"Probability Theory and Related Fields" Vol. 107 (1997) 359-381

LIMIT THEOREMS FOR BIVARIATE APPELL POLYNOMIALS Part. I: CENTRAL LIMIT THEOREMS *†‡

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Revised version (July 10, 1996)

Abstract

Consider the stationary linear process $X_t = \sum_{u=-\infty}^{\infty} a(t-u)\xi_u$, $t \in \mathbf{Z}$, where $\{\xi_u\}$ is an i.i.d. finite variance sequence. The spectral density of $\{X_t\}$ may diverge at the origin (long-range dependence) or at any other frequency. Consider now the quadratic form $Q_N = \sum_{t,s=1}^{N} b(t-s)P_{m,n}(X_t, X_s)$, where $P_{n,m}(X_t, X_s)$ denotes a non-linear function (Appell polynomial). We provide general conditions on the kernels b and a for $N^{-1/2}Q_N$ to converge to a Gaussian distribution. We show that this convergence holds if b and a are not too badly behaved. However, the good behavior of one kernel may compensate for the bad behavior of the other. The conditions are formulated in the spectral domain.

1 Introduction

Central and non-central limit theorems for sums of random variables with long-range dependence have been the subject of many papers (see for example Taqqu (1975), Dobrushin and Major (1979), Breuer and Major (1983), Ho and Sun (1987), Giraitis and Surgailis (1990), Ho and Hsing (1995)). These random variables can arise from Gaussian time series, linear processes or, more generally, non-linear functions of these processes. We consider here not single sums, but quadratic forms. Our goal is to obtain general conditions for

$$Q_N := \sum_{t,s=1}^{N} b(t-s)P_{m,n}(X_t, X_s) \tag{1.1}$$

to satisfy the central limit theorem (CLT), that is, for $N^{-1/2}Q_N$ to converge to a Gaussian distribution as $N \to \infty$. Here $P_{m,n}(X_t, X_s)$ is a multivariate Appell polynomial (Appell polynomials are defined below) and (X_t) is a linear stationary sequence

$$X_t = \sum_{u\in\mathbf{Z}} a(t-u)\xi_u \tag{1.2}$$

*The second author was partially supported by the NSF Grants DMS-9404093 and NCR-9404931 at Boston University. The first author was supported by the NSF Grant NCR-9404931. He would like to thank Boston University for its hospitality.

†*AMS Subject classification.* 60F05, 62M10. *Short title:* CLT for Bivariate Appell Polynomials

‡*Keywords and phrases:* Appell polynomials, central limit theorem, long-range dependence, quadratic forms, time series.

in i.i.d. finite variance random variables $\xi_u,\ u \in \mathbf{Z}$, with mean zero. Our conditions cover a wide spectrum. They are formulated in the spectral domain, that is, on $\widehat{b}$ and on the spectral density $f = 2\pi|\widehat{a}|^2$, where a "hat" denotes the Fourier transform. (For conditions formulated in the time domain, see Giraitis and Taqqu (1996)). Some of the assumptions on f allow for long-range dependence ($f(0) = \infty$), others for discontinuities at other frequencies. We will see that sometimes the good behavior of one kernel, say $\widehat{b}$ can compensate for the behavior of f and vice versa.

The asymptotic behavior of quadratic forms of the type (1.1) is of great importance in determining whether least squares estimators are asymptotically Gaussian. For example, Fox and Taqqu (1986) used the results on quadratic forms in the case $m = n = 1$ (Fox and Taqqu (1987)) to show that the Whittle estimator of the long-range dependence parameter in Gaussian time series is asymptotically normal. This result, as well as the subsequent work of Giraitis and Surgailis (1990), has had a wide impact on the statistical literature (see for example Dahlhaus (1989), Robinson (1994*a*, 1994*b*, 1995)). Introducing different kernels b is useful when considering misspecified models, integrated periodograms, etc ... (Klüppelberg and Mikosch(1995, 1996)). In many real-life situations, e.g. in the Ethernet studies (Leland, Taqqu, Willinger, Wilson (1993, 1994)), the observations are often not Gaussian. There are two standard ways to construct (finite variance) non-Gaussian time series with long-range dependence. One is to use a linear sequence X_t as in (1.2) with non-Gaussian innovations ξ_u, the other is to consider (in addition) some non-linear functions (typically, polynomials) of these sequences. Here, we restrict ourselves to polynomials and include both scenarios by focusing on $P_{m,n}(X_t, X_s)$.

When $m = n = 1$, the asymptotic normality of $N^{-1/2}Q_N$ has been established by Fox and Taqqu (1987) if X_t is Gaussian and by Giraitis and Surgailis (1990) if X_t is linear as in (1.2). We provide a new result in the case $m = n = 1$. We show for example that if either $\widehat{b}$ or $f = 2\pi|\widehat{a}|^2$ is smooth (Lipschitz) and $\int_{-\pi}^{\pi}(\widehat{b}f)^2dx < \infty$ then

$$N^{-1/2}Q_N = N^{-1/2}\sum_{t,s=1}^{N} b(t-s)(X_tX_s - EX_tX_s)$$

converges to a Gaussian distribution as $N \to \infty$.

It is more delicate to derive the asymptotic distribution of Q_N when $m \geq 1, n \geq 1$. This was done by Terrin and Taqqu (1990, 1991) who focused on situations where the limit is non-Gaussian. The goal of this paper, however, is to provide conditions for $N^{-1/2}Q_N$ to converge to a *Gaussian* distribution for all possible choices of such $m \geq 1$ and $n \geq 1$. These conditions, moreover, do not assume, that the kernels b and a in (1.1) and (1.2) respectively, are regularly varying. Thus, they allow for the spectral density of the sequence X_t to diverge at any frequency in the spectrum. Long-range dependence is associated with a divergence at the origin.

In the regularly varying case (see (2.2) and (2.3) below), this paper, together with a companion paper that deals with non-central limit theorems, provides a complete answer to the limiting behavior of the quadratic forms Q_N for all possible values of m and n and all possible exponents of the regularly varying functions. In the next section we introduce the Appell polynomials and state the main theorems. Multivariate extensions are stated in the last section.

2 Statement of the main results

We start with the definition of the multivariate Appell polynomials (see also (1987), (1986)) $P_{n_1,...,n_k}(X_{t_1},...,X_{t_k})$, $n_1,...,n_k = 0,1,....$ They can be defined by the recurrence relations

$$\frac{\partial}{\partial x_j} P_{n_1,...,n_k}(x_1,...,x_k) = n_j P_{n_1,...,n_j-1,...,n_k}(x_1,...,x_k),$$

$$EP_{n_1,...,n_k}(X_{t_1},...,X_{t_k}) = 0.$$

The first relation indicates that these polynomials behave like power functions. The second relation provides the constants of integration and relates the polynomials to the joint distribution of the X_t's. The multivariate Appell polynomials can also be defined by the generating function

$$\sum_{n_1,...,n_k=0}^{\infty} \frac{z_1^{n_1}}{n_1!} \cdots \frac{z_k^{n_k}}{n_k!} P_{n_1,...,n_k}(x_1,...,x_k) = \frac{\exp(\sum_1^k z_j x_j)}{E\exp(\sum_1^k z_j X_{t_j})}.$$

This expression, understood as a formal expansion, can be used even if X_t has only a finite number of moments. Hence, if X_t has $n_1 + \ldots + n_k$ moments,

$$P_{n,...,n_k}(x_1,\ldots,x_k) = \frac{\partial^{n_1+\ldots n_k}}{\partial^{n_1} z_1 \ldots \partial^{n_k} z_k} \frac{\exp(\sum_1^k z_j x_j)}{E\exp(\sum_1^k z_j X_{t_j})}\bigg|_{z_1=\ldots=z_k=0}$$

with, for example, $\frac{\partial^{n_1}}{\partial z_1^{n_1}} E\exp(\sum_1^k z_j X_{t_j})|_{z_1=\ldots=z_k=0} = EX_{t_1}^{n_1}$.

