SUP-NORM CONVERGENCE RATES OF WAVELET EXPANSIONS IN BESOV SPACES

Mark A. Kon[1]
Boston University

Louise Arakelian Raphael
Howard University

Abstract

New results for uniform pointwise convergence rates of wavelet expansions in Besov spaces are presented.

KEYWORDS: approximation, Besov spaces, convergence, Sobolev spaces, wavelets

1 INTRODUCTION

This paper announces new approximation properties of wavelet expansions on Besov spaces. Optimal pointwise convergence rates for Besov spaces are established by using embedding theorems ([1], [5]) and results for Sobolev spaces ([12], [13]). More specifically, we show that conditions for convergence rates of expansions of functions in L_2 Sobolev spaces, together with Besov-Sobolev embedding theorems, lead to specific supremum rates of convergence for functions in L_2 Besov spaces.

Besov spaces are the natural spaces for wavelets for two reasons. First, Besov spaces describe the smoothness properties of functions more precisely than Sobolev spaces. Second, and more importantly, there is an essential connection between Besov norms and wavelet expansion coefficients [8]. More precisely, Besov spaces can be characterized in terms of wavelet coefficients, a property which has no direct correspondence in Sobolev spaces.

As expected, the convergence rate of a wavelet expansion for functions in an L_2 Sobolev space $H_2^s(\mathbb{R}^d)$ depends on the smoothness of the expanded function. In [12], uniform pointwise approximation properties are formulated in Sobolev spaces using conditions related to vanishing moment properties for the wavelet (or similar

[1]Research partially supported by the National Science Foundation.

conditions on the scaling function). More precisely, for all projections P_n associated with a multiresolution analysis (MRA),

$$\|f - P_n f\|_\infty \leq C 2^{-n(s-d/2)} \|f\|_{H_2^s(\mathbb{R}^d)} \tag{1.1}$$

whenever $d/2 < s < \sigma$, where s is the smoothness index, d dimension and σ is a fixed parameter.

The Sobolev parameter σ is actually sharp and depends on the MRA used in the expansion [13]. It is related to the behavior of the wavelet's Fourier transform $\widehat{\psi}(\xi)$ near the origin. It can be defined in a number of equivalent ways involving: the operator $I - P_0$, the Fourier transform of the scaling function $\phi(x)$ or of the wavelet $\psi(x)$, or the scaling function's symbol $m_0(\xi)$. For function spaces with smoothness s greater than σ the rate of convergence "freezes" and fails to improve, no matter how large s is [12]. Such behaviors are again expected for wavelet expansions. The key to these proofs is an L_1 bound on the reproducing kernel of the MRA.

For overviews of basic results for wavelet expansions in Sobolev and Besov spaces see the text [8]. For results in nonlinear approximation see [5] and [7]. For pointwise convergence results see [9], [10], and [11]. For results on shift invariant spaces see [2]. Older results on convergence for wavelets are found in [14], [15].

This paper is organized as follows. Basic definitions are given in section 2. Section 3 contains known results on convergence rates of wavelet expansions for Besov spaces from [8]. In section 4 we give our main pointwise convergence rates results for Besov spaces using the Sobolev results proved in [13]. We state the Sobolev results used in section 5.

Finally section 6 gives optimal convergence rates in Besov spaces for Haar, Meyer, Battle-Lemarié and Daubechies wavelets.

2 BASIC DEFINITIONS

A multiresolution analysis (MRA) is defined as an increasing sequence of subspaces $\{V_n\}$ of $L_2(\mathbb{R}^d)(d \geq 1)$ such that

$$f(x) \in V_n \text{ iff } f(2x) \in V_{n+1},$$

the intersection of the spaces is $\{0\}$, the closure of their union is all of L_2, and V_0 is invariant under integer translations. It is also generally assumed (though we do not require it here) that there exists a function $\phi(x)$ (the *scaling function*) whose integer translates form an orthogonal basis (ONB) for V_0. For detailed definitions and theory of an MRA we refer to [4] or [9].

