrel{def}{=} \{\int_{\mathbb{R}^d} |f(x)|^p dx\}^{1/p} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_\infty \stackrel{def}{=} \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} |f(x)| < \infty$$

Let $L^p_{\rm loc}$ be the space of all functions f with $\{\int_{B_1(x)} |f(y)|^p dy\}^{1/p} < \infty, \quad x \in \mathbb{R}^n$, where $B_1(x)$ is the unit ball of $\mathbb{R}^d$ at x. Also let $L^p_{\text{un loc}}$ be uniformly local L^p-space of functions $f \in L^p_{\rm loc}$ for which

$$\sup_{x\in\mathbb{R}^d}\{\int_{B_1(x)} |f(y)|^p dy\}^{1/p} < \infty.$$

For an $f \in L^1$, its Fourier transform $\mathcal{F}f$ is defined by $\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} f(x)e^{-i\xi x}dx, \quad \xi \in \mathbb{R}^d$. By the usual procedure this operator extends to $\mathcal{F}\colon S' \to S'$ on the Schwartz space S' of tempered distributions on $\mathbb{R}^d$ (see [Ste]).

Define C_0^∞ to be the infinitely differentiable functions on $\mathbb{R}^d$ with compact support. L^p_s denotes the generalized Sobolev space (space of Bessel potentials) on $\mathbb{R}^d$ with norm given by $\|f\|_{p,s} \stackrel{def}{=} \|J^s f\|_p, \quad 1 \le p < \infty, \ -\infty < s < \infty$, where J^s is the Bessel potential operator.

When $s < 0$ the Bessel potential J^s is the integral operator $J^s f = f * G_s, \quad f \in L^p, \ p \ge 1$, whose kernel G_s is given by $\mathcal{F}G_s(\xi) = (1+|\xi|^2)^{-\frac{|s|}{2}}$.

It is known that $G_s(x) \ge 0, \ x \in \mathbb{R}^d, \ 0 < s < d$, and $G_s(x) = \gamma_s|x|^{-d+s} + o(|x|^{-d+s})$, as $|x| \to 0$, where $0 < s < d$, and γ_s is a positive constant depending only on s. Moreover, $G_s(x) \le \delta_s e^{-c|x|}, \quad x \in \mathbb{R}^d$ where $\delta_s > 0$ depends only on s, and c is an absolute constant. These results follow from statements on pp. 132-133 in [Ste] by rescaling.

Let $L^p_{s,\rm loc}$ be the local Bessel potential space, consisting of all functions f on $\mathbb{R}^n$ with $\|f\|_{p,s,\phi} \stackrel{def}{=} \|\phi f\|_{p,s} < \infty$ for all $\phi \in C_0^\infty$.

The next definition is important in what follows. It concerns the Kato class K_d on Euclidean space $\mathbb{R}^d$, useful in many problems in the theory of partial differential equations.

Definition 2.1 (see [Si3], p. 453) A real-valued measurable function V on $\mathbb{R}^d$ belongs to K_d if and only if

$$\lim_{\alpha\downarrow 0}\left[\sup_x \int_{|x-y|\le\alpha} |x-y|^{-d+2}|V(y)|dy\right] = 0, \qquad d \ge 3$$

$$\lim_{\alpha\downarrow 0}\left[\sup_x \int_{|x-y|\le\alpha} \ln\{|x-y|^{-1}\}|V(y)|dy\right] = 0, \qquad d = 2$$

$$\sup_x \int_{|x-y|\le 1} |V(y)|dy < \infty, \qquad d = 1.$$

If $d \ge 3$, the norm in K_d is given by

$$\|V\|_{K_d} = \sup_x \int_{|x-y|\le 1} |x-y|^{-d+2}|V(y)|dy;$$

for $d = 2$ it is

$$\|V\|_{K_2} = \sup_x \int_{|x-y|\le \frac{1}{2}} \ln\{|x-y|^{-1}\}|V(y)|dy.$$

For $d = 1$ the Kato norm is the uniform local L^1-norm

$$\|V\|_{K_1} = \sup_x \int_{|x-y|\le 1} |V(y)|dy.$$

With the norm $\|\cdot\|_{K_d}$, the Kato class is an (incomplete) normed linear space.

Our next goal is to characterize the Kato class using the Bessel potential J^{-2}. We will need the following variation on the classical definition of approximation of the identity. Let a family of kernels $\{W_\alpha\},\ \alpha > 0$ be such that (a) $\int W_\alpha dx = 1,\ \alpha = 0$; (b) $\sup_\alpha \int |W_\alpha|dx < \infty$; (c) $\lim_{\alpha\to 0} \mathcal{F}W_\alpha(\xi) = 1$ for almost every $\xi \in \mathbb{R}^d$; (d) $\lim_{M\to\infty} \sup_\alpha \int_{|x|>M} |W_\alpha(x)|dx = 0$. Define a family of linear operators on L^∞ by $\overline{W}_\alpha f = f * W_\alpha,\ f \in L^\infty$, $\alpha > 0$.

Lemma 2.1 *Let $V \in L^1_{\text{loc}}$. Then the following are equivalent:*
(1) $V \in K_d$.
(2) $|V| = J^2 h$, where h is a bounded uniformly continuous function.
(3) $|V| = J^2 h$, where $h \in L^\infty$ and $\lim \|h - \overline{W}_\alpha h\|_\infty = 0$ for some (or any) approximation of the identity $\{\overline{W}_\alpha\}$.

Proof: It is known that

$$V \in K_d \Leftrightarrow \lim_{E\to\infty} \|(E-\Delta)^{-1}|V|\ \|_\infty = 0 \tag{2.1}$$

(see [Si3], Proposition A.2.3). We have

$$J^{-2}|v| = (I-\Delta)^{-1}|V| = [(I-\Delta)^{-1}|V| - (E-\Delta)^{-1}|V|] + (E-\Delta)^{-1}|V|.$$

Let $\alpha > 0$, and define the rescaled Bessel potential kernel of order 2 by $(G_2)_\alpha(x) = \alpha^{-d}G_2(\alpha^{-1}x),\ x \in \mathbb{R}^d$. Set $(J^{-2})_\alpha f = f * (G_2)_\alpha$.

Using known properties of the Fourier transform,

$$(I - E^{-1}\Delta)^{-1}|V| = (J^{-2})_{E^{-1/2}}|V|. \tag{2.2}$$

The resolvent equation (see [ReS]) gives

$$\begin{aligned}(I-\Delta)^{-1}|V| - (E-\Delta)^{-1}|V| &= (I-\Delta)^{-1}(I-E^{-1}\Delta)^{-1}|V| - \\ &-(I-\Delta)^{-1}(E-\Delta)^{-1}|V|.\end{aligned} \tag{2.3}$$

It follows from (2.2) and (2.3) that for $h = J^{-2}|V|$,

$$h - (J^{-2})_{E^{-1/2}}h = (E-\Delta)^{-1}|V| - J^{-2}(E-\Delta)^{-1}|V|. \tag{2.4}$$

Suppose $V \in K_d$. Then by (2.1), (2.2), and (2.4) we have $h \in L^\infty$. Moreover, (2.1) and (2.4) give

$$\lim_{\alpha\to 0} \|h - (J^{-2})_\alpha h\|_\infty = 0. \tag{2.5}$$

Now uniform continuity of h follows by standard arguments, using the fact that $\tilde{W}_\alpha = (J^{-2})_\alpha$ generates an approximation of the identity. This proves the implication (1) $\Rightarrow$ (2). The implication 3 $\Rightarrow$ (2) is similar.

Suppose condition (2) holds. Then we have

$$\|h - (J^{-2})_\alpha h\|_\infty \le \sup_x \int |h(x-u) - h(x)|(J^{-2})_\alpha(u)du$$

$$\le c \sup_{|u|\le\epsilon} \sup_{x\in\mathbb{R}^d} |h(x+u) - h(x)| + c\int_{|u|>\epsilon} (J^{-2})_\alpha(u)du.$$

Now the uniform continuity of h implies (2.5).

Using the fact that $L^\infty \subset K_d$ and also condition (2.1), we obtain

$$\lim_{E\to\infty} \|J^{-2}(E-\Delta)^{-1}|V|\|_\infty = 0.$$

Now (2.4) and (2.5) imply

$$\lim_{E\to\infty} \|(E-\Delta)^{-1}|V|\|_\infty = 0$$

and hence $V \in K_d$ (see (2.1)). Therefore, (2) $\Rightarrow$ (1).

The last step consists of proving (2) $\Leftrightarrow$ (3). Suppose (2) holds. It is well known that for standard approximations of the identity $\{v * P_\beta\}$, $\beta > 0$ where $P_\beta(x) = \beta^{-d}\ell(\beta^{-1}x)$ and $\ell \in C_0^\infty$ with $\int \ell dx = 1$, we have $\|v - v * P_\beta\|_\infty \to 0$ as $\beta \to 0$. Now from

$$\|v - v * W_\alpha\|_\infty \le \|v - v * P_\beta\|_\infty + \|v * W_\alpha - v * W_\alpha * P_\beta\|_\infty$$

$$+\|v * P_\beta - v * P_\beta * W_\alpha\|_\infty \le C\|v - v * P_\beta\|_\infty + \|v\|_\infty \|P_\beta - P_\beta * W_\alpha\|_1$$

we get that for every $\epsilon > 0$ there exists a C^∞-function τ for which

$$\|v - v * W_\alpha\|_\infty \le \frac{\epsilon}{2} + \|v\|_\infty \|\tau - \tau * W_\alpha\|_1.$$

Hence the implication (2) $\Rightarrow$ (3) will be proven if we can show that

$$\|\tau - \tau * W_\alpha\|_1 \to 0 \quad \text{as} \quad \alpha \to 0 \quad \text{for any} \quad \tau \in L^1. \tag{2.6}$$

Suppose supp $(\tau) \subset B_r(0)$, where $B_r(0)$ denotes the ball of radius r centered at 0. Then for every $M > r$

$$\|\tau-\tau*W_\alpha\|_1 \leq \int_{|x|\leq M} |\tau(x)-\tau*W_\alpha(x)|dx+\int_{|x|>M} |\tau*W_\alpha(x)|dx = I_1(M,\alpha)+I_2(M,\alpha). \quad (2.7)$$

It is clear that

$$I_1(M,\alpha) \leq M^{1/2}\|\tau - \tau * W_\alpha\|_2 \leq C_M\{\int |1-\mathcal{F}W_\alpha(\xi)|^2|\mathcal{F}\tau(\xi)|^2 d\xi\}^{1/2}.$$

Hence, by properties (b) and (c) of approximations of the identity and by the Lebesgue dominated convergence theorem,

$$\lim_{\alpha\to 0} I_1(M,\alpha) = 0 \quad (2.8)$$

for every $M > 0$.