It is convenient to use the *Wick power* notation : : and set

$$:\underbrace{X_{t_1},\ldots,X_{t_1}}_{n_1},...,\underbrace{X_{t_k},\ldots,X_{t_k}}_{n_k} := P_{n_1,...,n_k}(X_{t_1},...,X_{t_k});$$

(the indices in P correspond to the number of times that the variables in ": :" are repeated). The (univariate) polynomial $P_n(X_t) =: \underbrace{X_t \ldots X_t}_{n}:$ is called the Appell polynomial. If X_t is Gaussian, then $P_n(X_t)$ is the Hermite polynomial. Among all Appell polynomials, the Hermite polynomials are the only ones that are orthogonal. The highest degree term of $P_{n_1,...,n_k}(X_{t_1},...,X_{t_k})$ is $X_{t_1}^{n_1}...X_{t_k}^{n_k}$. One always has

$$P_1(X) =: X := X - EX,$$

$$P_{1,1}(X_1, X_2) =: X_1, X_2 := (X_1 - EX_1)(X_2 - EX_2) - E(X_1 - EX_1)(X_2 - EX_2);$$

and, if $EX_1 = EX_2 = 0$,

$$: X_1 X_2^2 := X_1 X_2^2 - X_1 EX_2^2 - 2X_2 EX_1 X_2 - EX_1 X_2^2.$$

The multivariate Appell polynomials $P_{n_1,...,n_k}(X_{t_1},\cdots,X_{t_k})$ or Wick powers are linear combinations of products of the random variables $X_{t_1},\cdots,X_{t_k}$ with coefficients expressed in terms of their joint cumulants. For a general formula, see Relation (4.12) in Giraitis and Surgailis (1986). We assume below that $X_t,\ t \in \mathbf{Z}$ is a linear stationary sequence given by (1.2), where $a(t) = \int_{[-\pi,\pi]} e^{itx}\widehat{a}(x)dx, \quad t \in \mathbf{Z}$ are real-valued weights, $\sum a^2(t) < \infty$ and $\xi_u,\ u \in \mathbf{Z}$ is i.i.d. sequence with mean 0 and finite $2(m+n)$-th moment, $E\xi_u^2 = 1$. We denote by $f(x),\ |x| \le \pi$ the spectral density of the linear sequence (X_t),

$$f(x) = 2\pi|\widehat{a}(x)|^2, \tag{2.1}$$

and by

$$r(t) := EX_tX_0 = \int_{[-\pi,\pi]} e^{itx} f(x)\, dx, \quad t \in \mathbf{Z},$$

its covariance function. The weights $b(\cdot)$ in (1.1) are assumed to have a Fourier transform $b(t) = \int_{[-\pi,\pi]} e^{itx}\widehat{b}(x)\, dx$, where $\widehat{b}(x), x \in [-\pi,\pi]$ is a real even integrable function. The Fourier transforms f and $\widehat{b}$ of r and b respectivelly, can be, in particular, regularly varying, that is,

$$f(x) = |x|^{-\alpha} L_1(|x|^{-1}), \quad x \in [-\pi,\pi] \quad (\alpha < 1) \tag{2.2}$$

$$\widehat{b}(x) = |x|^{-\beta}\ L_2(|x|^{-1}), \quad x \in [-\pi,\pi] \quad (\beta < 1) \tag{2.3}$$

where $L_1(\cdot),\ L_2(\cdot)$ are slowly varying functions at infinity. The conditions $\alpha < 1$ and $\beta < 1$ ensure that f and $\widehat{b}$ are in L^1 and hence that r and b are well defined. The theorems below present sufficient conditions for Q_N to satisfy the central limit theorem. Theorem 2.1 gives a general condition for the CLT (see (2.5) below). While not very explicit, this condition does not require the assumptions (2.2) and (2.3) on f and $\widehat{b}$. It is, moreover, close to the linear growth condition of $\operatorname{Var} Q_N$ as $N \to \infty$. The proof does not use the method of moments. It uses instead the method of approximations (see Giraitis and Surgailis (1990)) which consists of reducing the problem to the study of univariate sums. The spectral domain theorems, Theorems 2.2, 2.3 and 2.4, give more explicit conditions. They make either L^p type assumptions on f and $\widehat{b}$, or use (2.2) and (2.3) and provide conditions on α and β.

For each integer $k \ge 1$, let r^k denote the k-th power of r, $R_N^{(k)} = (r^k(t-s))_{t,s=\overline{1,N}}$, $B = (b(t-s))_{t,s=\overline{1,N}}$, the corresponding Toeplitz matrices and let

$$f^{(*k)}(y) := \int_{[-\pi,\pi]^{k-1}} f(y - x_1 - \ldots - x_{k-1}) f(x_1) \ldots f(x_{k-1})\, d^{k-1}x, \quad k \ge 1 \tag{2.4}$$

be the k-tuple convolution of f with itself. (We assume here and below that the functions $f,\ \widehat{b}$ and $\widehat{a}$ are periodically extended to the whole real line with period 2π.) Recall that we also assume $E|\xi_0|^{2(m+n)} < \infty$, where $m \ge 1$ and $n \ge 1$ are given in (1.1).

Theorem 2.1 (Basic convergence theorem). *If*

$$N^{-1} Tr(R_N^{(m)} B_N R_N^{(n)} B_N) \to (2\pi)^3 \int_{[-\pi,\pi]} f^{(*m)}(x) f^{(*n)}(x) \widehat{b}^2(x)\, dx \tag{2.5}$$

as $N \to \infty$, then the distribution of $Q_N/\sqrt{N}$ tends to the normal distribution $N(0,\sigma^2)$ with variance

(2.6)

$$\sigma^2 := \sum_{\gamma \in \Gamma(m+n,m+n)} d(\gamma)(2\pi)^{r+3} \int_{H(\gamma)} \prod_{v=1}^{2} [\widehat{b}(x_{v,1} + \ldots + x_{v,m}) \widehat{a}(x_{v,1}) \ldots \widehat{a}(x_{v,m+n})]\, d^{2(m+n)}x,$$

where the diagrams γ, the domain of integration $H(\gamma)$ and the product of cumulants $d(\gamma)$ are defined in Section 3.

We will say that "the CLT holds" if $N^{-1/2}Q_N$ tends to a $N(0,\sigma^2)$ with σ^2 given by (2.6). The variance σ^2 of the limiting Gaussian distribution is expressed in terms of "diagrams", that is, partitions of sets of indices. In order, however, not to encumber the presentation, the definition of these diagrams is postponed to Section 3. A multivariate extension of Theorem 2.1 is given in Theorem 6.1.

We now present a number of theorems where the assumption (2.5) of Theorem 2.1 is satisfied. The following theorem provides conditions for the CLT to hold when either $\widehat{b}$ or f is smooth.

Denote by Λ_β, $0<\beta<1$, the class of Lipschitz functions on $[-\pi,\pi]$. By definition, $g\in\Lambda_\beta$ if

$$|g(x)-g(y)| \le K|x-y|^\beta, \quad x,y\in[-\pi,\pi] \quad (\exists K<\infty). \tag{2.7}$$

Theorem 2.2 *Let either $\widehat{b}\in\Lambda_\beta$, $\beta>1/2$ or $f\in\Lambda_\beta$, $\beta>1/2$. Assume that*

$$\widehat{b}f^{(*m)}\in L^2, \widehat{b}f^{(*n)}\in L^2. \tag{2.8}$$

Then (2.5) *holds and so does the CLT.*

Remark. The CLT for Q_N also holds if $f^{(*m)}, f^{(*n)}\in\Lambda_\beta$ with $\beta>1/2$ and (2.8) is satisfied. The proof is similar to that of Theorem 2.2.