Let W_i be the orthogonal complement of V_i in V_{i+1}, i.e., $W_i = V_{i+1} \ominus V_i$, so that $V_{i+1} = V_i \oplus W_i$. From existence of ϕ it follows that there is a set of basic wavelets $\{\psi^\lambda(x)\}_{\lambda\in\Lambda}$ (with Λ a finite index set) such that $\psi^\lambda_{jk}(x) \equiv 2^{jd/2}\psi^\lambda(2^j x - k)$ ($j \in \mathbb{Z}, k \in \mathbb{Z}^d$) form an orthonormal basis for W_j for fixed j, and form an orthonormal basis for $L^2(\mathbb{R}^d)$ as λ, j, k vary. Our results will hold for any wavelet set $\{\psi^\lambda\}_\lambda$ related to W_0 whose translations and dilations form an orthonormal basis for $L_2(\mathbb{R}^d)$, regardless of how they are constructed (see [4], Ch. 10; [15]; [9]).

It follows from the above definitions that there exist numbers $\{h_k\}_{k\in\mathbb{Z}^d}$ such that the scaling equation

$$\phi(x) = 2^d \sum_{k=-\infty}^{\infty} h_k \phi(2x-k) \tag{2.1}$$

holds. We define

$$m_0(\xi) \equiv \sum_{k=-\infty}^{\infty} h_k e^{-ik\xi} \tag{2.2}$$

to be the *symbol* of the MRA. Note m_0 satisfies $\widehat{\phi}(\xi) = m_0(\xi/2)\widehat{\phi}(\xi/2)$, where $\widehat{\ }$ denotes Fourier transform where our convention for the Fourier transform is

$$\widehat{\phi}(\xi) \equiv \mathcal{F}(\phi)(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbb{R}^d} \phi(x) e^{-i\xi\cdot x} dx$$

where $\xi \cdot x$ denotes inner product.

Definition 2.1: Define P_n to be the L_2 orthogonal projection onto V_n, with kernel (when it exists) $P_n(x,y)$. We define $P_0 = P$.

Given $f \in L_2$,
(i) the *multiresolution approximation* of f is the sequence $\{P_n f\}_n$;
(ii) the *wavelet expansion* of f is

$$\sum_{j;k;\lambda} a^\lambda_{jk} \psi^\lambda_{jk}(x) \sim f, \tag{2.3a}$$

with a^λ_{jk} the L_2 expansion coefficients of f, and $\sim$ denoting convergence in L_2;
(iii) the *scaling wavelet expansion* of f is

$$\sum_f b_k \phi_k(x) + \sum_{j\geq 0;k;\lambda} a^\lambda_{jk} \psi^\lambda_{jk}(x) \sim f, \tag{2.3b}$$

where the b_k, a^λ_{jk} are L_2 expansion coefficients, and $\phi_k(x) = \phi(x-k)$.

We say such sums converge in any given sense (e.g., pointwise, in L_p, etc.) if the sums are calculated in such a way that at any stage in the summation there is a uniform bound on the range (largest minus smallest) of j values for which we have only a partial sum over k, λ.

Definition 2.2: The *Sobolev space* $H_2^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, is defined by

$$H_2^s(\mathbb{R}^d) \equiv \left\{ f \in L_2(\mathbb{R}^d) : \|f\|_{H_2^s} \equiv \sqrt{\int |\widehat{f}(\xi)|^2(1+|\xi|^2)^s d\xi} < \infty \right\}.$$

The *homogeneous Sobolev space* is:

$$\widetilde{H}_2^s(\mathbb{R}^d) \equiv \left\{ f \in L_2(\mathbb{R}^d) : \|f\|_{\widetilde{H}_2^s} \equiv \sqrt{\int |\widehat{f}(\xi)|^2|\xi|^{2s} d\xi} < \infty. \right\}$$

Note the spaces contain the same functions (by virtue of the fact that $\widetilde{H}_2^s$ is restricted to L^2). Only the norms differ, and the second space is incomplete as defined (its completion contains non-L_2 functions which grow at ∞).

We denote the space $\mathcal{F}H_2^s$ to be the Fourier transforms of functions in H_2^s, with the analogous definition for $\mathcal{F}\widetilde{H}_2^s$.