If $\epsilon > 0$ is given, then there exists an M_1 such that for $M > M_1$ we have

$$\begin{aligned}\sup_\alpha I_2(M,\alpha) &\leq \|\tau\|_\infty \sup_\alpha \int_{|x|>M} dx \int_{|u|\leq r} |W_\alpha(x-u)|du \\ &\leq \|\tau\|_\infty r \sup_\alpha \int_{|u|>M-r} |W_\alpha(v)|dv \leq \epsilon. \quad (2.9)\end{aligned}$$

Now (2.6) follows from (2.7), (2.8), and (2.9). Therefore, (2) implies (3) and Lemma 2.1 is proved.

Two examples of approximations of the identity are given by (1) $\overline{W}_\alpha f = f*(W)_\alpha$, where $W \in L^1$, $\int W dx = 1$, $(W)_\alpha(x) = \alpha^{-n}W(\alpha^{-1}x)$; (2) $\tilde{W}_\alpha f = J^{-\alpha} f$, $\alpha > 0$. The second example will be important here. Lemma 2.1 in this case gives

$$V \in K_d \Leftrightarrow \lim_{\alpha\to 0} \|h - J^{-\alpha}h\|_\infty = 0, \quad (2.10)$$

where $h = J^{-2}|V|$.

We equip K_d with the new norm

$$\|V\|^*_{K_d} = \|J^{-2}|V|\|_\infty. \quad (2.11)$$

It can be shown that the norms $\|\cdot\|_{K_d}$ and $\|\cdot\|^*_{K_d}$ are equivalent (we leave the details to the reader). Assertion (2.10) can be written as

$$V \in K_d \Leftrightarrow \lim_{\alpha\to 0} \||V| - J^{-\alpha}|V|\|_{K_d} = 0. \quad (2.12)$$

DEFINITION 2.2 We say that a function V belongs to the local Kato class $K_{d,\text{loc}}$ iff $VX_B \in K_d$ for all balls B in $\mathbb{R}^d$, where χ_B denotes the characteristic function of B.

In [Si3] there is the following characterization of the class $K_{d,\text{loc}}$:
$V \in K_{d,\text{loc}} \Leftrightarrow (E-\Delta)^{-1}(gV)$ is a continuous function for every $E>0$ and every bounded Lebesgue measurable function g with compact support.

The following compares some properties of K_d and $K_{d,\text{loc}}$. The definitions of the classes K_d and $K_{d,\text{loc}}$ involve only absolute values of functions. Hence $V \in K_d \Leftrightarrow |V| \in K_d$ and $V \in K_{d,\text{loc}} \Leftrightarrow |V| \in K_{d,\text{loc}}$. If $V \in K_d$ then $V^+, V^- \in K_d$ and therefore $J^{-2}V$ is a uniformly continuous function, by Lemma 2.1. Moreover, part (3) of Lemma 2.1 holds for $J^{-2}V$. However, we do not know whether uniform continuity of $J^{-2}V$ implies $V \in K_d$.

Suppose $V \in K_{d,\text{loc}}$. This is equivalent to $gV \in K_d$ for every $g \in L^\infty$ having compact support. By Lemma 2.1, $V \in K_{d,\text{loc}} \Leftrightarrow J^{-2}|gV|$ is uniformly continuous for all $g \in L^\infty$ with compact support. But it is clear that we may delete the absolute value in the characterization above and that the uniform continuity can be replaced by continuity, because for L^1 functions f with compact support continuity and the uniform continuity of $J^{-2}f$ are equivalent. Hence we obtain the characterization of $K_{d,\text{loc}}$ formulated above in the equivalent form $V \in K_{d,\text{loc}} \Leftrightarrow$ $J^{-2}(gV)$ is a continuous function for all functions $g \in L^\infty$ having compact support.

As examples of functions belonging to the Kato class, let us only mention the fact that for $d \geq 3$, we have $L(\frac{d}{2},1) \subset K_d$, where $L(\frac{d}{2},1)$ is the Lorentz space with norm

$$\|f\|_{L(\frac{d}{2},1)} = \int_0^\infty t^{\frac{2}{d}-1} f^*(t)dt.$$

Here f^* is the non-increasing rearrangement of $|f|$ on $(0,\infty)$ (see [Si3], examples B and C on page 456, for details).

The next definitions concern the Schrödinger semigroup (see [Si3] for more details). The Schrödinger operator H is an operator on $\mathbb{R}^d$ defined by $H = H_0 + V$, where $H_0 = -\frac{1}{2}\Delta$ and V is denoted the potential. By a Schrödinger semigroup is meant the semigroup $e^{-tH},\ t>0$. It is known that if $V^- \in K_d$, $V^+ \in K_{d,\text{loc}}$ then the quadratic form $H_0 + V$ is closed on the intersection of domains of $(-\Delta)^{1/2}$ and $(V^+)^{1/2}$, and so defines a semibounded self-adjoint operator H. The corresponding semigroup $e^{-tH},\ t>0$ consists of bounded operators on L^2 (see [Si3], Theorem A.2.7 or [Da2] section 1.8).

We note that e^{-tH} has important connections with functional integrals. Suppose $b(t)$ is an $\mathbb{R}^d$-valued Gaussian process with

$$E_x(b_j(t)) = x_j, \quad E_x([b_j(t)-x_j][b_k(s)-x_k]) = \delta_{jk}\min(s,t),\ x \in \mathbb{R}^d,$$

where E_x denotes expectation corresponding to this process (see, e.g. [Fr]). The importance of Gaussian processes in the theory of Schrödinger semigroups can be seen from the Feynman-Kac formula, which gives the path-integral representation of the semigroup

$$(e^{-tH}f)(x) = E_x(\exp(-\int_0^t V(b(s))ds)f(b(t))) \tag{2.13}$$

(see [Si3], p. 459), and from the fact that

$$V \in K_d \Leftrightarrow \lim_{t\to 0}\sup_x E_x(\int_0^t |V(b(s))|ds) = 0 \tag{2.14}$$

(see [Si3], Proposition A.2.6).

We will now need several lemmas which give norm estimates for operator families of the form $H^m e^{-tH}$ with $t > 0$, $m \geq 0$ in terms of constants characterizing Kato class potentials.

The following lemma is known. A variation of it appears in [Si3].

LEMMA 2.2 *Suppose a potential V satisfies $V^- \in K_d$, $V^+ \in K_{d,\mathrm{loc}}$ and a number $\tau > 0$ is given such that*

$$G_\tau = \sup_x E_x(\int_0^\tau |2V^-(b(s))|ds) < 1$$

Then

$$\|e^{-tH}\|_{p\to q} \leq (1-G_\tau)^{-1}(\pi t)^{-(1-\theta)\frac{d}{2}} \exp\left(\frac{(1+\theta)t}{2\tau}\ln\frac{1}{1-G_\tau}\right), \tag{2.15}$$

where $1-\theta = p^{-1} - q^{-1}$.

The next assertion is a corollary of Lemma 2.2.

COROLLARY 2.1 *Suppose a potential V satisfies $V^- \in K_d$, $V^+ \in K_{d,\mathrm{loc}}$ and a number $\tau > 0$ is given such that*

$$\sup_x E_x\left(\int_0^\tau |2V^-(b(s))|ds\right) < \frac{1}{2}. \tag{2.16}$$

Then

$$\|e^{-tH}\|_{p\to q} \leq c_1(\pi t)^{-(1-\theta)\frac{d}{2}} \exp\left(c_2\frac{(1+\theta)t}{\tau}\right),$$

where $1 \leq p \leq q \leq \infty$, $1-\theta = p^{-1} - q^{-1}$, and c_1, c_2 are absolute constants.

Consider the following families of kernels on $\mathbb{R}^n$:

(a) $\mathcal{D}_\alpha(x) = \chi_\alpha(x)|x|^{-d+2}, \quad \alpha > 0$

where

$$\chi_\alpha(x) = \begin{cases} 1, |x| \leq \alpha \\ 0, |x| > \alpha, \end{cases}$$

(b) $Q_t(x) = \int_0^t (2\pi s)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2s}\right) ds,\ t > 0,$

(c) $Y_E(x) = E^{\frac{d}{2}-1} G_2(E^{\frac{1}{2}} x), \quad E > 0,$

where G_2 denotes the Bessel potential kernel defined above.

All of these kernels are closely connected with Kato class potentials; for simplicity we assume $d \geq 3$; the cases $d = 1, 2$ are similar. The kernel $\mathcal{D}_\alpha$ arises in the definition of the Kato class; in fact, we have $V \in K_d \Leftrightarrow \lim_{\alpha\to 0} \|\, |V| * \mathcal{D}_\alpha\|_\infty = 0$. The kernel Q_t arises in path integral representations (see (2.14)). We have

$$E_x \left(\int_0^t |V(b(s))| ds \right) = (|V| * Q_t)(x), \qquad t > 0$$

and $V \in K_d \Leftrightarrow \lim_{t\to 0} \|\, |V| * Q_t\|_\infty = 0$ (see [Si3], Proposition A.2.6). Finally the kernel Y_E is connected with formulas (2.1) and (2.5). Precisely, it is the kernel of the operator $(E - \Delta)^{-1}$, and it follows from (2.1) that $V \in K_d \Leftrightarrow \lim_{E\to\infty} \|\, |V| * Y_E\|_\infty = 0$.

All the above conditions equivalent to $V \in K_d$ are interconnected. It is not difficult to prove that

$$c_2 \|\, |V| * \mathcal{D}_{\sqrt{t}}\|_\infty \leq \|\, |V| * Q_t\|_\infty \leq c_1 \|\, |V| * \mathcal{D}_{\sqrt{t}}\|_\infty, \tag{2.17}$$

$$c_4 \|\, |V| * \mathcal{D}_E^{-1/2}\|_\infty \leq \|\, |V| * Y_E\|_\infty \leq c_3 \|\, |V| * \mathcal{D}_E^{-1/2}\|_\infty. \tag{2.18}$$

with positive constants c_1, c_2, c_3, and c_4. The last inequalities hold because it can be easily seen from the definitions that $Q_t(x) \geq c_2 \mathcal{D}_{\sqrt{t}}(x)$, $Y_E(x) \geq c_4 \mathcal{D}_{E^{-1/2}}(x)$. These give the left-hand estimates in (2.17) and (2.18). The right-hand inequalities in (2.17) and (2.18) obtain from the fact that the kernels Q_t and Y_E have exponential decay at infinity, and from the condition $V \in K_d$.