Corollary 2.1 *If either f or $\widehat{b}$ is in Λ_β with $\beta>1/2$, then $\int_{[-\pi,\pi]}(\widehat{b}f)^2dx<\infty$ implies the CLT for the quadratic form $Q_N:=\sum_{t,s=1}^N b(t-s)(X_tX_s-EX_tX_s)$ with*

$$\sigma^2=(E\xi_0^2)^2 16\pi^3\int_{[-\pi,\pi]}(\widehat{b}(x)f(x))^2dx+\kappa_4(\xi_0)\left(2\pi\int_{[-\pi,\pi]}\widehat{b}(x)f(x)dx\right)^2,$$

where $\kappa_4(\xi_0)=E\xi_0^4-2(E\xi_0^2)^2$ denotes the fourth cumulant of ξ_0.

We now turn to integrability conditions involving f and $\widehat{b}$ separately.

We will show, when proving the next two theorems, that Condition (2.5) and therefore the CLT holds, if either:

$$\left\{\begin{array}{l} f^{(*m)}\in L^{p_1}, \quad f^{(*n)}\in L^{p_2}, \quad \widehat{b}\in L^q \quad (p_1,p_2,q\ge 1)\\ p_1^{-1}+p_2^{-1}+2q^{-1}\le 1\end{array}\right. \tag{γ1}$$

or

$$\left\{\begin{array}{l} f^{(*m)}\le C|x|^{-\alpha_1}, \quad f^{(*n)}\le C|x|^{-\alpha_2}, \quad |\widehat{b}(x)|\le C|x|^{-\beta}, \quad x\in[-\pi,\pi]\\ \alpha_1+\alpha_2+2\beta<1 \quad (\alpha_1,\alpha_2,\beta<1).\end{array}\right. \tag{γ2}$$

These conditions are not easy to apply because they involve $f^{(*m)}$ and $f^{(*n)}$ instead of f. Theorems 2.3 and 2.4 below provide easily verifiable conditions for (γ1) and (γ2) to hold.

Let

$$d_k(\alpha):=1-k(1-\alpha),$$

and

$$d_k^+(\alpha):=\left\{\begin{array}{ll}\alpha, & \text{if} \quad k=1\\ \max(d_k(\alpha),0), & \text{if} \quad k\ne 1.\end{array}\right. \tag{2.9}$$

Theorem 2.3 *If*

$$f \in L^p, \quad \widehat{b} \in L^q \quad (1 \leq p, q \leq \infty), \tag{2.10}$$

and

$$d_m^+(p^{-1}) + d_n^+(p^{-1}) + 2q^{-1} \leq 1, \tag{2.11}$$

then (2.5) *holds and so does the CLT.*

In the case $m > 1, n = 0$, the result remains true under an additional assumption, that $\widehat{b}(x)$ is continuous at $x = 0$. The variance σ^2 in (2.6) *becomes $\sigma^2 = (2\pi)^3 (\widehat{b}(0))^2 \psi_m(0)$, where ψ_m is the spectral density of the sequence $(P_m(X_t))$.*

In the case $m = 1$, $n = 0$, the result remains true if, in addition, both $f(x)$ and $\widehat{b}(x)$ are continuous at $x = 0$. The variance σ^2 in (2.6) *becomes $\sigma^2 = (2\pi)^3 (\widehat{b}(0))^2 f(0)$.*

Although this paper focuses on cases where $m \geq 1$ and $n \geq 1$, the L^p conditions of Theorem 2.3 extend, as indicated, to the case $m \geq 1$, $n = 0$. When $m > 1, n = 0$ we have $d_n^+(\alpha) = 1$ and hence (2.11) implies $d_m^+(p^{-1}) = 0$ and $q^{-1} = 0$, that is $p \geq m/(m-1)$ and $q = \infty$. To ensure that the conclusion of Theorem 2.3 holds when $n = 0$, we assume not only that $\widehat{b}(x)$ is bounded ($\widehat{b} \in L^\infty$) but also that it is continuous at $x = 0$. The result is useful if $\widehat{b}(0) \neq 0$. If $\widehat{b}(0) = 0$, then the limiting variance σ^2 is 0 which indicates that this case requires further investigation. A similar comment can be made in the case $m = 1, n = 0$, which implies in addition that $p = \infty$ and f bounded.

Theorem 2.4 *If*

$$f(x) \leq C|x|^{-\alpha}, \quad |\widehat{b}(x)| \leq C|x|^{-\beta}, \quad x \in [-\pi, \pi] \tag{2.12}$$

where $\alpha < 1$, $\beta < 1$, and

$$d_m^+(\alpha) + d_n^+(\alpha) + 2\beta < 1, \tag{2.13}$$

then (2.5) *holds and so does the CLT.*

Note that (2.11) and (2.13) are sufficient conditions for the integral in the right hand side of (2.5) to be finite. For example if $f \in L^p$, then $f^{(*m)} \in L^{p_m}$ with $p_m^{-1} = d_m^+(p^{-1})$. Similarly, if we exclude the case $d_m(\alpha) = 0$, then $|f(x)| \leq C|x|^{-\alpha}$ implies $|f^{(*m)}(x)| \leq C|x|^{-\alpha_m}$ where $\alpha_m = d_m^+(\alpha)$ (see Lemma 5.2). Conditions (2.11) and (2.13), it turns out, are also sufficient conditions for the convergence (2.5) to obtain. The assumptions in Theorem 2.3 and 2.4 are formulated in the spectral domain. For assumptions in the time domain, see Giraitis and Taqqu (1996). If α and β in Theorem 2.4 are either both positive or both negative, then the result follows directly from Theorem 2.3. Theorem 2.4 includes, however, a much stronger statement since it allows β to compensate an $\alpha > 0$ and vice-versa. When $m = n = 2$, for example, the condition (2.13) becomes $\beta < 1/2$ if $\alpha < 1/2$ but, if $\alpha > 1/2$, the condition (and hence the CLT) still obtains if β is not too large, namely, $\beta < 1/2 - (2\alpha - 1) = 3/2 - 2\alpha$. When $m = n = 1$, the condition reduces to $\alpha + \beta < 1/2$ which was obtained by Fox and Taqqu (1987). We show in a companion paper that if Condition (2.12) is strenghtened to (2.2) and (2.3) and if Condition (2.13) is replaced by $d_m^+(\alpha) + d_n^+(\alpha) + 2\beta > 1$, then instead of the CLT we have a non-central limit theorem where the limit is given by a (non-Gaussian) Wiener–Itô integral. The results in Theorems 2.1, 2.3 and 2.4 in the linear case $m = n = 1$ were obtained by Surgailis and Giraitis (1990), Avram (1988) and Fox and Taqqu (1987) respectively.

Remark. Under the assumptions of the previous theorems, the finite-dimensional distributions of $\{Q_{[Nt]}/\sqrt{N},\ t \geq 0\}$ converge to those of $\{\sigma B(t),\ t \geq 0\}$ where $B(t)$ is standard Brownian motion. The proof of this more general statement is similar to the proof of the one-dimensional theorems.

The rest of the paper contains the proofs and additional results. Diagrams are defined in the next section. Theorem 2.1 (basic convergence theorem) is proved in Section 4 and Theorems 2.2, 2.3 in Section 5. Section 6 contains multivariate extensions. The value of a constant C that appears in the proofs may change from line to line.