Definition 2.3: A multiresolution analysis (MRA) or corresponding family of wavelets ψ^λ yields *pointwise order of approximation* (or *pointwise order of convergence*) $s > 0$ in H_2^r if for any $f \in H_2^r$, the n^{th} order approximation $P_n f$ satisfies

$$\|P_n f - f\|_\infty = O(2^{-ns}), \tag{2.4}$$

as n tends to infinity, if $r - d/2 > 0$ (if $r - d/2 \leq 0$ the left side of (2.4) is in fact infinite for some f).

It yields *best pointwise order of approximation (or convergence)* $s > 0$ in H_2^r if s is the largest positive number such that (2.4) holds for all $f \in H_2^r$. If the supremum s of the numbers for which (2.4) holds is not attained, then we denote the best pointwise order of convergence by s^-. By convention best order of approximation 0 means that the supremum in (2.4) fails to go to 0; thus $s \geq 0$ by our definitions.

The MRA yields *optimal pointwise order of approximation (or convergence)* s if s is the best pointwise order of approximation for sufficiently smooth f, i.e. for $f \in H_2^r$ for sufficiently large r (i.e., for $r > R$ for some $R > 0$). Thus this order of convergence is the best possible order in any Sobolev space. We say $s = \infty$ if the best order of approximation in H_2^r becomes arbitrarily large for large r.

The motivation for our next definitions is found in [12] where the finiteness of each integral in the definition guarantees the multiresolution approximation yields pointwise order of approximation $s - d/2$ in $H_2^s(\mathbb{R}^d)$.

Definition 2.4: We define for $s, c \geq 0$

$$I_s(c) \equiv \int_{1\geq|\xi|\geq c} \left(1-(2\pi)^{d/2}|\widehat{\phi}(\xi)|\right)|\xi|^{-2s}d\xi \tag{2.5}$$

$$K_s(c) \equiv \sup_\lambda \int_{1\geq|\xi|\geq c} |\widehat{\psi}^\lambda(\xi)|^2|\xi|^{-2s}d\xi$$

$$M_s(c) \equiv \int_{1\geq|\xi|\geq c} (1-|m_0(\xi)|^2)|\xi|^{-2s}d\xi.$$

In this announcement an often-used consequence of [12] is the existence of a least upper bound σ (best Sobolev parameter) on s, depending only on the MRA, which motivates the following definition.

Definition 2.5: The *best Sobolev parameter* σ of an MRA is

$$\sigma = \sup\{s > 0 | (I-P) : \widetilde{H}_2^s(\mathbb{R}^d) \to L_\infty \text{ is bounded}\}. \tag{2.6}$$

By convention $\sigma = 0$ if the set in the supremum is empty.

It can be shown that if the best Sobolev parameter $\sigma \neq 0$, then $\sigma > d/2$ (where d is the dimension) and the set $\Sigma \equiv \{s > 0 | (I-P) : \widetilde{H}_2^s \to L^\infty \text{ is bounded}\}$ satisfies $\Sigma = (d/2, \sigma]$ or $\Sigma = (d/2, \sigma)$.

Moreover if the best Sobolev parameter $\sigma \neq 0$, then

$$\sigma = \sup\{s > 0 | I_s(0) < \infty\} \tag{2.7}$$

$$= \sup\{s > 0 | K_s(0) < \infty\}$$

$$= \sup\{s > 0 | M_s(0) < \infty\}.$$

Definition 2.6: A function $f(x)$ on $\mathbb{R}^d$ is *radial* if f depends on $|x|$ only. A real valued radial function is *radial decreasing* if $|f(x)| \leq |f(y)|$ whenever $|x| \geq |y|$.

A function $\phi(x)$ is in the radially bounded class [RB] if $|\phi(x)| \leq \eta(|x|)$, with $\eta(\cdot)$ a decreasing function on $\mathbb{R}^+$, and $\eta(|x|)$ integrable in x.

With a slight abuse of terminology, a kernel $P(x,y)$ is in [RB] if $|P(x,y)| \leq \eta(|x-y|)$, where $\eta(|x|)$ is, as above, decreasing in $|x|$ and integrable in x.