¿From Corollary 2.1 we obtain the following norm estimate for the Schrödinger semigroup corresponding to a potential V with $V^- \in K_d,\ V^+ \in K_{d,\mathrm{loc}}$:

$$\|e^{-tH}\|_{p\to p} \leq \delta_1 \exp\left(\frac{\delta_2 t}{\rho_1(V^-)(\frac{1}{4})} \right), \qquad (1 \leq p \leq \infty), \tag{2.19}$$

where $\delta_1 > 0$ and $\delta_2 > 0$ are absolute constants and

$$\rho_1(V^-)(y) = \sup\{\alpha > 0\colon \|(V^-) * Q_\alpha\|_\infty < y\}, \quad y > 0.$$

By (2.17) and (2.19),

$$\|e^{-tH}\|_{p\to p} \leq \gamma_1 \exp\left(\frac{\gamma_2 t}{\rho_2(V^-)(\gamma_3)} \right), \quad (1 \leq p \leq \infty) \tag{2.20}$$

where γ_1, γ_2 and γ_3 are absolute positive constants, and

$$\rho_2(V^-)(y) = \Big[\sup\{\alpha > 0\colon \|(V^-) * \mathcal{D}_\alpha\|_\infty < y\}\Big]^2, \quad y > 0. \tag{2.21}$$

This is the estimate we will use in the sequel.

It is known that $\inf \operatorname{spec}_{L^2}(H) = -\lim_{t\to\infty} t^{-1} \ln \|e^{-tH}\|_{p\to p} \quad (1 \le p \le \infty)$ (see [Si3], p. 470). It follows from (2.20) that

$$|\inf \operatorname{spec}_{L^2}(H)| \le \frac{\tau_1}{\rho_2(V^-)(\tau_2)}, \tag{2.22}$$

where $\tau_1 > 0$ and $\tau_2 > 0$ are absolute constants.

The potentials $V \in K_d$ are small perturbations of the operator $H_0 = -\frac{1}{2}\Delta$ in L^1:

Lemma 2.3 (see [AS],[Si3]) *the following are equivalent:*
(1) $V \in K_d$
(2) For all $\epsilon > 0$ there exists $a_\epsilon > 0$ so that for all ϕ in the L^1-domain of Δ,

$$\|V\phi\|_1 \le \epsilon\|H_0\phi\|_1 + a_\epsilon\|\phi\|_1.$$

From the proof of this lemma in [Si3], p. 455, it follows that

$$a_\epsilon = E\|(E - \frac{1}{2}\Delta)^{-1}|V|\;\|_\infty,$$

where E is such that

$$\|(E - \frac{1}{2}\Delta)^{-1}|V|\;\|_\infty < \epsilon.$$

Hence, by (2.18),

$$a_\epsilon \le \tau_3\epsilon\frac{1}{\rho_2(V)(\tau_4\epsilon)}, \tag{2.23}$$

where $\tau_3 > 0$ and $\tau_4 > 0$ are absolute constants and $\rho_2(V)$ is defined for a Kato class potential V similarly to (2.21) with $|V|$ instead of V^-.

The following assertions are standard. For the sake of completeness we include schemes of their proofs.

Lemma 2.4 *Suppose $V \in K_d$. Then $\|H^m e^{-tH}\|_{1\to q} \le \lambda(m, q, t, V) < \infty$ for all $m \ge 0$, $1 \le q \le \infty$, $t > 0$, where the function λ depends only on m, q, t and on the number $\rho_2(V)(\tau)$ where τ is an absolute positive constant. Moreover, if a family of potentials $V_\alpha \in K_d$ is given such that $\rho_2(V_\alpha)(\tau) \ge a$ for all α with some $a > 0$, then $\sup_\alpha \|H_\alpha^m e^{-tH_\alpha}\|_{1\to q} < \infty$ for all $m \ge 1$, $t > 0$, and $1 \le q \le \infty$.*

LEMMA 2.5 *Suppose* $V^- \in K_d$, $V^+ \in K_{d,\mathrm{loc}}$. *Then* $\|H^m e^{-tH}\|_{p\to q} \le \lambda(m,p,q,t,V)$ *for all* $m \ge 1$, $p \le q \le \infty$, $1 < p < \infty$, $t > 0$, *where the function* λ *depends only on* m,p,q,t, *and* $\rho_2(V^-)(\tau)$, *where* τ *is an absolute positive constant. Moreover, if a family of potentials* V_α *satisfies* $V_\alpha^- \in K_d$, $V_\alpha^+ \in K_{d,\mathrm{loc}}$ *and* $\rho_2(V_\alpha^-)(\tau) \ge a$ *for all* α *with some* $a > 0$ *then* $\sup_\alpha \|H_\alpha^m e^{-tH_\alpha}\|_{p\to q} < \infty$ *for all* $m \ge 1$, $p \le q \le \infty$ $q < p < \infty$, $t > 0$.

PROOF OF LEMMA 2.4 With no loss of generality, we assume $z_V = \inf \operatorname{spec}_{L^2} H = 0$, since the infimum of the L^p-spectrum of H on the real line does not depend on p ([Si3], Section B.5), and if $z_V \neq 0$, we may pass to a new operator $\overline{H} = H - z_V$.

We will use the theory of holomorphic semigroups (see [Ka],[Yo]); our proof of Lemma 2.4 follows the proof of Theorem 2.4 on p. 499 in [Ka]. It is known that the semigroup e^{-tH_0}, with $H_0 = -\frac{1}{2}\Delta$, is holomorphic, as follows from the explicit representation of the heat diffusion semigroup (see [Ka], p. 495). We have

$$\|(\zeta - H_0)^{-1}\|_{1\to 1} \le M|\zeta|^{-1} \tag{2.24}$$

for every ζ with $\pi \ge |\arg\zeta| > \alpha$, where α is a small fixed number. Moreover, by (2.24) and Lemma 2.3,

$$\|V(\zeta - H_0)^{-1}\|_{1\to 1} \le \epsilon\|H_0(\zeta - H_0)^{-1}\|_{1\to 1} + a_\epsilon M|\zeta|^{-1}$$

for every $\epsilon > 0$.

From

$$\|H_0(\zeta - H_0)^{-1}\|_{1\to 1} \le \|I - \zeta(\zeta - H_0)^{-1}\|_{1\to 1} \le 1 + M,$$

it follows that

$$\|V(\zeta - H_0)^{-1}\|_{1\to 1} \le a_\epsilon M|\zeta|^{-1} + \epsilon(1+M). \tag{2.25}$$

By the second Neumann series for the resolvent,

$$(\zeta - H)^{-1} = (\zeta - H_0 - V)^{-1} = \sum_{n=0}^{\infty} (\zeta - H_0)^{-1}[V(\zeta - H_0)^{-1}]^n$$

(see [Ka], p.444), we have from (2.24) and (2.25) that

$$\|(\zeta - H)^{-1}\|_{1\to 1} \le \frac{M}{|\zeta|(1 - a_\epsilon M|\zeta|^{-1} - \epsilon(1+M))} = \frac{M(1-\epsilon(1+M))^{-1}}{|\zeta| - Ma_\epsilon(1-\epsilon(1+M))^{-1}}$$

for all

$$\epsilon < (1+M)^{-1}, \quad |\zeta| > \frac{Ma_\epsilon}{1-\epsilon(1+M)}.$$

Hence, if we fix ϵ satisfying the above condition and let

$$|\zeta| > \frac{2Ma_\epsilon}{1-\epsilon(1+M)} \quad \text{and} \quad \pi \geq |\arg\zeta| > \alpha$$

then

$$\|(\zeta - H)^{-1}\|_{1\to 1} \leq \frac{\overline{M}}{|\zeta|}, \tag{2.26}$$

where λ, α, and $\overline{M}$ depend only on the relative boundedness constants for the potential V.

We now represent our semigroup by the Dunford-Taylor integral

$$e^{-tH} = \frac{1}{2\pi i}\int_\Gamma e^{-\zeta t}(\zeta - H)^{-1}d\zeta, \tag{2.27}$$

where the contour of integration Γ is a curve running from infinity with arg $\zeta = \alpha$ to its intersection with the circle $|\zeta| = M$, then along the arc of this circle, lying inside the set $\{\zeta\colon \alpha \leq |\arg\zeta| \leq \pi\}$, and then back to infinity with arg $\zeta = -\alpha$ (c.f. [Ka], p. 489). Taking derivatives with respect to t in (2.27), we have

$$H^m e^{-tH} = \frac{1}{2\pi i}\int_\Gamma e^{-\zeta t}\zeta^m(\zeta - H)^{-1}d\zeta. \tag{2.28}$$

Applying (2.26) in (2.28) and using (2.22), (2.23), Corollary 2.1 and the formula

$$H^m e^{-tH} = e^{-\frac{1}{2}H}H^m e^{-\frac{1}{2}H}, \tag{2.29}$$

we complete the proof of Lemma 2.4.

Proof of Lemma 2.5 Estimate (2.20) shows that the semigroup

$$A_t = \exp\left(-\frac{\gamma_2 t}{\rho_2(V^-)(\gamma_3)}\right)e^{-tH}, \ t > 0 \tag{2.30}$$

is uniformly bounded in all L^p-spaces with $1 \leq p < \infty$. We will use an adaptation of the proof of Theorem X.55 in [ReS, Vol. 1] to show that the semigroup in L^p with a fixed p admits analytic continuation to an analytic L^p-operator valued semigroup defined in a sector

$$S_p\colon\ |\arg(t + i\tau)| < \frac{\pi}{2}\left(1 - \left|\frac{2}{p} - 1\right|\right).$$

This allows the use of the well-developed theory of analytic semigroups (see [Yo], [Ka]). The main result of this theory [Yo], which gives several equivalent conditions for analyticity, allows us to find an appropriate contour and to express the extended semigroup as a Dunford-Taylor integral (2.27). Resolvent estimates in L^p similar to those in (2.26) also obtain from the general theorem in [Yo]. After that we can differentiate (2.27) with respect to t any number of times to obtainL^p norm estimates for $\frac{d^m}{dt^m}A_t, \qquad m \geq 1, \quad t > 0.$

Using (2.30) we obtain L^p-norm estimates for the operators $H^m e^{-tH}$, involving the quantities

$$\exp\left(\frac{\gamma_2 t}{\rho_2(V^-)(\gamma_3)}\right) \tag{2.31}$$

and their derivatives with respect to t. This reasoning shows that the L^p-norm estimate for $H^m e^{-tH}$ depends on V only through quantities of the type (2.31) and their t-derivatives.

Thus, Lemma 2.5 is proved for $q = p$. The general case follows from formula (2.29).