3 Diagrams

Denote by $\Gamma(k_1, k_2)$, $k_1, k_2 \geq 1$, the set of all partitions (diagrams) $\gamma = (V_1, \ldots, V_r) \in \Gamma(k_1, k_2)$, $r \geq 1$, of the table

$$W = \left(\begin{array}{c} (1,1), \ldots, (1,k_1) \\ (2,1), \ldots, (2,k_2) \end{array} \right)$$

by non-empty subsets $V_i \subset W$ such that $\bigcup_{i=1}^{r} V_i = W$ and every V_i, $i = 1, \ldots, r$ contains elements from both rows of the table W. (For example, $V_1 = \{(1,1),(2,1),(2,2)\}$.) Note that $|V_1| + \ldots + |V_r| = k_1 + k_2$, where $|V_i|$ denotes the number of elements $V_i \subset W$. The partitions $\gamma = (V_1, \ldots, V_r) \in \Gamma(k_1, k_2)$ enter in integrals of the form $\int_{H(V)} d^{|V|}x$ and $\int_{H(\gamma)} d^{k_1+k_2}x$ which we shall now define. Let $V = \{(i_1, j_1), \ldots (i_{|V|}, j_{|V|})\}$ be one of the subsets of a partition γ. Writing

$$\int_{H(V)} d^{|V|}x$$

means that we are taking the integral over the hyperplane

$$H(V) = \{(x_{i_1,j_1}, \ldots, x_{i_{|V|},j_{|V|}}) \in [-\pi,\pi]^{|V|} : x_{i_1,j_1} + \ldots + x_{i_{|V|},j_{|V|}} = 0 \bmod 2\pi\}.$$

Thus, when integrating over $H(V)$, one can set $y_1 = x_{i_1,j_1}, \ldots, y_{|V|-1} = x_{i_{|V|-1},j_{|V|-1}}$ and $x_{i_{|V|},j_{|V|}} = -y_1 - \ldots - y_{|V|-1} \bmod 2\pi$. Since a partition $\gamma = (V_1, \ldots, V_r) \in \Gamma(k_1, k_2)$ is made out of disjoint sets of indices $V_1, \ldots, V_r$, we can define

$$\int_{H(\gamma)} d^{k_1+k_2}x = \int_{H(V_1)} d^{|V_1|}x \int_{H(V_2)} d^{|V_2|}x \ldots \int_{H(V_r)^*} d^{|V_r|}x.$$

We do not have full separation of variables because of the presence of $H(V_r)^*$. Integration over $H(V_r)^*$ means that we are integrating over $H(V_r)$ and also on the hyperplane

$$x_{1,1} + \ldots + x_{1,k_1} = 0 \mod 2\pi.$$

Note that the condition $x_{2,1} + \ldots + x_{2,k_2} = 0 \mod 2\pi$ is automatically satisfied when integrating over $H(\gamma)$. For example, if $k_1 = k_2 = 3$ and $\gamma = (V_1, V_2)$ with $V_1 = \{(1,1),(2,1),(2,2)\}$ and $V_2 = \{(1,2),(1,3),(2,3)\}$ then

$$\int_{H(\gamma)} F(x_{1,1}, \ldots, x_{2,3}) d^6x = \int_{H(V_1)} \int_{H(V_2)^*} F(x_{1,1}, \ldots, x_{2,3}) d^6x$$

$$= \int_{[-\pi,\pi]^4 \cap \{y_1 + y_1' + y_2' = 0 \bmod 2\pi\}} F(y_1, y_2, -y_1 - y_2, y_1', y_2', -y_1' - y_2') dy_1 dy_2 dy_1' dy_2'$$

$$= \int_{[-\pi,\pi]^3} F(y_1, y_2, -y_1 - y_2, y_1', -y_1 - y_1', y_1) dy_1 dy_2 dy_1',$$

if F is a function which is periodic with period 2π in each of its arguments. Such integrals appear in the expression of the limiting variance σ^2 given in (2.6). That expression involves also products of cumulants $d(\gamma)$, which are defined as follows: $d(\gamma) := \kappa_{|V_1|} \dots \kappa_{|V_r|}$ where $\kappa_j = \kappa_j(\xi_0)$ is the j^{th} cumulant of the random variable ξ_0 and where $\gamma = (V_1, \dots, V_r)$.

4 Proof of the basic convergence theorem

Proof of the Theorem 2.1. We shall first approximate a and b by a bounded a_K and b_K respectively. Hence set for arbitrary $K > 0$,

$$\widehat{g}_1^{(K)}(x_1, \dots x_m) = (2\pi)^{1/2} |\widehat{b}_K(x_1 + \dots + x_m)|^{1/2} \widehat{a}_K(x_1) \dots \widehat{a}_K(x_m), \tag{4.1}$$

$$\widehat{g}_2^{(K)}(x_1, \dots x_n) = (2\pi)^{1/2} (\widehat{b}_K(x_1 + \dots + x_n))^{\langle 1/2 \rangle} \widehat{a}_K(x_1) \dots \widehat{a}_K(x_n), \tag{4.2}$$

where $\widehat{b}_K(\cdot) = \widehat{b}\mathbb{I}(|\widehat{b}| \le K)$, $\widehat{a}_K(\cdot) = \widehat{a}\mathbb{I}(|\widehat{a}| \le K)$, $\mathbb{I}(A)$ is the indicator function of a set $A \subset \mathbf{R}$ and where $(\widehat{b}_K)^{\langle 1/2 \rangle} = |\widehat{b}_K|^{1/2} \mathrm{sgn}\, \widehat{b}_K$ denotes a signed power. We then approximate the quadratic form Q_N (1.1) by the sum

$$S_N^{(K)} = \sum_{t=1}^{N} Y_K(t) \tag{4.3}$$

where
(4.4)

$$Y_K(t) := \sum_{u_1, \dots, u_{m+n} \in \mathbf{Z}} g_1^{(K)}(t - u_1, \dots, t - u_m) g_2^{(K)}(t - u_{m+1}, \dots, t - u_{m+n}) : \xi_{u_1} \dots \xi_{u_{m+n}} :$$

and $g_j^{(K)}(t_1, \dots, t_{l_j}) = \int_{[-\pi,\pi]^{l_j}} e^{i(t_1 x_1 + \dots + t_{l_j} x_{l_j})} \widehat{g}_j^{(K)}(x_1, \dots, x_{l_j})\, d^{l_j} x,\ j = 1, 2;\ l_1 = m,\ l_2 = n.$

Theorem 2.1 is a direct consequence of the following two lemmas.

Lemma 4.1 *If the condition* (2.5) *of Theorem 2.1 holds, then*

$$\lim_{K \to \infty} \limsup_N N^{-1} \mathrm{Var}\, (Q_N - S_N^{(K)}) = 0.$$

Lemma 4.2 *For any fixed* $K > 0$,

$$N^{-1/2} S_N^{(K)} \overset{d}{\Rightarrow} N(0, \sigma_K^2) \quad (N \to \infty) \tag{4.5}$$

with finite variance σ_K^2*, where* σ_K^2 *denotes the right side of* (2.6) *in which* $\widehat{a}$, $\widehat{b}$ *are replaced by* $\widehat{a}_K$ *and* $\widehat{b}_K$*, respectively. Moreover, as* $K \to \infty$

$$\sigma_K^2 \to \sigma^2. \tag{4.6}$$

PROOF OF LEMMA 4.1. Set $k = m + n$ and let

$$S := \sum_{u_1,\dots u_k \in \mathbf{Z}} c(u_1, \dots, u_k) : \xi_{u_1} \dots \xi_{u_k} :,$$

with weights $c(u_1, \dots, u_k)$ satisfying the condition $c(\cdot) \in l^2(\mathbf{Z}^k)$. Then (see Giraitis and Surgailis (1986), Theorem 4 (iv)),

$$\text{(4.7)} \qquad \operatorname{Var} S = ES^2 = \sum_{\gamma \in \Gamma(k,k)} \sum_{u_1,\dots,u_r \in \mathbf{Z}} c(u_{11}, \dots, u_{1k}) c(u_{21}, \dots, u_{2k})\, d(\gamma)$$

where $\Gamma(k,k)$ and $d(\gamma)$ are as in (2.6) and $u_{ij} \equiv u_l$ for $(i,j) \in V_l$, $V_l \in \gamma = (V_1, \dots, V_r)$. Using the Cauchy inequality it is easy to verify that

$$\text{(4.8)} \qquad \operatorname{Var} S \le C(k, \xi_0) \sum_{u_1,\dots,u_k \in \mathbf{Z}} c^2(u_1, \dots, u_k)$$

where the constant $C(k, \xi_0)$ depends only on k and the distribution of ξ_0. Since

$$Q_N = \sum_{u_1,\dots,u_{m+n} \in Z} d_{1,N}(u_1, \dots, u_{m+n}) : \xi_{u_1} \dots \xi_{u_{m+n}} :$$

where

$$d_{1,N}(u_1, \dots, u_{m+n}) = \sum_{t,s=1}^{N} b(t-s) a(t-u_1) \dots a(t-u_m) a(s-u_{m+1}) \dots a(s-u_{m+n})$$