We say $\phi(x) \in [RB(N)]$ if, $\phi(x) \in [RB]$ and if we can choose an $\eta(x)$ as above such that

$$\int \eta(|x|)|x|^N dx < \infty.$$

3 BASIC RESULTS FOR BESOV SPACES

To define Besov spaces on $\mathbb{R}^n$, we proceed as follows [1]. (For alternative definitions of Besov spaces see [5],[6],[8].)

Definition 3.1: Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multiindex with α_i non-negative integers, and define for a function f on $\mathbb{R}^n$

$$D^n f(x) = \frac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_n^{\alpha_n}} f(x).$$

We define the *Schwarz space of rapidly decreasing functions* on $\mathbb{R}^n$ by

$$\mathcal{S}(\mathbb{R}^n) = \{f(x) : \|(1+|x|)^b D^\alpha f\|_\infty < \infty \ \forall b > 0\}$$

where $\|\cdot\|_\infty$ denotes the essential supremum norm.

It can be shown that there exists a function $\phi \in \mathcal{S}(\mathbb{R}^n)$ whose support supp ϕ satisfies

$$\text{supp } \phi = \{\xi | 2^{-1} \le |\xi| \le 2\},$$

and such that $\phi(x) > 0$ for $2^{-1} < |\xi| < 2$, and

$$\sum_{k=-\infty}^{\infty} \phi(2^{-k}\xi) = 1 \ (\xi \neq 0).$$

We then define functions $\phi_k(x)$ and $\psi(x)$ such that

$$\mathcal{F}\phi_k(\xi) = \phi(2^{-k}\xi) \quad (k = 0, \pm 1, \pm 2, \ldots) \tag{3.1a}$$

$$\mathcal{F}\psi(\xi) = 1 - \sum_{k=1}^{\infty} \phi(2^{-k}\xi). \tag{3.1b}$$

Letting $\mathcal{S}'$ denote the dual space, we define for $f \in S'$, $s \in \mathbb{R}$, $1 \le p, q \le \infty$,

$$\|f\|_p^{s,q} \equiv \|\psi * f\|_p + \left(\sum_{k=1}^{\infty} (2^{sk}\|\phi_k * f\|_p)^q\right)^{1/q},$$

and the *Besov space* $B_p^{s,q}$ to be all functions for which this norm is finite.

We cite a basic result on convergence rates of wavelet expansions in Besov spaces $B_p^{s,q}(\mathbb{R})$ $1 \le p \le \infty$.

Theorem 3.2 [8]: *Let ϕ be a scaling function generating an MRA whose integer translates form an orthonormal system, $\phi \in [RB(N+1)]$ for some integer $N \ge 0$.*

Assume that ϕ is $N+1$ times weakly differentiable, and that the derivative $\phi^{(N+1)}$ satisfies

$$\text{ess sup}_x \sum_k |\phi^{(N+1)}(x-k)| < \infty.$$

Then for any $0 < s < N+1,\ 1 \le p, q \le \infty$, and any function $f \in L_p$, the following conditions are equivalent:

(a) $f \in B_p^{s,q}(\mathbb{R})$
(b) $\|P_j f - f\|_p = 2^{-js}\epsilon_j$ for $j = 0, 1, \ldots,$ with $\{\epsilon_j\}_j \in l_q$
(c) $\|\beta\|_p < \infty$ and $\|\alpha_j\|_{l_p} = 2^{-j(s+1/2-1/p)}\epsilon'_j,\ j = 0, 1, \ldots$ where $\{\epsilon'_j\} \in l_q$, where $\beta = \{b_k\}_{k=-\infty}^{\infty}$ and $\alpha_j = \{a_{jk}\}_{k=-\infty}^{\infty}$ [see (2.3a,b)].

4 MAIN RESULTS FOR BESOV SPACES

We derive optimal pointwise convergence rates for some classes of Besov spaces. Our principal techniques are our convergence rates results for Sobolev spaces [12],[13] and embedding theorems between Sobolev and Besov spaces [1],[5]. The following theorems are extensions of the Sobolev results [13] to sup-norm convergence rates for wavelet expansions of functions in Besov spaces. The Sobolev results used are cited in the next section.