Remark. If V is a potential from K_d and $V_j \in K_d$ with $j \geq 1$ is a sequence of potentials such that $\lim \|V - V_j\|_{K_d} = 0$ then it is easy to see that $\inf_j \rho_2(V_j)(y) > 0$ for every $y > 0$.

We will need the following known result (see [Si3], Theorems B.10.1 and B.10.2).

LEMMA 2.6 *(a) Let* $V, V_j \in K_d$ *and let* $\lim_{j\to\infty} \|V - V_j\|_{K_d} = 0$. *Then for any* $t > 0,\ 1 \leq p \leq \infty$ *we have* $\lim_{j\to\infty} \|e^{-tH_j} - e^{-tH}\|_{p\to p} = 0$.

(b) Let $V^-, V_j^- \in K_d;\ V^+, V_j^+ \in K_{d,\mathrm{loc}}$. *Suppose that*

$$\lim_{t\to 0} \sup_{j,x} E_x\left(\int_0^t V_j^-(b(s))ds\right) = 0$$

and for any ball B_r *centered at 0 and of radius* r *we have* $\lim_{j\to\infty} \|\chi_{B_r}(V_j - V)\|_{K_d} = 0$. *Then for any* $t > 0,\ r > 0$ *and* $1 \leq p \leq \infty$ $\lim_{j\to\infty} \|\chi_{B_r}(e^{-tH_j} - e^{tH})\|_{p\to p} = 0$.

An L^p-eigenfunction ϕ of the Schrödinger operator H is a function ϕ from the L^p-domain of H such that there exists a complex number λ for which $H\phi = \lambda\phi$. Such a ϕ is always an L^p-eigenfunction of e^{-tH}, and so belongs to all L^q with $q \geq p$ and is continuous (see [Si3]). As mentioned earlier, a goal of this paper is to study smoothness of the eigenfunctions of $H = -\frac{1}{2}\Delta + V$ in connection with that of the potential V. The notion of non-triviality of L^p-eigenfunctions of H is required. Recall that non-triviality means that for every $x \in \mathbb{R}^d$ there exists an L^p-eigenfunction ϕ of H such that $\phi(x) \neq 0$.

3 The Leibniz and chain rules for Sobolev spaces

In this section we obtain generalizations of the classical Leibniz and chain rules for the case where, instead of the ordinary partial derivatives, Bessel and Riesz potentials are considered. The results have been stated in the introduction (Theorems 1.4 and 1).

Proof of Theorem 1.4. The main ideas of our proof are due to Coifman-Meyer [CM1], [CM2] and Kato-Ponce [KP]. Following Coifman-Meyer [CM1], we write

$$\begin{aligned} J^s(fg)(x) &= \int\int e^{i<x,\xi_1+\xi_2>}(1+|\xi_1+\xi_2|^2)^{s/2}\mathcal{F}g(\xi_1)\mathcal{F}f(\xi_2)d\xi_1 d\xi_2 \\ &= \int\int e^{i<x,\xi_1+\xi_2>}(1+|\xi_1+\xi_2|^2)^{s/2}\phi\left(\frac{|\xi_1|}{|\xi_2|}\right)\mathcal{F}g(\xi_1)\mathcal{F}f(\xi_2)d\xi_1 d\xi_2 \\ &+ \int\int e^{i<x,\xi_1+\xi_2>}(1+|\xi_1+\xi_2|^2)^{s/2}\phi\left(\frac{|\xi_2|}{|\xi_1|}\right)\mathcal{F}g(\xi_1)\mathcal{F}f(\xi_2)d\xi_1 d\xi_2 \\ &= I_1(s,f,g)(x) + I_2(s,f,g)(x), \end{aligned} \tag{3.1}$$

where ϕ is a fixed C_0^∞-function on $[0,\infty)$, which is non-negative and non-increasing and such that $(a)\quad \phi(0)=1;\qquad (b)\ \ \phi(t)+\phi(t^{-1})=1,\ t>0.$

We need:

THEOREM 3.1 (Coifman-Meyer [CM2]) *Let $\lambda(\xi_1,\xi_2)$ be such that*

$$\left|\frac{\partial^{m+k}\lambda}{\partial\xi_1^m\partial\xi_2^k}(\xi_1,\xi_2)\right| \leq \frac{c_{m,k}}{(|\xi_1|+|\xi_2|)^{m+k}}, \tag{3.2}$$

where $m\geq 0,\ k\geq 0,$ and $m+\kappa\leq[\frac{n}{2}]+1$. Define a bilinear operator Λ by

$$\Lambda(f,g)(x) = \int\int e^{i<x,\xi_1+\xi_2>}\lambda(\xi_1,\xi_2)\mathcal{F}f(\xi_1)\mathcal{F}g(\xi_2)d\xi_1 d\xi_2.$$

Then Λ maps $L^{\tau_1}\times L^{\tau_2}$ into L^τ, where

$$\frac{1}{\tau}=\frac{1}{\tau_1}+\frac{1}{\tau_2},\quad 1\leq\tau<\infty,\quad 1<\tau_1,\tau_2\leq\infty,\ \tau_1<\infty.$$

For every $\alpha\geq 0$ rewrite the expression defining $I_1(s,f,g)(x)$ as:

$$\begin{aligned} I_1(s,f,g)(x) &= \int\int e^{i<x,\xi_1+\xi_2>}(1+|\xi_1+\xi_2|^2)^{s/2}\frac{(1+|\xi_1|^2)^{\alpha/2}}{(1+|\xi_2|^2)^{\frac{s+\alpha}{2}}} \\ &\times\phi\left(\frac{|\xi_1|}{|\xi_2|}\right)\mathcal{F}(J^{-\alpha}g)(\xi_1)\mathcal{F}(J^{s+\alpha}f)(\xi_2)d\xi_1 d\xi_2. \end{aligned} \tag{3.3}$$

When s and α are both even integers we may apply Theorem 3.1 to obtain

$$\|I_1(s,f,g)\|_p \le c\|J^{s+\alpha}(f)\|_{p_1}\|J^{-\alpha}(g)\|_{q_1}, \tag{3.4}$$

where c depends on p, s, α, p_1, q_1 and the constants $c_{m,k}$ in (3.2). It is easy to see that

$$I_1(s,f,g)(x) = J^s I_1(f,g)(x), \tag{3.5}$$

where

$$I_1(f,g)(x) = \int\int e^{i<x,u>}\phi\left(\frac{|\xi_1|}{|\xi_2|}\right)\cdot \mathcal{F}g(\xi_1)\mathcal{F}f(\xi_2)d\xi_1 d\xi_2. \tag{3.6}$$

Using (3.5) and (3.6), we may interpolate between two versions of (3.4), where α and g are fixed, and prove (3.4) for all $s \ge 0$ and for all $\alpha = 2m,\ m = 0, 1, \ldots$. We write the obtained inequality as $\|I_1(f, J^\alpha h)\|_{p,s} \le c\|h\|_{q_1}\|f\|_{p_1,s+\alpha}$.

Fix $s \ge 0,\ \alpha_1 = 2m_1,\ \alpha_2 = 2m_2$ and a function $g \in C_0^\infty$. Consider the analytic family of operators $T_z(f) = I_1(f, J^z h)$, where $z = x + iy,\ 2m_1 \le x \le 2m_2$, and $J^z(h)$ is the Bessel potential of a complex order defined by $\mathcal{F}J^z(h)(\xi) = (1 + |\xi|^2)^{z/2}\mathcal{F}h(\xi)$.

Reasoning as above we conclude

$$\|T_{\alpha_1+iy}(f)\|_{p,s} \le c(y)\|h\|_{q_1}\|f\|_{p_1,s+\alpha_1},$$

$$\|T_{\alpha_2+iy}(f)\|_{p,s} \le c(y)\|h\|_{q_1}\|f\|_{p_1,s+\alpha_2},$$

where $c(y)$ has polynomial growth in y. Applying Stein's interpolation theorem for analytic families of operators (see [SW], p. 205) we get $\|I_1(f, J^\alpha h)\|_{p,s} \le c\|h\|_{q_1}\|f\|_{p_1,s+\alpha}$, or

$$\|I_1(\alpha,f,g)\|_p \le c\|J^{-\alpha}(g)\|_{q_1}\|J^{s+\alpha}(f)\|_{p_1} \tag{3.7}$$

for $\alpha_1 \le \alpha \le \alpha_2$, and hence for all $\alpha, 0 \le \alpha < \infty$. Similar reasoning for I_2 gives

$$\|I_2(s,f,g)\|_p \le c\|J^{-\beta}(f)\|_{p_2}\|J^{s+\beta}(g)\|_{q_2}. \tag{3.8}$$

Now by (3.1), (3.7) and (3.8) Theorem 1.4 is proved.

Next we prove the generalized chain rule:

Proof of Theorem 1.5:

We need the following characterization of Sobolev spaces of order less than 1 due to Strichartz [Str]. Namely, for $0 < s < 1$ and $1 < p < \infty$, $f \in L^p_s(\mathbb{R}^d)$ if and only if $f \in L^p$ and

$$S_s f(x) = \left(\int_0^\infty \left[\int_B |f(x+ry) - f(x)|dy\right]^2 \frac{dr}{r^{1+2s}}\right)^{1/2} \tag{3.9}$$

is in L^p, where B is the unit ball in $\mathbb{R}^d$. In that case, $\|f\|_{L^p_s} \approx \|f\|_p + \|S_s f\|_p$.