(multilinearity of Wick powers) and

$$S_N^{(K)} = \sum_{u_1,\dots,u_{m+n} \in Z} d_{2,N}^{(K)}(u_1, \dots, u_{m+n}) : \xi_{u_1} \dots \xi_{u_{m+n}} :$$

where

$$d_{2,N}^{(K)}(u_1, \dots, u_{m+n}) := \sum_{t,s=1}^{N} \delta(t-s) g_1^{(K)}(t-u_1, \dots, t-u_m) g_2^{(K)}(s-u_{m+1}, \dots, s-u_{m+n}),$$

the inequality (4.8) gives

$$\text{(4.9)} \qquad \operatorname{Var}(Q_N - S_N^{(K)}) \le C(k, \xi_0) \sum (d_{1,N}(u_1, \dots, u_{m+n}) - d_{2,N}^{(K)}(u_1, , \dots, u_{m+n}))^2.$$

Expressing b and a in terms of their Fourier transforms $\widehat{b}$ and $\widehat{a}$, we have

$$d_{1,N} = \int_{[-\pi,\pi]^{m+n}} e^{-i(x_1 u_1 + \dots + x_{m+n} u_{m+n})} \left[\int_{[-\pi,\pi]} D_N(y + x_1 + \dots + x_m) D_N(-y + x_{m+1} + \dots + x_{m+n}) \widehat{b}(y)\, dy \right]$$

$$\text{(4.10)} \qquad \cdot e^{i\frac{(N+1)}{2}(x_1 + \dots + x_{m+n})} \widehat{a}(x_1) \dots \widehat{a}(x_{m+n}) d^{m+n} x,$$

and

$$d_{2,N} = \int_{[-\pi,\pi]^{m+n}} e^{-i(u_1x_1+\ldots+u_{m+n}x_{m+n})}\frac{1}{2\pi}\left[\int_{[-\pi,\pi]} D_N(y+x_1+\ldots+x_m)D_N(-y+x_{m+1}+\ldots+x_{m+n})\,dy\right]$$

$$\cdot\, e^{i\frac{(N+1)}{2}(x_1+\ldots+x_{m+n})}\widehat{g}_1^{(K)}(x_1,\ldots,x_m)\widehat{g}_2^{(K)}(x_{m+1},\ldots,x_{m+n})\,d^{m+n}x, \tag{4.11}$$

since $\sum_{t=1}^N e^{ixt} = e^{ix(N+1)/2}D_N(x)$, where $D_N(x) = \sin(Nx/2)/\sin(x/2)$ denotes the Dirichlet kernel. Using the definitions (4.10) and (4.11) of $\widehat{g}_1^{(K)}$ and $\widehat{g}_2^{(K)}$ and Parseval's identity one can verify that

$$i_N := N^{-1}\sum(d_{1,N} - d_{2,N}^{(K)})^2 = \|p_1 - p_2^{(K)}\|_N^2 \tag{4.12}$$

where

$$\|p\|_N^2 = \int_{[-\pi,\pi]^4} p(z_1,z_2,y_1)\overline{p(z_1,z_2,y_2)}\,d\mu_N,$$

(4.13)

$$\mu_N(A) := N^{-1}\int_A D_N(z_1-y_1)D_N(z_1-y_2)D_N(z_2-y_1)D_N(z_2-y_2)dz_1dz_2dy_1dy_2$$

is a (signed) measure on $[-\pi,\pi]^4$, and where p_1 and $p_2^{(K)}$ are defined as follows: $p_1^{(K)}(z_1,z_2,y) := \widehat{b}_K(y)[f_K^{(*m)}(z_1)f_K^{(*n)}(z_2)]^{1/2}$ $(0 < K \le \infty)$, $p_1(z_1,z_2,y) := p_1^{(\infty)}(z_1,z_2,y)$, $p_2^{(K)}(z_1,z_2,y) := |\widehat{b}_K(z_1)|^{1/2}(\widehat{b}_K(z_2))^{\langle 1/2\rangle}[f_K^{(*m)}(z_1)f_K^{(*n)}(z_2)]^{1/2}$. Here $f_K^{(*k)}(\cdot)$ denotes the right side of (2.4) in which f is replaced by $f_K(x) := 2\pi|\widehat{a}_K(x)|^2$. Now (4.12) implies that

$$i_N \le 2(\|p_1^{(K)} - p_2^{(K)}\|_N^2 + \|p_1 - p_1^{(K)}\|_N^2).$$

Choosing $K > 0$ such that the set $\{x \in [-\pi,\pi] : \widehat{b}(x) = K\}$ has Lebesgue measure zero, we get

$$\lim_{K\to\infty}\lim_{N\to\infty}\|p_1^{(K)} - p_2^{(K)}\|_N^2 = 0 \tag{4.14}$$

and

$$\lim_{K\to\infty}\limsup_N \|p_1 - p_1^{(K)}\|_N^2 = 0. \tag{4.15}$$

We omit the proofs of (4.14) and (4.15) since they are similar to those of relations (2.14) and (2.15) in Giraitis and Surgailis (1990). This establishes the lemma. □

Proof of Lemma 4.2. The proof is in three steps.

(i) First we show that

$$N^{-1}\mathrm{Var}\ S_N^{(K)} \to \sigma_K^2 \quad (N\to\infty) \tag{4.16}$$

where σ_K^2 denotes the right side of (2.6) in which $\widehat{a}$, $\widehat{b}$ are replaced by $\widehat{a}_K$ and $\widehat{b}_K$, respectively. Relations (4.3), (4.7) and Parseval's equality imply that

$$\mathrm{Var}\ S_N^{(K)} = \sum_{t,t'=1}^N EY_K(t)Y_K(t')$$

$$= \sum_{\gamma=(V_1,\ldots,V_r)\in\Gamma(m+n,m+n)} d(\gamma)(2\pi)^r \int_{H(V_1)} d^{|V_1|}x \int_{H(V_2)} d^{|V_2|}x \ldots \int_{H(V_r)} d^{|V_r|}x |D_N(x_{1,1}+\ldots+x_{1,m+n})|^2$$

$$\cdot \prod_{j=1}^{2} \widehat{g}_1^{(K)}(x_{j,1},\ldots,x_{j,m})\widehat{g}_2^{(K)}(x_{j,m+1}\ldots x_{j,m+n}). \tag{4.17}$$

By the definition of $\int_{H(V)} d^{|V|}x$ and of the partition γ, we have

$$(x_{1,1}+\ldots+x_{1,m+n}) \bmod 2\pi \equiv -(x_{2,1}+\ldots+x_{2,m+n}) \bmod 2\pi.$$