The basic questions regarding embedding take the form: Given a fixed Besov space $B_p^{s,q}$ where s is the smoothness parameter, is it true that every Sobolev space H_2^r is contained in this space for sufficiently large r? Conversely given a Sobolev space H_2^s and fixed p, q, is it true that $B_p^{r,q}$ is contained in this space for sufficiently large r?

The answer is yes and follows from Sobolev embedding theorems. For such embedding theorems see [1],[5]. We now state only those embedding results that we need.

Theorem 4.1: *For $s \in \mathbb{R}$ and $1 \le p \le \infty$ we have*

$$B_p^{s,1} \subset H_p^s \subset B_p^{s,\infty}.$$

In particular, $B_2^{s,1} \subset H_2^s \subset B_2^{s,\infty}$. This yields the following.

Theorem 4.2: *Given a multiresolution approximation $\{P_n\}$, let r be the best order of approximation of P_n in H_2^s, and σ the best Sobolev paramneter of the given MRA (Definition 2.5).*
If $\sigma = 0$, then $r = 0$.
If $\sigma > \frac{d}{2}$ and
(i) $0 \le s \le 2$, then $r = 0$;

(ii) $\frac{d}{2} < s < \sigma$, *then* $r = s - \frac{d}{2}$;

(iii) $s = \sigma$, *then* $r = \begin{cases} \sigma - \frac{d}{2} & \text{if } I_\sigma(0) < \infty \\ (\sigma - \frac{d}{2})^- & \text{if } I_\sigma(0) = \infty \end{cases}$;

(iv) $s > \sigma$, *then* $r = \begin{cases} \sigma - \frac{d}{2} & \text{if } I_{\sigma+\frac{1}{2}}(c) = 0(\frac{1}{c}),\ c \to 0 \\ (\sigma - \frac{d}{2})^- & \text{otherwise} \end{cases}$,

then the order of approximation in the Besov space $B_2^{s,1}$ *is at least* r*, while in* $B_2^{s,\infty}$ *it is at most* r*.*

The proof follows immediately from the Besov embedding Theorem 4.1 and the convergence rates results for $H_2^s(\mathbb{R}^d)$ stated in Theorem 5.1. We note that in (iii) and (iv) above, $I_s(c)$ can be replaced by $K_s(c)$ or $M_s(c)$ (Definition 2.4).

In order to obtain results on optimal orders of convergence we need the following:

Lemma 4.3: *Let* $2 \leq p \leq \infty$*, and* $q \geq 1$*. Given* $s \in \mathbb{R}$*, for sufficiently large* s_1*, we have*

$$H_2^{s_1} \subset B_p^{s,q}. \tag{4.1a}$$

Similarly, if $1 \leq p \leq 2$*, then for sufficiently large* s_1*,*

$$B_p^{s_1,q} \subset H_2^s. \tag{4.1b}$$

Next, to obtain exact optimal sup-norm orders of convergence for functions in Besov spaces we compare the scales $\{H_2^s\}_s$ and $\{B_p^{sq}\}_s$, for fixed p and q. To this end, note that by the above inclusions we have:

Corollary 4.4: *For any* $1 \leq q \leq \infty$*, the scales of spaces* $\{H_2^s\}_{s\geq 0}$ *and* $\{B_2^{s,q}\}_{s\geq 0}$ *are intertwined; that is, for any fixed* s *and sufficiently large* s_1*,* $H_2^{s_1} \subset B_2^{s,q}$*, and* $B_2^{s_1,q} \subset H_2^s$*.*

The next corollary allows us to find precise optimal convergence rates in the scale $\{B_2^{s,q}\}_{s\geq 0}$ for any fixed q. Since this scale intertwines with the scale $\{H_2^s\}_{s\geq 0}$, the optimal rate must be the same, namely:

Corollary 4.5: *If the best Sobolev parameter* $\sigma \neq 0$*, then in the scale* $\{B_2^{s,q}\}_{s\geq 0}$ *of* L_2 *Besov spaces, the wavelet collection* ψ^λ *[or scaling function* ϕ*] has optimal pointwise order of approximation given by*

(i) $\sigma - d/2$ *if* $K_{\sigma+1/2}(c) = O(1/c)$*, and*

(ii) $(\sigma - d/2)^-$ *otherwise.*

(here K *can be replaced by* I *or* M*; see Definition 2.4).*

The proof follows immediately from Corollaries 4.4 and 5.2.