We first prove Theorem 1 for the case $0 \le s < 1$. Define a function $\phi: \mathbb{C} \to \mathbb{C}$ to be *locally Lipschitz* if for every $M > 0$ there exists a $C > 0$ such that if $x, y \in \mathbb{C}$ and both have norm less than M, then

$$|\phi(x) - \phi(y)| \le C|x - y|. \tag{3.10}$$

LEMMA 3.1 *Let $\phi: \mathbb{C} \to \mathbb{C}$ be a function which is locally Lipschitz. Then for $1 < p < \infty$ and $0 \le s < 1$, ϕ preserves $L^p_{s,\mathrm{loc}} \cap L^\infty$ under composition.*

PROOF: If $s = 0$ the result is trivial, and so for $0 < s < 1$, assume that $f \in L^p_{s,\mathrm{loc}} \cap L^\infty$ and $\psi \in C_0^\infty(\mathbb{R}^d)$. Define $\|\phi\|_L$ to be the smallest constant C so that (3.10) holds for all $x, y \in \mathbb{R}$ with magnitude smaller than $\|f\|_\infty$. Defining also the norm

$$\|F(y,r)\|_B^2 \equiv \int_0^\infty \left[\int_B |F(y,r)| dr\right]^2 \frac{dr}{r^{1+2s}},$$

we have (x is a fixed parameter in the $\|\cdot\|_B$ norms below)

$$\begin{aligned}
&|S_s(\phi(f)\psi)(x)| = \|\phi(f(x+ry))\psi(x+ry) - \phi(f(x))\psi(x)\|_B \\
&\le \|\phi(f(x+ry))\psi(x+ry) - \phi(f(x))\psi(x+ry)\|_B + \|\phi(f(x))\psi(x+ry) - \phi(f(x))\psi(x)\|_B \\
&\le \|\phi\|_L f(x+ry)\psi(x+ry) - f(x)\psi(x+ry)\|_B + \|\phi\|_\infty \|\psi(x+ry) - \psi(x)\|_B \\
&\le \|\phi\|_L \|f(x+ry)\psi(x+ry) - f(x)\psi(x)\|_B + \|\phi\|_L \|f(x)\psi(x) - f(x)\psi(x+ry)\|_B \\
&+ \|\phi\|_\infty \|\psi(x+ry) - \psi(x)\|_B = \|\phi\|_L \|f(x+ry)\psi(x+ry) - f(x)\psi(x)\|_B \\
&+ (\|\phi\|_L |f(x)| + \|\phi\|_\infty)\|\psi(x) - \psi(x+ry)\|_B.
\end{aligned} \tag{3.11}$$

Note that since $f \in L^p_{s,\mathrm{loc}}$, it follows from the Strichartz characterization of L^p_s that the first term above is in L^p, and since $\psi \in L^p_s$

$$\left(\int_0^\infty \left[\int_B |\psi(x) - \psi(x+ry)| dy\right]^2 \frac{dr}{r^{1+2s}}\right)^{1/2} \in L^p,$$

so that since $f \in L^\infty$, the right side of (3.11) is in L^p. Thus we conclude that $S_s(\phi(f)\psi) \in L^p$. Since clearly $\phi(f)\psi \in L^p$ as well, it follows that $\phi(f)\psi \in L^p_s$. Since ψ was an arbitrary C_0^∞ function, it follows that $\phi(f) \in L^p_{s,\mathrm{loc}}$, as desired. Since clearly $\phi(f) \in L^\infty$, Lemma 3.1 is proved.

We now proceed with the proof of Theorem 1.5. It is easy to check that given any integer $n < s$, we have $f \in L^p_{s,\mathrm{loc}}$ if and only if $D^\alpha f \in L^p_{s-n,\mathrm{loc}}$ for all multiindices α with $|\alpha| = n$, and $f \in L^p_{\mathrm{loc}}$. We use this fact later.

Assume that ϕ is as in the statement of Theorem 1.5. Let $f \in L^p_{s,\mathrm{loc}} \cap L^\infty$, and $\psi \in C_0^\infty$. For $0 \le s < 1$, Theorem 1.5 is the content of Lemma 3.1 above. For $s = 1$, we have for ν a multiindex with $|\nu| = 1$: $D^\nu \phi(f) = \phi'(f) D^\nu f$. Since f is bounded and ϕ' is locally bounded, it follows that $\phi'(f) \in L^\infty$, and since $f \in L^p_{s,\mathrm{loc}}$, it follows that $D^\nu f \in L^p_{\mathrm{loc}}$, so that $D^\nu \phi(f) \in L^p_{\mathrm{loc}}$. It is also clear that $\phi(f) \in L^\infty$, which proves the theorem in the case $s = 1$.

Now assume $s > 1$. Note that for ν a multiindex with $|\nu| = 1$, we have for $\psi \in C_c^\infty$, $D^\nu(\phi(f)\psi) = \phi'(f) D^\nu f \psi + \phi(f) D^\nu \psi$. We will assume henceforth without loss that $\psi \ge 0$, and that in general all mollifiers ψ are positive. If we can show that $D^\nu(\phi(f)\psi) \in L^p_{s-1}$ for all $|\nu| = 1$, we are finished by the remark preceding the statement of the theorem. Notice that since $\sqrt{\psi} f \in L^p_s \cap L^\infty$, it follows by interpolation that $\sqrt{\psi} f \in L^{p/\theta}_{\theta s}$ for all $\theta, 0 \le \theta \le 1$ (this obtains by interpolation between BMO and Sobolev spaces; see [Tr1],[Tr2]).

We will define here a slight variant on the greatest integer function, whereby $\{s\}$ = greatest integer *strictly* smaller than s. This coincides with the standard definition except that $\{n\} = n - 1$ for integer n.

We have, writing $s = \{s\} + \eta$, for $\psi \in C_c^\infty$, $\psi \ge 0$, by Theorem 1.4,

$$\begin{aligned} &\|J^{s-1}(\psi \phi'(f) D^\nu f)\|_p = \|J^{s-1}((\sqrt{\psi}\phi'(f))(\sqrt{\psi} D^\nu f))\|_p \\ &\le \|J^{\{s\}}(\sqrt{\psi}\phi'(f))\|_{ps/\{s\}} \|J^{\eta-1}\sqrt{\psi} D^\nu f\|_{ps/\eta} + \|\sqrt{\psi}\phi'(f)\|_\infty \|J^{s-1}(\sqrt{\psi} D^\nu f)\|_p. \end{aligned} \tag{3.12}$$

We need to show that the right side above is bounded. The second term is trivially bounded, since $f \in L^p_{s,\mathrm{loc}}$. To bound the first pair of norms on the right, note first that for γ a multiindex and $|\gamma| = \{s\}$, we have by the product rule for derivatives

$$D^\gamma(\sqrt{\psi}\phi'(f)) = \sum_{|\delta| \le \{s\}} d_\delta D^\delta \sqrt{\psi} D^{\gamma - \delta}(\phi'(f)), \tag{3.13}$$

where d_δ are combinatorial coefficients; above γ and δ are multiindices. We consider each of the terms on the right side above. Note that for any multiindex α,

$$D^\alpha \phi'(f) = \sum_{i=1}^{|\alpha|} \phi^{(i+1)}(f) \sum_{\Sigma \beta^j = \alpha} c_{\beta^1 \ldots \beta^i} \prod_{j=1}^{i} f^{(\beta^j)}(x),$$

where β^j are multiindices, and $f^{(\beta^j)} \equiv D^{\beta^j} f$. To show boundedness of $\|J^{\{s\}}(\sqrt{\psi}\phi'(f))\|_{ps/\{s\}}$, it suffices to show $\|D^\alpha(\sqrt{\psi}\phi'(f))\|_{ps/\{s\}}$ is bounded for all α with $|\alpha| \le \{s\}$. To this end, we need only consider terms of the form

$$\phi^{(i+1)}(f)_h \prod_{j=1}^{i} f_h^{(\beta^j)}(x) \tag{3.14}$$

where henceforth we define $f_h^{(\beta^j)} \equiv f^{(\beta^j)}(x)h(x)$ for a C_0^∞ function h (with a similar notation for $\phi^{(i+1)}(f)$). In (3.14), we will assume that $\sum_j |\beta^j| = \{s\}$ and show that the product is in $L^{ps/\{s\}}$. Note that $\phi^{(i)}(f) \in L^\infty$, while $f \in L^p_{s,\mathrm{loc}} \cap L^\infty$. By interpolation, $L^p_{s,\mathrm{loc}} \cap L^\infty \subset L^{sp/|\beta^j|}_{|\beta^j|,\mathrm{loc}}$, so that $f^{(\beta^j)} \in L^{sp/|\beta^j|}_{\mathrm{loc}}$. Thus by Hölder's inequality

$$\|\phi^{(i)}(f)_h \prod_{j=1}^{i} f_h^{(\beta^j)}\|_{ps/\{s\}} \leq \|\phi\|_\infty \|h\|_\infty \prod_{j=1}^{i} \|f_h^{(\beta^j)}\|_{ps/|\beta^j|} < \infty,$$

since $\sum_{j=1}^{i} \frac{|\beta^j|}{ps} \leq \frac{\{s\}}{ps}$. This proves that

$$\|J^{\{s\}}(\sqrt{\psi}\phi'(f))\|_{ps/\{s\}} \leq \infty.$$

However, note also that since $f \in L^p_{s,\mathrm{loc}} \cap L^\infty$ we have by interpolation that $f \in L^{sp/\eta}_{\eta,\mathrm{loc}}$, so (recalling that $|\nu| = 1$) $\sqrt{\psi} D^\nu f \in L^{sp/\eta}_{\eta-1}$ (note that this is a negative Sobolev space), and so $J^{\eta-1}(\sqrt{\psi} D^\nu f) \in L^{sp/\eta}$.

Therefore we conclude that the right side of (3.12) is finite, and so $\psi\phi'(f)(D^\nu f) \in L^p_{s-1}$. Since this is true for arbitrary ψ and arbitrary ν with $|\nu| = 1$, we conclude $\psi D^\nu \phi(f) \in L^p_{s-1}$, so that $\phi(f) \in L^p_{s,\mathrm{loc}}$, as desired. Since $\phi(f) \in L^\infty$, the proof is complete.

We remark that if in the statement of the above theorem, s is an integer, then the condition that ϕ has s^{th} derivative which is Lipschitz can be replaced by the condition that it has $s-1^{th}$ derivative which is Lipschitz. Indeed, in that case the induction argument in the proof can be applied directly without the use of (3.9).

4 Proofs of Smoothing Results

In this section we prove the theorems in the introduction on smoothing by the Schrödinger semigroup e^{-tH} and on the smoothness of eigenfunctions of H. First we prove the global results.

Proof of Theorem 1.2 Let $V \in K_d$, and define a sequence of potentials

$$V_j = J^{-2/j}(V), \quad j \geq 2. \tag{4.1}$$

These potentials exist because $V \in K_d \Rightarrow V \in L^1_{u,\text{loc}}$. By Lemma 2.1 (see remarks after its proof) we have $\lim_{j\to\infty} \|V - V_j\|_{K_d} = 0$.

Consider a sequence of Schrödnger operators

$$H_j = -\frac{1}{2}\Delta + V_j, \quad j \geq 2 \tag{4.2}$$

and the corresponding sequence of semigroups $e^{-tH_j}, \quad t > 0, \quad j \geq 2$. In the next part of the proof we use a finite induction which gives the smoothing result for Schrödinger semigroups by raising the smoothness step by step by small increments.