Moreover,

$$N^{-1}\int_{[-\pi,\pi]} D_N^2(u)h(u)\,du \to 2\pi h(0) \quad (N\to\infty)$$

if h is continuous at $u=0$. These two facts and the definitions (4.1), (4.2) of $\widehat{g}_1^{(K)}, \widehat{g}_2^{(K)}$ imply the convergence (4.16). (ii) Next we prove (4.6), namely, $\lim_{K\to\infty}\sigma_K^2 = \sigma^2$. Since

$$\mathrm{Var}(S_N^{(K)} - S_N^{(K')}) \le 2[\mathrm{Var}\,(S_N^{(K)} - Q_N) + \mathrm{Var}\,(S_N^{(K')} - Q_N)],$$

$$\mathrm{Var}(S_N^{(K)} + S_N^{(K')}) \le [3((\mathrm{Var}\,(S_N^{(K)} - Q_N) + \mathrm{Var}\,(S_N^{(K')} - Q_N) + \mathrm{Var}\,(2Q_N))],$$

$$\mathrm{Var}(Q_N) \le 2[\mathrm{Var}\,(Q_N - S_N^{(K)}) + \mathrm{Var}\,(S_N^{(K)})],$$

we obtain by Lemma 4.1 and (4.16) that

$$|\sigma_K^2 - \sigma_{K'}^2| \le \limsup_N N^{-1}[\mathrm{Var}(S_N^{(K)} - S_N^{(K')})\mathrm{Var}(S_N^{(K)} + S_N^{(K')})]^{1/2} \to 0 \quad (K,K'\to\infty)$$

which establishes (4.6). (iii) Finally, we prove the CLT (4.5). From the definitions of σ_K^2, (4.1), (4.2) and the Cauchy inequality, we get

$$\sigma_K^2 \le \mathrm{const}\, \|\widehat{g}_1^{(K)}\|^2_{L^4([-\pi,\pi]^m)} \|\widehat{g}_2^{(K)}\|^2_{L^4([-\pi,\pi]^n)}. \tag{4.18}$$

Since trigonometric polynomials are dense in L^4, for any $\epsilon > 0$ one can find a constant $M = M(\varepsilon) > 0$ and numbers $p_j(t_1,\ldots,t_{l_j})$ equal to 0 for $|t_1|,...,|t_{l_j}| > M$, $l_1 = m, l_2 = n$ such that

$$\|\widehat{g}_j^{(K)} - \widehat{p}_j\|_{L^4([-\pi,\pi]^{l_j})} < \varepsilon, \quad j=1,2.$$

Since

$$Y_{K,\varepsilon}(t) := \sum_{u_1,\ldots,u_{m+n}\in\mathbf{Z}} p_1(t-u_1,\ldots,t-u_m)p_2(t-u_{m+1},\ldots,t-u_{m+n}) : \xi_{u_1}\ldots\xi_{u_{m+n}} :$$

is a stationary $2M$ dependent sequence, the partial sums $S_N^{(K,\varepsilon)} := \sum_{t=1}^N Y_{K,\varepsilon}(t)$ satisfy the CLT, namely,

$$N^{-1/2}S_N^{(K,\varepsilon)} \Rightarrow N(0,\sigma^2_{K,\varepsilon}) \quad (N\to\infty)$$

for some $\sigma^2_{K,\varepsilon} \geq 0$ (Billingsley (1986)). To obtain the CLT (4.5) one has only to verify that $\sigma^2_{K,\varepsilon} \to \sigma^2_K \quad (\varepsilon \to 0)$. This last relation holds because

$$|\sigma^2_{K,\varepsilon} - \sigma^2_K| \leq \limsup_N N^{-1}|[\mathrm{Var}(S_N^{(K)}) - \mathrm{Var}(S_N^{(K,\epsilon)})]|$$

$$\leq \limsup_N N^{-1}[(\mathrm{Var}(S_N^{(K)} - S_N^{(K,\epsilon)})(\mathrm{Var}(S_N^{(K)} + S_N^{(K,\epsilon)})]^{1/2},$$

and because, by (4.16) and (4.18),

$$\limsup_N N^{-1}\mathrm{Var}(S_N^{(K)} - S_N^{(K,\epsilon)}) \leq C(\|\widehat{g}_1^{(K)} - \widehat{p}_1\|^2_{L^4}\|\widehat{q}_2\|^2_{L^4}$$

$$+\|\widehat{p}_1\|^2_{L^4}\|\widehat{g}_2^{(K)} - \widehat{p}_2\|^2_{L^4}) \to 0; \quad (\epsilon \to 0)$$

$$\limsup_N N^{-1}\mathrm{Var}(S_N^{(K)} + S_N^{(K,\epsilon)}) \leq 2\limsup_N N^{-1}[\mathrm{Var}(S_N^{(K)}) + \mathrm{Var}(S_N^{(K,\epsilon)})]$$

$$\leq C(\|\widehat{g}_1^{(K)}\|^2_{L^4}\|\widehat{g}_2^{(K)}\|^2_{L^4} + \|\widehat{p}_1\|^2_{L^4}\|\widehat{p}_2\|^2_{L^4}) \leq C$$

as $N \to \infty$ (cf. (3.8) in Giraitis and Surgailis (1990)). This proves (4.5) and completes the proof of both Lemma 4.2 and Theorem 2.1. □

Remark. It follows, as in Q4) of Giraitis and Surgailis (1990) that (2.5) also holds if

$$\text{(4.19)} \qquad \begin{cases} f^{(*k)}, \quad f^{(*k)}\widehat{b}, \quad \widehat{b} \in L^2 \quad (p_1, p_2, q \geq 1) \text{ for } \quad k = m, n \quad \text{ and} \\ \int_{[-\pi,\pi]} |f^{(*k)}(x)\widehat{b}(x-y)|^2 dx \to \int_{[-\pi,\pi]} |f^{(*k)}(x)\widehat{b}(x)|^2 dx \quad (y \to 0), \quad k = m, n. \end{cases}$$

5 Proof of the spectral domain theorems

We use the following lemma in the proof of Theorem 2.2. (Λ_β is defined in (2.7)).

Lemma 5.1 *Let $\widehat{g} \in \Lambda_\beta$, $\beta > 1/2$. Set $g(j) = \int_{[-\pi,\pi]} e^{ijx}\widehat{g}(x)dx$ and put $S_{n,m}(\widehat{g})(x) = (2\pi)^{-1}\sum_{j=n}^m e^{-ijx}g(j)$. Then uniformly in $n < 0$, $m > 0$,*

$$\text{(5.1)} \qquad \sup_x |\widehat{g}(x) - S_{n,m}(\widehat{g})(x)| \leq CK(1 + \ln(\max(|n|,|m|))(\frac{1}{\min(|n|,|m|)})^\beta,$$

where K is the Lipschitz constant from (2.7), and C does not depend on g.

If $\min(|n|,|m|) = 0$ then $\sup_x |\widehat{g}(x) - S_{n,m}(\widehat{g})(x)| \leq C\sup_x |\widehat{g}(x)|(1 + \ln\max(|n|,|m|))$.

Proof of Lemma 5.1. Let $T_k(x)$ denote a trigonometric polynomial of order k which is the best approximation to the continuous function $\widehat{g}$. Set $\hat{h} = \widehat{g} - T_{\min(|n|,|m|)}$. Then, as in Zygmund (1979) (13.25), p.120,

$$\widehat{g} - S_{n,m}(\widehat{g}) = \hat{h} - S_{n,m}(\hat{h}) + T_{\min(|n|,|m|)} - S_{n,m}(T_{\min(|n|,|m|)}) = \hat{h} - S_{n,m}(\hat{h}).$$