We note that our convergence result for $B_2^{s,q}(\mathbb{R}^d)$ is with respect to sup norm; while for the case $p = 2$, Theorem 3.2 gives convergence rates for $B_2^{s,q}(\mathbb{R})$ with respect to the L_2 norm.

5 POINTWISE CONVERGENCE THEOREMS FOR SOBOLEV SPACES

This section cites the results which are used to prove our main results on Besov spaces. These Sobolev results extend previous results [8] to necessary and sufficient conditions for given convergence rates for expansions in Sobolev spaces on $\mathbb{R}^d$. While these theorems deal with pointwise sup-norm (i.e. L_∞) convergence, they can be extended to convergence results of the same nature for L_p spaces.

The main results in [12] state that under mild assumptions on the MRA (the scaling function or wavelet has a radially decreasing L_1 majorant) for $f \in H_2^s(\mathbb{R})$, the rate of convergence to 0 of the error $\|f - P_n f\|_\infty$ has sharp order $2^{-n(s-d/2)}$. For the sake of brevity, we refer the reader to [12].

We assume that one of the following conditions holds:

(i) The projection P onto V_0 satisfies $|P(x,y)| \leq F(x-y)$ for some $F \in$ [RB].
(ii) The scaling function $\phi \in$ [RB].
(iii) For a wavelet family ψ^λ, $\psi^\lambda(x)(\ln(2+|x|)) \in [RB]$ for all λ.

By representing the kernel of $P(x,y)$ in terms of sums of products involving ϕ or ψ^λ, it is shown in [10] that *(ii)*$\Rightarrow$ *(i)* and *(iii)*$\Rightarrow$*(i)*.

The above conditions are required to prove the main result about convergence rates of wavelet expansions for $H_2^s(\mathbb{R}^d)$ (see [13]).

For approximation rates in H_2^s, we give the following summary of convergence rates in all H_2^s in terms of properties of the projections P_n, or integrals involving the wavelets or scaling functions.

Theorem 5.1 [13]: *Given a multiresolution approximation $\{P_n\}$,*
(o) If $\sigma = 0$, there is no positive order of approximation for the MRA $\{P_n\}$ in any H_2^s, $s \in \mathbb{R}$.
If (o) does not hold then $\sigma > d/2$ and:
(i) For $0 \leq s \leq d/2$, the best pointwise order of approximation in H_2^s is 0;
(ii) If $d/2 < s < \sigma$, the best pointwise order of approximation in H_2^s is $r = s - d/2$;
(iii) If $s = \sigma$, the best pointwise order of approximation in H_2^s is

$$r = \begin{cases} \sigma - d/2 & \text{if } I_\sigma(0) < \infty \\ (\sigma - d/2)^- & \text{if } I_\sigma(0) = \infty \end{cases};$$

(iv) If $s > \sigma$, the best pointwise order of approximation in H_2^s is

$$r = \begin{cases} \sigma - d/2 & \text{if } I_{\sigma+1/2}(c) = O(1/c) \ (c \to 0) \\ (\sigma - d/2)^- & \text{otherwise} \end{cases};$$

(v) In (iii) and (iv) above, $I_s(c)$ can be replaced by $K_s(c)$ or by $M_s(c)$.

Equivalent conditions to those in *(iv)* exist in the form $I_{\sigma+\alpha/2}(c) = O(c^{-\alpha})$ for any (or all) $\alpha > 0$. An interpretation of (iv) is that if $s > \sigma$, then there exists $g \in H_1^s(\mathbb{R}^d)$ such that for all $\epsilon > 0$, $\sup_j 2^{j(\sigma+\epsilon-d/2)} \|g - P_j g\|_\infty = \infty$. This says the convergence rate cannot be improved for functions belonging even to very smooth Sobolev spaces, i.e., convergence rates are wavelet dependent.