Assume first that $s \geq 2$ and let $\delta_j = 2/j, \ j \geq 2$. It is clear that

$$\frac{1}{2}\Delta e^{-tH_j} f = -H_j e^{-tH_j} f + V_j e^{-tH_j} f, \ f \in L^p. \tag{4.3}$$

Using (4.3) and the known inequality $\|\lambda\|_{p,\tau+2} \leq c\|\Delta\lambda\|_{p,\tau}, \ \ \tau > 0$, which obtains from Lemma 2 on page 133 of [Ste], we get for any $f \in L^p, \ j \geq 2$, and $0 \leq k \leq j-1$

$$\begin{aligned} \|e^{-tH_j} f\|_{p,s+(k+1)\delta_j} &\leq c\|H_j e^{-tH_j} f\|_{p,s+(k+1)\delta_j-2} + c\|V_j e^{-tH_j} f\|_{p,s+(k+1)\delta_j-2} \\ &\leq c\|H_j e^{-tH_j} f\|_{p,s} + c\|V_j e^{-tH_j} f\|_{p,s+(k+1)\delta_j-2} = I_1 + I_2. \end{aligned} \tag{4.4}$$

The constant c in (4.4) is independent of f, j, k.

To estimate I_2 we use the generalized Leibniz rule (Theorem 1.4). By Theorem 1.4, Lemma 2.1 and formulas (4.1), (2.11),

$$\begin{aligned} I_2 &\leq c\|V_j\|_{\infty,-2+\delta_j} \|e^{-tH_j} f\|_{p,s+k\delta_j} + c\|e^{-tH_j} f\|_\infty \|V_j\|_{p,s+(k+1)\delta_j-2} \\ &\leq c\|V\|_{K_d} \|e^{-tH_j} f\|_{p,s+k\delta_j} + c\|V\|_{p,s} \|e^{-tH_j} f\|_\infty, \end{aligned} \tag{4.5}$$

where c does not depend on j and k. Now by (4.4) and (4.5),

$$\|e^{-tH_j} f\|_{p,s+(k+1)\delta_j} \leq A_{s,j} + B\|e^{-tH_j} f\|_{p,s+k\delta_j}, \tag{4.6}$$

where $A_{s,j} = c\|H_j e^{-tH_j} f\|_{p,s} + c\|V\|_{p,s}\|e^{-tH_j} f\|_\infty, \ \ B = c\|V\|_{K_d}$. By induction then,

$$\|e^{-tH_j} f\|_{p,s+2} \leq A_{s,j} + A_{s,j}B + \ldots + A_{s,j}B^{j-1} + B^j \|e^{-tH_j} f\|_{p,s}. \tag{4.7}$$

In the case $s = 0$,

$$\begin{aligned} &\|e^{-tH_j} f\|_{p,2} \leq c\|H_j e^{-tH_j} f\|_p + c\|V_j e^{-tH_j} f\|_p \\ &\leq c\|H_j e^{-tH_j} f\|_p + c\|V\|_p \|e^{-tH_j} f\|_\infty \leq A_{0,j}. \end{aligned} \tag{4.8}$$

If $0 < s < 2$, then we first reason as in (4.8), and then use the small step induction between 2 and $s+2$ exactly as above.

Thus, we have proved estimate (4.6) for all $s \geq 0$, with c independent of j and such that for every fixed $s = s_0$, c can be chosen uniformly for all $s \leq s_0$. Assume $s \geq 0$ is given and choose the constant c as above. Suppose moreover that

$$\|V\|_{K_d} < c^{-1}. \tag{4.9}$$

Then (4.7) gives $\|e^{-tH_j}f\|_{p,s+2} \leq \frac{A_{s,j}}{1-B} + \|e^{-tH_j}f\|_{p,s}$. By induction with respect to s,

$$\|e^{-tH_j}f\|_{p,s+2} \leq \frac{c_s}{1-B}\|H_je^{-tH_j}f\|_{p,s} + \frac{c_s}{1-B}\|V\|_{p,s}\|e^{-tH_j}f\|_\infty + \|e^{-tH_j}f\|_p, \tag{4.10}$$

where c_s depends on s and not on j.

Now repeat the reasoning above for $H_je^{-tH_j}$ instead of e^{-tH_j}. The starting equality is $\frac{1}{2}\Delta H_je^{-tH_j} = -H_j^2e^{-tH_j} + V_jH_je^{-tH_j}$. The resulting inequality will be

$$\begin{aligned}\|H_je^{-tH_j}f\|_{p,s+2} \leq \frac{c_s}{1-B} & \| \quad H_j^2e^{-tH_j}f\|_{p,s} \\ + & \quad \frac{c_s}{1-B}\|V\|_{p,s}\|H_je^{-tH_j}f\|_\infty + \|H_je^{-tH_j}f\|_p.\end{aligned}$$

Similarly, for higher powers of H_j,

$$\begin{aligned}\|H_j^me^{-tH_j}f\|_{p,s+2} \leq & \frac{c_s}{1-B}\|H_j^{m+1}e^{-tH_j}f\|_{p,s} \\ + & \frac{c_s}{1-B}\|V\|_{p,s}\|H_j^me^{-tH_j}f\|_\infty + \|H_j^me^{-tH_j}f\|_p.\end{aligned} \tag{4.11}$$

Finally, using induction several times,

$$\|e^{-tH_j}f\|_{p,s+2} \leq \frac{c_s'}{(1-B)^{\tau_s}}\|V\|_{p,s}\sum_{\ell=0}^{\tau_s-1}\|H_j^\ell e^{-tH_j}f\|_\infty + \frac{c_s'}{(1-B)^{\tau_s}}\sum_{\ell=0}^{\tau_s}\|H_j^\ell e^{-tH_j}f\|_p, \tag{4.12}$$

where $c_s' > 0$ does not depend on j, and τ_s is defined by $\tau_s = \left[\frac{s}{2}\right] + 2$. Here $[a]$ denotes the integer part of a.

By (4.12) and Lemma 2.5, the right side of (4.12) is uniformly bounded with respect to j for every fixed s and $f \in L^p$. Using the reflexivity of L_p^{s+2}, we obtain that for every $f \in L^p$ there exists a subsequence $\{j'\}$ such that $e^{-tH_{j'}}$ converges weakly in L_{s+2}^p to some $f_0 \in L_{s+2}^p$. But from Lemma 2.6(a) and Lemma 2.1 we also have $e^{-tH_{j'}}f \to e^{-tH}f$ in L^p. Hence $f_0 = e^{-tH}f$, and since $\|e^{-tH}f\|_{p,s+2} \leq \limsup_{j'\to\infty}\|e^{-tH_{j'}}f\|_{p,s+2}$, we obtain from (4.12) and Lemmas 2.1 and 2.5 that

$$\|e^{-tH}f\|_{p,s+2} \leq A(p,s,t,V)\|f\|_p \tag{4.13}$$

for potentials V satisfying $\|V\|_{K_d} < a_s$, with a_s some small number, where A above is a constant which depends only on its arguments.

To eliminate the restriction on V in (4.13), we note $\mathcal{P}_\gamma e^{-tH}\mathcal{P}_\gamma^{-1} = e^{-t\mathcal{P}_\gamma H \mathcal{P}_\gamma^{-1}}, \ \ \gamma > 0$ where $\mathcal{P}_\gamma f(x) = f(\gamma x)$. Moreover, $\mathcal{P}_\gamma^{-1}\overline{H}_\gamma \mathcal{P}_\gamma = \gamma^2 H$ where $\overline{H}_\gamma$ is defined by $\overline{H}_\gamma = -\frac{1}{2}\Delta + \gamma^2\mathcal{P}_\gamma V = -\frac{1}{2}\Delta + \overline{V}_\gamma$, and hence

$$\|e^{-\gamma^2 tH} f\|_{p,s+2} \leq \|\mathcal{P}_\gamma^{-1} e^{-t\overline{H}_\gamma}\mathcal{P}_\gamma f\|_{p,s+2} \tag{4.14}$$

for every $f \in L^p$. If $V \in K_d$ then

$$\|\overline{V}_\gamma\|^*_{K_d} = \|\gamma^2 \mathcal{P}_\gamma V\|^*_{K_d} = \gamma^2 \|J^2 \mathcal{P}_\gamma V\|_\infty$$

$$= \gamma^2 \|\mathcal{P}_\gamma^{-1} J^2 \mathcal{P}_\gamma V\|_\infty = \|\frac{\gamma^2}{I - \mathcal{P}_\gamma^{-1}\Delta\mathcal{P}_\gamma} V\|_\infty = \|\frac{\gamma^2}{I - \gamma^2\Delta} V\|_\infty = \|\frac{1}{\gamma^{-2} I - \Delta} V\|_\infty.$$

The last expression tends to zero as $\gamma \to \infty$ (see (2.1)). Hence, for small γ_0 we have $\|\overline{V}_\gamma\|_{K_d} < c^{-1}, \ \ \gamma < \gamma_0$. It follows from (4.14) and from the case already considered in Theorem 1.3 that $\|e^{-\gamma^2 tH} f\|_{p,s+2} < \infty, \ t > 0, \ f \in L^p, \ \gamma < \gamma_0$. This proves Theorem 1.2.

Proof of sufficiency in Theorem 1.1:

Assume

$$V \in K_d \cap L^p_s. \tag{4.15}$$

Our first goal is to obtain local estimates for potentials satisfying (4.15). Define a sequence $V_j = J^{-1/j} V, \quad j \geq 2$, and consider the corresponding sequence H_j of Schrödinger operators and the sequence e^{-tH_j} of Schrödinger semigroups. Assume a C_0^∞-function ϕ is given. Instead of (4.3) we will use its local version, namely,

$$\frac{1}{2}\Delta(\phi e^{-tH_j} f) = \frac{1}{2}\Delta(\phi) e^{-tH_j} f + \sum_{i=1}^n \frac{\partial\phi}{\partial x_i}\frac{\partial}{\partial x_i} e^{-tH_j} f$$

$$-\phi H_j e^{-tH_j} f + \phi V_j e^{-tH_j} f, \quad f \in L^p. \tag{4.16}$$

As in the proof of Theorem 1.2, first assume $s \geq 2$. Then, reasoning as there, we have for $j \geq 2, \ 0 \leq k \leq 2j - 1$:

$$\|\phi e^{-tH_j} f\|_{p,s+(k+1)1/j} \leq c\|\Delta(\phi) e^{-tH_j} f\|_{p,s}$$

$$+c\sum_{i=1}^n \|\frac{\partial\phi}{\partial x_i}\frac{\partial}{\partial x_i} e^{-tH_j} f\|_{p,s+(k+1)1/j-2} + c\|\phi H_j e^{-tH_j} f\|_{p,s}$$

$$+c\|\phi V_j e^{-tH_j} f\|_{p,s+(k+1)1/j-2} = I_1 + I_2 + I_3 + I_4, \tag{4.17}$$

with an absolute constant c. It is clear that

$$I_2 \leq c\sum_{i=1}^{n}\|\frac{\partial}{\partial x_i}\left(\frac{\partial\phi}{\partial x_i}e^{-tH_j}f\right)\|_{p,s+(k+1)1/j-2} + c\sum_{i=1}^{n}\|\frac{\partial^2\phi}{\partial x_i^2}e^{-tH_j}f\|_{p,s}$$

$$\leq c\sum_{i=1}^{n}\|\frac{\partial\phi}{\partial x_i}e^{-tH_j}f\|_{p,s+(k+1)1/j-1} + c\sum_{i=1}^{n}\|\frac{\partial^2\phi}{\partial x_i^2}e^{-tH_j}f\|_{p,s}. \tag{4.18}$$