Denote $E_k(\widehat{g}) = \sup_x |\widehat{g} - T_k|$. Then, using $\hat{g} \in \Lambda_\beta$ and applying Theorem (13.6) of Zygmund (1979), p.115, to both the real and imaginary part of $\widehat{g}$, we get

$$E_k(\widehat{g}) \leq CK(\frac{1}{k})^\beta \quad \text{if} \quad k > 0,$$

and clearly, $E_0(\widehat{g}) \leq \sup_x |\hat{g}(x) - T_0| \leq \sup_x |\widehat{g}(x)|$ since T_0 is the best approximation. Further, note, that $|S_{n,m}(\hat{h})(y)| = |(1/2\pi)\int_{[-\pi,\pi]}\sum_{j=n}^m e^{ij(x-y)}\hat{h}(x)dx| \leq E_{\min(|n|,|m|)}(\widehat{g})\int_{[-\pi,\pi]}|\sum_{j=n}^m e^{ij(x-y)}|dx \leq C(1+\ln(\max(|n|,|m|))E_{\min(|n|,|m|)}(\widehat{g})$, where C does not depend on $\widehat{g}$ an d n, m. Thus,

$$|\hat{h} - S_{n,m}(\hat{h})| \leq |\hat{h}| + |S_{n,m}(\hat{h})| \leq CE_{\min(|n|,|m|)}(\widehat{g})(1 + \ln\max(|n|,|m|))$$

$$\leq CK(1 + \ln\max(|n|,|m|))(\frac{1}{\min(|n|,|m|)})^\beta$$

if $\min(|n|,|m|) > 0$. Obviously, if $\min(|n|,|m|) = 0$, then $|\hat{h} - S_{n,m}(\hat{h})| \leq C(1 + \ln\max(|n|,|m|))\sup_x |\widehat{g}(x)|$. Lemma 5.1 is proved. □

Proof of Theorem 2.2. Let $\widehat{b} \in \Lambda_\beta$, $\beta > 1/2$. (The case $f \in \Lambda_\beta$, $\beta > 1/2$ can be considered similarly, taking into account that $f \in \Lambda_\beta$ implies $f^{(*m)}, f^{(*n)} \in \Lambda_\beta$).

We show that the assumption (2.5) is satisfied. We have

$$i_N := N^{-1}Tr(R_N^{(m)}B_NR_N^{(n)}B_N) = N^{-1}\sum_{t_1,s_1,t_2,s_2=1}^{N} b(t_1 - s_1)b(t_2 - s_2)r^m(t_1 - t_2)r^n(s_1 - s_2)$$

$$= N^{-1}\sum_{s_1,t_2=1}^{N}\sum_{v_1=1-s_1}^{N-s_1} b(v_1)r^m(v_1 + s_1 - t_2)\sum_{v_2=1-t_2}^{N-t_2} b(v_2)r^n(v_2 - s_1 + t_2).$$

$$= N^{-1}\sum_{s_1,t_2=1}^{N} 2\pi S_{1-s_1,N-s_1}(\widehat{B}_{m,s_1-t_2})(0)2\pi S_{1-t_2,N-t_2}(\widehat{B}_{n,t_2-s_1})(0),$$

where S is as in Lemma 5.1, and where for fixed u,

$$\widehat{B}_{m,u}(y) = \int_{[-\pi,\pi]}\widehat{b}(y - x)e^{iux}f^{(*m)}(x)dx$$

has Fourier coefficients $b(v)r^m(v + u)$. The assumption $\widehat{b} \in \Lambda_\beta$ implies that

$$|\widehat{B}_{m,u}(y) - \widehat{B}_{m,u}(y')| \leq K\left(\int_{[-\pi,\pi]} f^{(*m)}(x)dx\right)|y - y'|^\beta$$

that is, $\widehat{B}_{m,u} \in \Lambda_\beta$ uniformly in $u \in \mathbf{Z}$. By Lemma 5.1, for $1 \leq s \leq N$,

$$S_{1-s,N-s}(\widehat{B}_{m,u})(0) = \widehat{B}_{m,u}(0) + O\left(\ln N\left(\frac{1}{(1+|s|)^\beta} + \frac{1}{(1+|N-s|)^\beta}\right)\right) \tag{5.2}$$

where the "O" term is uniform in u. Observe that $\widehat{B}_{m,u}(0) = \int_{[-\pi,\pi]}\widehat{b}(x)e^{iux}f^{(*m)}(x)dx$. (One has also $\widehat{B}_{m,u}(0) = (2\pi)^{-1}\sum_{v\in\mathbf{Z}} b(v)r^m(v + u))$. For convenience of notation, we set $R_m(u) = \widehat{B}_{m,u}(0)$ and denote the "O term" by $R^*_{N,m}(s)$. Since $\widehat{b}$ and $f^{(*m)}$ are even, we have $R_m(u) = R_m(-u)$. Applying (5.2) we obtain:

$$i_N = N^{-1}(2\pi)^2[\sum_{s,t=1}^{N}[R_m(s-t)R_n(s-t) + R_m(s-t)R^*_{N,n}(t) + R_n(s-t)R^*_{N,m}(s) + R^*_{N,m}(s)R^*_{N,n}(t)].$$

From (2.8) it follows that $R_m \in l^2, R_n \in l^2$. Therefore $N^{-1}\sum_{s,t=1}^{N} R_m(s-t)R_n(s-t) = \sum_{|k|<N}(1-\frac{|k|}{N})R_m(k)R_n(k) \to \sum_{s\in\mathbf{Z}} R_m(s)R_n(s) = 2\pi\int_{[-\pi,\pi]} f^{(*m)}(x)f^{(*n)}(x)\widehat{b}^2(x)\,dx$. We also have $N^{-1}\sum_{s,t=1}^{N}[|R_m(s-t)R^*_{N,n}(t)| + |R_n(s-t)R^*_{N,m}(s)| + |R^*_{N,m}(s)R^*_{N,n}(t)|] = o(1)$ as $N \to \infty$. Indeed,

$$\sum_{s,t=1}^{N}|R_m(s-t)R^*_{N,n}(t)| \le \|R_m\|_{l^2}(\sum_{s=1}^{N}1)^{1/2}\sum_{s=1}^{N}|R^*_{N,n}(s)| \le CN^{1/2}(\ln N)N^{1-\beta} = o(N).$$

Thus, (2.5) holds. □

Proof of Theorem 2.3. Let $m \ge 1, n \ge 1$. Assuming (2.10) and (2.11), we show that (2.5) holds. Clearly,

$$i_N := N^{-1}Tr(R_N^{(m)}B_NR_N^{(n)}B_N) = \int_{[-\pi,\pi]^4} F(x_1,\dots,x_4)d\mu_N$$

where

$$F \equiv F(x_1,\dots,x_4) := f^{(*m)}(x_1)f^{(*n)}(x_2)\widehat{b}(x_3)\widehat{b}(x_4)$$

and where μ_N is defined in (4.13). Let $K > 0$. We divide the integral i_N into two parts

$$i_N = \int_{T_K} Fd\mu_N + \int_{T_K^c} Fd\mu_N =: i_{N,K} + r_{N,K}$$

where $[-\pi,\pi]^4 \supset T_K := \{f^{(*m)}(x_1) \le K\} \times \{f^{(*n)}(x_2) \le K\} \times \{|\widehat{b}(x_3)| \le K\} \times \{|\widehat{b}(x_4)| \le K\} =: T_{K,1} \times T_{K,2} \times T_{K,3} \times T_{K,4}$ and $T_K^c := [-\pi,\pi]^4 \setminus T_K$. Thus (2.5) will follow if we show that as $K \to \infty$,

$$\lim_N i_{N,K} \to (2\pi)^3 \int_{[-\pi,\pi]} f^{(*m)}(x)f^{(*n)}(x)\widehat{b}^2(x)\,dx =: i \tag{5.3}$$

and

$$\limsup_N |r_{N,K}| \to 0. \tag{5.4}$$

First we prove (5.3). Since F is bounded on T_K, Lemma 2 of Giraitis and Surgailis (1990) implies that

$$\Delta(K) := \lim_N i_{N,K} = (2\pi)^3 \int_{T_K} f^{(*m)}(x)f^{(*n)}(x)\widehat{b}^2(x)\,dx.$$