In terms of the Sobolev order s of the expanded function f and the best Sobolev parameter σ of the MRA, the diagram above gives rates for an MRA expansion in any Sobolev space. The rates on the boundary region $s = \sigma$ in *(iii)* above are not indicated in the diagram.

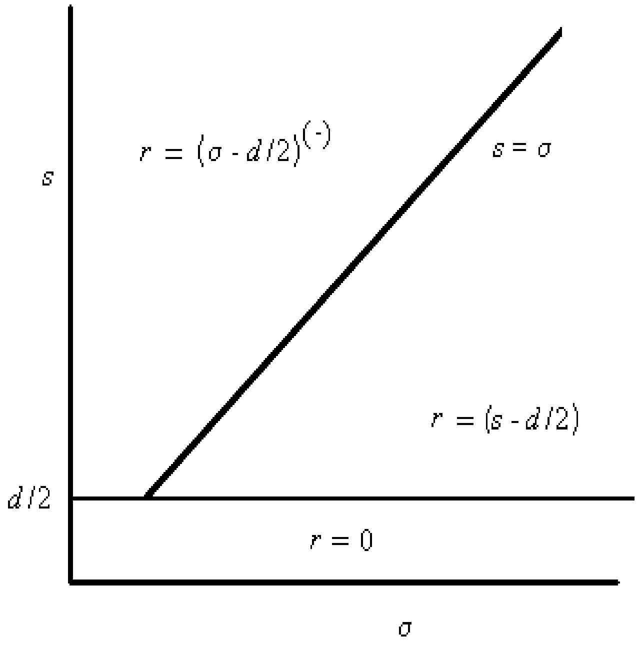

Figure 1: Approximation rate diagram; see Theorem 5.1 (*iii*) for rates on the boundary $s = \sigma$. The $(-)$ in $(\sigma - d/2)^{(-)}$ indicates that the superscript $-$ is present only in some cases.

We now state our result for optimal pointwise orders of convergence in Sobolev spaces, which denotes the highest order of approximation in sufficiently smooth Sobolev spaces.

Corollary 5.2: [13] *If the best Sobolev parameter* $\sigma \neq 0$*, then the wavelet collection* ψ^λ *[or scaling function* ϕ*] yields optimal pointwise order of approximation:*

(i) $\sigma - d/2$ *if* $K_{\sigma+1/2}(c) = O(1/c)$ *[where* K *can be replaced by* I *or* M*], and*

(ii) $(\sigma - d/2)^-$ *otherwise.*

This optimal order is attained for all functions f *with smoothness greater than* σ*, i.e., for* $f \in H_2^s$ *with* $s > \sigma$*.*

Corollary 5.2 gives "best possible" pointwise convergence rates, i.e., convergence rates for the smoothest possible functions. In fact this optimal rate is largely independent of how smoothness is defined, i.e., which particular scale of spaces we are working with. Such a statement is possible because when the smoothness parameter

s is sufficiently large, the most used scales of "smoothness spaces" satisfy inclusion relations. For example for s' large the space $H_2^{s'}$ is contained in the sup-norm Sobolev space H_∞^s and in other L_∞-type Sobolev spaces. Therefore the optimal rates of convergence given here are upper bounds for convergence rates in all H_∞^s spaces, no matter how smooth.

6 EXAMPLES

To illustrate these results we briefly mention applications to some well-known wavelet approximations in $H_2^s(\mathbb{R})$.

6.1 Haar wavelets

By Theorem 5.1, Haar expansions in H_2^s have best order of convergence

$$r = \begin{cases} 0, & s \leq 1/2 \\ s - 1/2, & 1/2 < s < 3/2, \\ 1^-, & s = 3/2 \\ 1, & s > 3/2 \end{cases}. \tag{7.1}$$

The optimal order in scale of Besov spaces is the same as the optimal order in the scale L_2 of Sobolev spaces H_2^s. Namely, the optimal approximation order for such expansions (i.e., for arbitrarily smooth functions) is 1.

6.2 Meyer wavelets

In the case of Meyer wavelets, $\widehat{\phi} \in C_0^\infty$ and $\sigma = \infty^-$. So we have order of convergence $s - 1/2$ in *every* Sobolev space H_2^s, $s > 1/2$, with a convergence order of 0 for $s \leq 1/2$. Thus $f \in \cap_s H^s$ in the intersection of all Sobolev spaces, the convergence is faster than any finite order r.