Moreover,

$$I_1 + I_2 \leq c\sum_{i=1}^{n}\|\frac{\partial\phi}{\partial x_i}e^{-tH_j}f\|_{p,s+(k+1)1/j-1} + c\sum_{i=1}^{n}\|\frac{\partial^2\phi}{\partial x_i^2}e^{-tH_j}f\|_{p,s}. \tag{4.19}$$

To estimate I_4, we consider a function $\Psi_\phi \in C_0^\infty$ such that the support of ϕ is contained in the interior of the set where Ψ_ϕ equals 1. Using Theorem 1.4,

$$\begin{aligned} I_4 &= c\|\phi\Psi_\phi V_j e^{-tH_j}f\|_{p,s+(k+1)1/j-2} \\ &\leq c\|\Psi_\phi V_j\|_{\infty,-2+1/j}\|\phi e^{-tH_j}f\|_{p,s+k/j} + c\|\Psi_\phi V_j\|_{p,s+(k+1)1/j-2}\|\phi e^{-tH_j}f\|_\infty \\ &\leq c(\|V\|_{1,\mathrm{loc}} + \|\chi_\phi V\|_{K_d})\|\phi e^{-tH_j}f\|_{p,s+k/j} + c\|\Psi_\phi V_j\|_{p,s}\|\phi e^{-tH_j}f\|_\infty. \end{aligned} \tag{4.20}$$

Here, χ_ϕ denotes the characteristic function of the ball centered at 0 and having radius $r+2$, where r is the radius of the smallest ball containing $\mathrm{supp}(\Psi_\phi)$. In the proof of the last estimate we ue the exponential decay of the kernels of Bessel potentials.

Let us denote $A_2(\phi) = c(\|V\|_{1,\mathrm{loc}} + \|\chi_\phi V\|_{k_d})$. Gathering (4.19) and (4.20), we have from (4.17)

$$\begin{aligned} \|\phi e^{-tH_j}f\|_{p,s+(k+1)/j} &\leq c\sum_{i=1}^{n}\|\frac{\partial\phi}{\partial x_i}e^{-tH_j}f\|_{p,s+(k+1)/j-1} \\ &+c\sum_{i=1}^{n}\|\frac{\partial^2\phi}{\partial x_i^2}e^{-tH_j}f\|_{p,s} + c\|\phi H_j e^{-tH_j}f\|_{p,s} \\ &+A_2(\phi)\|\phi e^{-tH_j}f\|_{p,s+k/j} + c\|\Psi_\phi V_j\|_{p,s}\|\phi e^{-tH_j}f\|_\infty. \end{aligned} \tag{4.21}$$

As $\tilde{W}_\alpha f = J_\alpha f$, $\alpha > 0$ is the approximation of the identity (see Section 2), $\lim_{j\to\infty}\|V - V_j\|_{p,s} = 0$. Therefore, $\lim_{j\to\infty}\|\Psi_\phi V - \Psi_\phi V_j\|_{p,s} = 0$ because Ψ_ϕ is a multiplier in the space L_s^p.

For large values of j in (4.21),

$$\begin{aligned} \|\phi e^{-tH_j}f\|_{p,s(k+1)/j} &\leq A_1(\phi) + A_2(\phi)\|\phi e^{-tH_j}f\|_{p,s+k/j} \\ &+c\sum_{i=1}^{n}\|\frac{\partial\phi}{\partial x_i}e^{-tH_j}f\|_{p,s+k+1/j-1}, \end{aligned} \tag{4.22}$$

where

$$A_1(\phi) = c\sum_{i=1}^{n} \|\frac{\partial^2\phi}{\partial x_i^2} e^{-tH_j} f\|_{p,s} + c\|\phi H_j e^{-tH_j} f\|_{p,s} + c\|\Psi_\phi V\|_{p,s}\|\phi e^{-tH_j} f\|_\infty.$$

For k such that $0 \le k \le j-1$ we have from (4.22)

$$\|\phi e^{-tH_j} f\|_{p,s+(k+1)/j} \le \overline{A}_1(\phi) + A_2(\phi)\|\phi e^{-tH_j} f\|_{p,s+k/j}, \tag{4.23}$$

where $\overline{A}_1(\phi) = A_1(\phi) + c\sum_{i=1}^{n} \|\frac{\partial\phi}{\partial x_i} e^{-tH_j} f\|_{p,s}$. Assume

$$A_2(\phi) < 1. \tag{4.24}$$

Then by induction between $k = 0$ and $k = j-1$ we get from (4.23)

$$\|\phi e^{-tH_j} f\|_{p,s+1} \le \frac{\overline{A}_1(\phi)}{1 - A_2(\phi)} + \|\phi e^{-tH_j} f\|_{p,s}. \tag{4.25}$$

Putting $\phi_i = \frac{\partial\phi}{\partial x_i}$, $1 \le i \le n$ instead of ϕ in (4.25),

$$\|\frac{\partial\phi}{\partial x_i} e^{-tH_j} f\|_{p,s+1} \le \frac{\overline{A}_1(\frac{\partial\phi}{\partial x_i})}{1 - A_2(\phi)} + \|\frac{\partial\phi}{\partial x_i} e^{-tH_j} f\|_{p,s},\ 1 \le i \le n.$$

Therefore, (4.23) gives

$$\|\phi e^{-tH_j} f\|_{p,s+k+1/j} \le A_1(\phi) + \frac{c\sum_{i=1}^{n}\overline{A}_1(\frac{\partial\phi}{\partial x_i})}{1 - A_2(\phi)}$$

$$+\sum_{i=1}^{n} \|\frac{\partial\phi}{\partial x_i} e^{-tH_j} f\|_{p,s} + A_2(\phi)\|\phi e^{-tH_j} f\|_{p,s+k/j}.$$

Using induction again, we get

$$\|\phi e^{-tH_j} f\|_{p,s+2} \le \frac{W(\phi)}{1 - A_2(\phi)} + \|\phi e^{-tH_j} f\|_{p,s}, \tag{4.26}$$

where

$$W(\phi) = A_1(\phi) + \frac{c\sum_{i=1}^{n}\overline{A}_1(\frac{\partial\phi}{\partial x_i})}{1 - A_2(\phi)} + \sum_{i=1}^{n} \|\frac{\partial\phi}{\partial x_i} e^{-tH_j} f\|_{p,s}.$$

The estimate (4.26) can be obtained in a similar way in the case $0 \le s \le 2$ (see the reasoning in the proof of Theorem 1.2).

From (4.26) we have

$$\|\phi e^{-tH_j} f\|_{p,s+2} \le \frac{c}{1 - A_2(\phi)}(\sum_{i=1}^{n} \|\frac{\partial^2\phi}{\partial x_i^2} e^{-tH_j} f\|_{p,s}$$

$$+\|\phi H_j e^{-tH_j} f\|_{p,s} + \|\Psi_\phi V\|_{p,s}\|\phi e^{-tH_j} f\|_\infty)$$

$$+\frac{c^2}{(1-A_2(\phi))^2}\sum_{i=1}^{n}\sum_{m=1}^{n}\{\|\frac{\partial^3\phi}{\partial x_i \partial x_m^2}e^{-tH_j}f\|_{p,s} + \|\frac{\partial\phi}{\partial x_i}H_j e^{-tH_j}f\|_{p,s}$$

$$+\|\Psi_\phi V\|_{p,s}\|\frac{\partial\phi}{\partial x_i}e^{-tH_j}f\|_\infty + \|\frac{\partial^2\phi}{\partial x_i \partial x_m}e^{-tH_j}f\|_{p,s}\}$$

$$+\frac{1}{1-A_2(\phi)}\sum_{i=1}^{n}\|\frac{\partial\phi}{\partial x_i}e^{-tH_j}f\|_{p,s} + \|\phi e^{-tH_j}f\|_{p,s}.$$

As $1 - A_2(\phi) < 1$, we have

$$\|\phi e^{-tH_j}f\|_{p,s+2} \leq \frac{c}{(1-A_2(\phi))^2}\{\sum_{\alpha:|\alpha|\leq 3}\|D^\alpha\phi\cdot e^{-tH_j}f\|_{p,s}$$

$$+\|\Psi_\phi V\|_{p,s}\sum_{\alpha:|\alpha|\leq 1}\|D^\alpha\phi\cdot e^{-tH_j}f\|_\infty + \sum_{\alpha:|\alpha|\leq 1}\|D^\alpha\phi H_j e^{-tH_j}f\|_{p,s}\}. \tag{4.27}$$

Reasoning exactly as above, but starting from $H_j^m e^{tH_j}$ instead of e^{-tH_j}, we obtain

$$\|\phi H_j^m e^{-tH_j}f\|_{p,s+2} \leq \frac{c}{(1-A_2(\phi))^2}\{\sum_{\alpha:|\alpha|\leq 3}\|D^\alpha\phi\cdot H_j^m e^{-tH_j}f\|_{p,s}$$

$$+\|\Psi_\phi V\|_{p,s}\sum_{\alpha:|\alpha|\leq 1}\|D^\alpha\phi H_j^m e^{-tH_j}f\|_\infty + \sum_{\alpha:|\alpha|\leq 1}\|D^\alpha\phi H_j^{m+1} e^{-tH_j}f\|_{p,s}\}. \tag{4.28}$$

Using (4.28) in (4.27), we obtain by induction

$$\|\phi e^{-tH_j}f\|_{p,s+2} \leq \frac{c}{(1-A_2(\phi))^{2\tau_s}}\{\sum_{m=0}^{\tau_s}\sum_{\alpha:|\alpha|\leq 3\tau_s}\|D^\alpha\phi H_j^m e^{-tH_j}f\|_p$$

$$+\|\Psi_\phi V\|_{p,s}\sum_{m=0}^{\tau_s}\sum_{\alpha:|\alpha|\leq 3\tau_s}\|D^\alpha\phi\cdot H_j^m e^{-tH_j}f\|_\infty\}, \tag{4.29}$$

where $\tau_s = \left[\frac{s}{2}\right] + 2$.

As $V_j \to V$ in K_d, the numbers $\rho_2(V_j^-)(\tau)$ arising in Lemma 2.5 are bounded away from zero. Applying Lemma 2.5, we get from (4.29)

$$\sup_{j\geq j_0}\|\phi e^{-tH_j}f\|_{p,s+2} \leq \eta\,(1-A_2(\phi))^{-2\tau_s}\,(1+\|\Psi_\phi V\|_{p,s})\|f\|_p \tag{4.30}$$

for every $f \in L^p$ with $\eta > 0$ independent of f. The constant η depends on V only through the number $\rho_2(V^-)(\tau)$, as in Lemma 2.5.

As the space L^p_{s+2} is reflexive, there exists a subsequence $\{j'\}$ such that

$$\phi e^{-tH_{j'}}f \to g \tag{4.31}$$

as $j' \to \infty$ weakly in L^p_{s+2} where g is a function in L^p_{s+2}. Using the fact that $V_{j'} \to V$ in K_d and also estimates (2.17), we conclude the conditions of Lemma 2.6 (b) are fulfilled. Applying this lemma, we get

$$\phi e^{-tH_j} f \to \phi e^{-tH} f \tag{4.32}$$

in L^p.

It follows from (4.31), (4.32) and (4.30) that $g = \phi e^{-tH} f$ and

$$\|\phi e^{-tH} f\|_{p,s+2} \le I < \infty, \tag{4.33}$$

where I denotes the expression on the right-hand side of (4.30).

Consider now a V such that $V \in L^p_{s,\mathrm{loc}},\ \ V^+ \in K_{d,\mathrm{loc}},\ \ V^- \in K_d$. Fix a function $\chi \in C_0^\infty$ for which $0 \le \chi \le 1$ and $\chi(y) = 1$ for all y with $|y| \le 1$. Denote $\chi_K(y) = \chi(y/k),\ y \in R^d,\ k \ge 1$. It is clear that $V_k \stackrel{def}{=} V\chi_k \in L^p_s \cap K_d, \quad (k \ge 1)$ and for every $\phi \in C_0^\infty$,

$$\|V\|_{1,\mathrm{loc}} + \|\chi_\phi V\|_{K_d} < c^{-1} \Rightarrow \|V_k\|_{1,\mathrm{loc}} + \|\chi_\phi V_k\|_{K_d} < c^{-1} \qquad (k \ge 1).$$

Denote by H_k the Schrödinger operator corresponding to the potential V_k. Now we may apply (4.33) and get

$$\|\phi e^{-tH_k} f\|_{p,s+2} \le \eta_k (1 - A_2(\phi))^{-2\tau_s} (1 + \|\Psi_\phi V_k\|_{p,s}) \|f\|_p. \tag{4.34}$$

The η_k are uniformly bounded because the numbers $\rho_2(V_k^-)(\tau)$ are bounded away from zero. As $\Psi_\phi V_k = \Psi_\phi V$ for sufficiently large values of k, we conclude from (4.32) that $\sup_k \|\phi e^{-tH_k} f\|_{p,s+2} < \infty$. Using reflexivity of L^p_{s+2} and Lemma 2.6 (b) as above, we obtain $\|\phi e^{-tH} f\|_{p,s+2} < \infty$. Hence, sufficiency in Theorem 1.1 holds in the case when estimate (4.24) is true.

We now eliminate the last restriction using a rescaling argument similar to one in the proof of Theorem 1.2.

There exists $\gamma_0 > 0$ such that for every $\gamma < \gamma_0$

$$\gamma^2 \|\mathcal{P}_\gamma V\|_{1,\mathrm{loc}} + \gamma^2 \|\mathcal{P}_\gamma(\chi_\phi V)\|_{K_d} < c^{-1}, \tag{4.35}$$

where $\mathcal{P}_\gamma f(x) = f(\gamma x)$, and c is the constant used above. We have $\mathcal{P}_\gamma(\chi_\phi) = \chi_{\mathcal{P}_\gamma(\phi)}$. Then from (4.35) and from the already-considered case we get

$$\|\mathcal{P}_\gamma(\phi) e^{-t\overline{H}_\gamma} f\|_{p,s+2} < \infty \qquad (f \in L^p), \tag{4.36}$$

where $\overline{H}_\gamma = -\frac{1}{2}\Delta + \gamma^2 \mathcal{P}_\gamma(V)$.

Reasoning as in (4.14), we obtain by (4.36)

$$\|\phi e^{-\gamma^2 tH} f\|_{p,s+2} \le \|\phi \mathcal{P}_\gamma^{-1} e^{-t\overline{H}_\gamma} \mathcal{P}_\gamma f\|_{p,s+2} \le c_\gamma \|\mathcal{P}_\gamma(\phi) e^{-t\overline{H}_\gamma} \mathcal{P}_\gamma f\|_{p,s+2} < \infty.$$

This proves the sufficiency part of Theorem 1.1.

Proof of necessity in Theorem 1.1.

Suppose $e^{-tH} f \in L^p_{s+2,\mathrm{loc}}$. Let $x \in \mathbb{R}^d$. Let f be a non-negative C_0^∞ function on $\mathbb{R}^d$ which is not identically 0. By the Feynman-Kac formula (2.13), it is easy to check that $e^{-tH} f$ is everywhere positive and nonvanishing. Indeed $e^{-tH} f$ is simply the expectation over Wiener space of a function on paths which is always positive (the exponential in (2.13)) multiplied by a function $f(b_x(t))$ which clearly is positive on a set of paths $b_x(t)$ of positive measure, namely those paths $b_x(t)$ which at time t are located in the support of f. Using a continuity argument for Brownian motion, we can show using an extension of this argument that $e^{-tH} f$ is uniformly bounded away from 0 in some neighborhood of any $x \in \mathbb{R}^d$.

Let $g \equiv e^{-tH} f$. Then $Hg = -\Delta g + Vg$, so that $V = g^{-1}(Hg + \Delta g)$. Note that $Hg = e^{-\frac{t}{2}H} H e^{-\frac{t}{2}H} f \in L^p_{s+2,\mathrm{loc}}$ by assumption, while $\Delta g \in L^p_{s,\mathrm{loc}}$. On the other hand, note that g is in L^∞, since the range of the semigroup consists of bounded functions (see [Si3], Theorem B.1.1). Therefore, $g \in L^p_{s,\mathrm{loc}} \cap L^\infty$, so that by Theorem 1 on compositions with functions in Sobolev spaces, $1/g \in L^p_{s,\mathrm{loc}} \cap L^\infty$.

To show that $V \in L^p_{s,\mathrm{loc}}$, it suffices to show that the product $Hg \cdot 1/g \in L^p_{s,\mathrm{loc}}$, and similarly for $\Delta g \cdot 1/g$. Consider the second product first. We have for $\psi \in C_0^\infty$, with the support of ψ sufficiently close to x,

$$\|J^s((\Delta g)\cdot(1/g)\psi)\|_p \le c\|J^s((\Delta g)h)\|_p \|(1/g)h\|_\infty + c\|J^{-2}((\Delta g)h)\|_\infty \|J^{s+2}((1/g)h)\|_p \quad (4.37)$$

where $h = \sqrt{\psi}$. Note that

$$J^{-2}(\Delta g)h = -J^{-2}(h(J^2 - 1)g) = -J^{-2} h J^2 g + J^{-2} hg. \quad (4.38)$$

To show $J^{-2}((\Delta g)h) \in L^\infty$, consider the first term on the right of (4.38). Note that for any $a(x) \in L^1(\mathbb{R}^d)$,

$$\int a J^{-2}(hJ^2 g)dx = \int (J^{-2}a) h (J^2 g) dx = \int J^2(h(J^{-2}a)) g dx.$$

Note however that $b = J^{-2} a \in L^1_2$, and $J^2(hb) = (1-\Delta)hb = hb - (\Delta h)b - h\Delta b - 2\nabla h \cdot \nabla b$. Clearly all terms on the right side above are in L^1, so that $J^2(h(J^{-2}a)) \in L^1$. Now, since $g \in L^\infty$, $\int a J^{-2}(hJ^2 g)dx < \infty$.

Since $a \in L^1$ is arbitrary, it follows that $J^{-2}(hJ^2 g) \in L^\infty$. Therefore, the first term on the right side of (4.38) is in L^∞. Clearly, since $hg \in L^\infty$, it follows that $J^{-2}hg \in L^\infty$. (Note that the above argument involving the function a can be stated more precisely if a is also chosen to be in $C_0^\infty \cap L^1$, with L^1 norm less than 1, from which it can be shown that the left hand side is uniformly bounded for all such a, leading to the same conclusion.)

Thus, by (4.38), $J^{-2}((\Delta g)h) \in L^\infty$. This shows that the left side of (4.37) is finite.

Similarly,

$$\|J^s((Hg)\cdot(1/g)\psi)\|_p \leq c\|J^s((Hg)h)\|_p\|(1/g)h\|_\infty + c\|(Hg)h\|_\infty\|J^s((1/g)h)\|_p,$$

and the right hand side of this equation is again finite, since $Hg = e^{-t/2H}He^{-t/2H}f$, which is bounded. Putting the above equations together we conclude $V \in L^p_{s,\mathrm{loc}}$, as desired.

The above reasoning shows that for the necessity in Theorem 1.1 it is enough to show that e^{-tH} maps C_0^∞ into $L^p_{s+2,\mathrm{loc}}$.

This proves necessity in Theorem 1.1, and the proof of the theorem is complete.

Proof of Theorem 1.3. Sufficiency follows from Theorem 1.1, proved above. To prove necessity, suppose the eigenfunctions $\{\phi_\lambda\}_{\lambda\in\Lambda}$ of the Schrödinger operator H are non-trivial on $\mathbb{R}^d$. Then for every point $x \in \mathbb{R}^d$ there exists an eigenfunction ϕ_λ such that $M > \phi_\lambda > \epsilon$ in some neighborhood $\mathcal{O}$ of x. Let $\phi_\lambda \in L^p_{s+2,\mathrm{loc}}$, $\lambda \in \Lambda$. Then $-\frac{1}{2}\Delta\phi_\lambda + V\phi_\lambda = \lambda\phi_\lambda$ implies

$$\tau V = \tau(\lambda + \frac{\Delta\phi_\lambda}{\phi_\lambda}) \tag{4.39}$$

where τ denotes any C_0^∞ function with support contained in $\mathcal{O}$.

Using the ideas in the proof of necessity in Theorem 1.1 and the existence of C_0^∞ resolutions of the identity, we see that (4.39) yields $V \in L^p_{s,\mathrm{loc}}$.

This proves Theorem 1.3.

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