Thus when $p = 2$, the optimal order in scale of Besov spaces is the same as the optimal order in the scale of Sobolev spaces.

6.3 Battle Lemarié wavelets

When $p = 2$, the optimal order in scale of Besov spaces is the same as the optimal order in the scale of Sobolev spaces. That is, the optimal order of convergence of the spline wavelets in Besov spaces is

$$r = \begin{cases} 0, & s \leq 1/2, \\ s - 1/2, & 1/2 < s < 5/2 \\ 2^-, & s = 5/2 \\ 2, & s > 5/2 \end{cases}.$$

6.4 Daubechies wavelets

For standard Daubechies wavelets of order 2, the optimal order in scale of Besov spaces is the same as the optimal order of Sobolev spaces namely,

$$r = \begin{cases} 0, & s \leq 1/2 \\ s - 1/2, & 1/2 < s < 5/2 \\ 2^{-}, & s = 5/2 \\ 2, & s > 5/2 \end{cases}.$$

Acknowledgments: The first author thanks the National Science Foundation for its support. The second author is especially appreciative to Ron DeVore for sharing his insights into Besov space embedding theorems.

References

1. Bergh, J. and J. Löfstrom, *Interpolation Spaces*, Springer, New York (1976).

2. deBoor, C., R. DeVore and A. Ron, The structure of shift invariant spaces and applications to approximation theory. *J. Functional Analysis* **119**, 37-78 (1994).

3. Cohen, A., R. DeVore, G., Kerkyacherian and D. Picard, *Maximal spaces and given rates of convergence for threshholding algorithms*, preprint [http://www.math.sc.edu/∼devore/publications/Research/Articles.html].

4. Daubechies, I., *Ten Lectures on Wavelets*, CBMS-NSF Series in Applied Mathematics, SIAM, (1992).

5. DeVore, R. Nonlinear Approximation. *Acta Numerica*, 51-151 (1998).

6. Devore R., and R. Lorentz, *Constructive Approximation*, Springer, New York (1993).

7. DeVore, R., P. Petrushev and V. Temlyakov, Nonlinear approximation by trigonometric sums. *J. of Fourier Analysis and Applications*, **1** 29-48 (1995).

8. Härdle, W, G. Kerkyacharian, D. Picard and A. Tsybakov, *Wavelets, Approximation, and Statistical Applications.* In *Lecture Notes in Statistics* **129** (1998).

9. Hernandez, E. and G. Weiss, *First Course in Wavelets*, CRC Press (1996).

10. Kelly, S., M. Kon and L. Raphael, Local convergence of wavelet expansions. *J. Functional Analysis* **126**, 102-138 (1994).

11. Kelly S., M. Kon, and L. Raphael, Pointwise convergence of wavelet expansions. *Bull. Amer. Math. Soc.* **30**, 87-94 (1994).

12. Kon, M. and L. Raphael, Convergence rates of multiresolution and wavelet expansions, in *Wavelet Transforms and Time-Frequency Signal Analysis*, CBMS Conference Proceedings, (L. Debnath, Ed.), Chapter 2, 37-65 (2001).

13. Kon, M. and L. Raphael, A characterization of wavelet convergence in Sobolev spaces, to appear in *Journal of Applicable Analysis I.*

14. Mallat, S., Multiresolution approximation and wavelets. *Trans. Am. Math. Soc.* **315**, 69-88 (1989).

15. Meyer, Y., *Ondelettes*, Hermann, Paris, (1990).

16. Strang, G. and Fix, G., A Fourier analysis of the finite element variational method. In *Constructive Aspects of Functional Analysis*, Edizioni Cremonese, Rome, (1973).

17. Walter, G., Approximation of the delta function by wavelets. *J. Approximation Theory* **71**, 329-343 (1992).

Mark A. Kon
Department of Mathematics
Boston University
Boston, MA 02215 USA
mkon@bu.edu

Louise Arakelian Raphael
Department of Mathematics
Howard University
Washington, DC 20059 USA
lar@scs.howard.edu