Note that

$$f^{(*k)} \in L^{p(k)}([-\pi,\pi]), \quad k \ge 1, \tag{5.5}$$

with $[p(k)]^{-1} = d_k^+(p^{-1})$. This follows from the assumption $f \in L^p([-\pi,\pi])$ by the repeated use of the Young inequality,

$$\|g_1 * g_2\|_{r'} \le \|g_1\|_{p'}\|g_2\|_{q'} \quad (1 \le p',q',r' \le \infty, \quad 1 + (r')^{-1} = (p')^{-1} + (q')^{-1}). \tag{5.6}$$

We may assume without loss of generality that Assumption (2.11) holds with equality since $f \in L^p \Rightarrow f \in L^{p'}$ with $p' \le p$. That assumption, together with the Hölder's inequality now yield $F(x,x,x,x) \in L^1([-\pi,\pi])$. Therefore by the Lebesque monotone convergence theorem $\Delta(K) \to i$ and (5.3) holds. To prove (5.4), it is sufficient to show that for $T_K^{c,j} := [-\pi,\pi] \times \ldots \times ([-\pi,\pi] \backslash T_{K,j}) \times \ldots \times [-\pi,\pi] \subset [-\pi,\pi]^4$,

$$r_{N,K}^{(j)} := \int_{T_K^{c,j}} |F| d|\mu_N| \le \delta(K) \quad (j = 1, ..., 4) \tag{5.7}$$

as $N \to \infty$, where $\lim_{K\to\infty} \delta(K) = 0$. Relation (5.7) implies (5.4) since $|r_{N,K}| \le \sum_{j=1}^4 r_{N,K}^{(j)}$. By symmetry it suffices to prove (5.7) for $j = 1$. Assume again without loss of generality that the equality holds in (2.11). Thus, using (5.5) and applying the Hölder's inequality on (5.7), we get

$$\begin{aligned} r_{N,K}^{(1)} \le & \Big(\int_{[-\pi,\pi]^4} |f^{(*m)}(x_1)|^{p(m)} \mathbb{I}(|f^{(*m)}(x_1)| > K)\, d|\mu_N|\Big)^{1/p(m)} \\ & \cdot \Big(\int_{[-\pi,\pi]^4} |f^{(*n)}(x_2)|^{p(n)} d|\mu_N|\Big)^{1/p(n)} \Big(\int_{[-\pi,\pi]^4} |\widehat{b}(x_3)|^q d|\mu_N|\Big)^{2/q}. \end{aligned}$$

It follows from the definition of μ_N and the estimate

$$|D_N(x)| = \left|\frac{\sin(Nx/2)}{\sin(x/2)}\right| \le C\frac{N}{1+N|x|}, \quad |x| \le \pi \tag{5.8}$$

that for any $g \in L^1([\pi,\pi])$ and $i = 1, ...4$, the bound

$$\int_{[-\pi,\pi]^4} |g(x_i)| d|\mu_N| \le C \int_{[-\pi,\pi]} |g(x)| dx \tag{5.9}$$

holds uniformly in $N \ge 1$ (cf. Giraitis and Surgailis (1990), proof of (5.6)). This implies

$$r_{N,K}^{(1)} \le C \|f^{(*m)} \mathbb{I}(f^{(*m)} > K)\|_{L^{p(m)}} \|f^{(*n)}\|_{L^{p(n)}} \|\widehat{b}\|_{L^q}^2 \to 0, \quad (K \to \infty)$$

establishing (5.7).

Now we consider the case $m \ge 1, n = 0$. We shall approximate the sum $Q_N = \sum_{t,s=1}^N b(t-s) P_m(X_t)$ by the univariate sum $S_N = 2\pi \widehat{b}(0) \sum_{t=1}^N P_m(X_t)$.

It is sufficient to show that

$$\mathrm{Var}(Q_N - S_N) = o(N) \tag{5.10}$$

and

$$N^{-1/2} S_N \overset{d}{\Rightarrow} N(0, \sigma^2) \quad (N \to \infty) \tag{5.11}$$

where σ^2 is defined by (2.6) and equals $\sigma^2 = (2\pi)^3 (\widehat{b}(0))^2 \psi_m(0)$.

The estimate (4.8) implies that

$$N^{-1}\mathrm{Var}(Q_N - S_N) \le CN^{-1} Tr\Big((B - 2\pi\widehat{b}(0) I_N) R_N^{(m)} (B - 2\pi\widehat{b}(0) I_N)\Big) = \int_{[-\pi,\pi]^4} F^*(x_1, \ldots, x_4) d\mu_N =: i_N^*$$

where

$$F^*(x_1, \ldots, x_4) := f^{(*m)}(x_1) \delta(x_2) (\widehat{b}(x_3) - \widehat{b}(0)) (\widehat{b}(x_4) - \widehat{b}(0)),$$

and $\delta(x)$ is the Dirac function, I_N is the identity matrix. (The use of the delta function simplifies the notation and relates the resulting expression to the case $n \neq 0$ considered earlier.) First we estimate the integral

$$i^*_{N,1} = \int_{\{|x_3| \le \epsilon\} \cup \{|x_4| \le \epsilon\}} F^* d\mu_N$$

for the fixed $\epsilon > 0$. Note that assumption (2.11) implies that $\hat{b} \in L^\infty$. Thus, $\hat{b}(x)$ is bounded function. The same assumption implies that $d^+_m(\alpha) = 0$, and this gives that $f \in L^p$ with $1 - 1/p \ge 1/m$. We assume below without restriction of generality that $m/p = m-1$. Then, the repeated use of the Young inequality (5.6) shows that

$$\sup_{|x| \le \pi} |f^{(*m)}(x)| = \|f^{(*m)}\|_{L^\infty} \le \|f\|^m_{L^p} < \infty \tag{5.12}$$

is a bounded function. Since $\hat{b}(x)$ is continuous at $x = 0$, $|\widehat{b}(x) - \widehat{b}(0)| < c(\epsilon)$ for $|x| < \epsilon$ and

$$|i^*_{N,1}| \le \text{const}\, c(\epsilon) \int_{[-\pi,\pi]^4} \delta(x_2) d|\mu_N| \le \text{const}\, c(\epsilon) \int_{[-\pi,\pi]} \delta(x)dx \to 0, \text{ as } \epsilon \to 0,$$

as in (5.9).

On the other hand, using (5.12)

$$\int_{|x_3| > \epsilon\ |x_4| > \epsilon} |F^*|\ d|\mu_N| \le \text{const} \int_{[-\pi,\pi]} \left[\int_{|y| > \epsilon} |D_N(x+y) D_N(y)| dy \right]^2 dx \le \text{const}\,(\ln N)^2 = o(N)$$

since $|D_N(y)| \le C(\epsilon)$ as $\epsilon \le |x| \le \pi$ and $\int_{[-\pi,\pi]} |D_N(x+y)|dy \le C \ln N$ as $|x| \le \pi$ by (5.8). Thus (5.10) holds.

We now prove (5.11). If $m = 1$, then (2.11) implies $f \in L^\infty$. Since we also assume in this case that $f(x)$ is continuous at $x = 0$, we get Var $\sum_{t=1}^N X_t \sim N 2\pi f(0)$ and hence (5.11) holds with $\sigma^2 = (2\pi)^3 (\widehat{b}(0))^2 f(0)$ (see Ibragimov and Linnik (1971)).

In the case $m \ge 2$, it is the spectral density $\psi_m(x)$ of the sequence $(P_m(X_t))$ which is continuous at $x = 0$. The proof of (5.11) can then be obtained in the same way as that of Relation (4.5), approximating the variables $P_m(X_t), t \in \mathbf{Z}$ by a m- dependent sequence, since

$$\text{Var}\, S_N \le C \int_{[-\pi,\pi]} |D_N(x)|^2 f^{(*m)}(x) dx \le C \|f\|^m_{L^p} \int_{[-\pi,\pi]} |D_N(x)|^2 dx = C 2\pi N \|f\|^m_{L^p}.$$

The proof of theorem is now complete. □

Lemma 5.2 *Let* $0 \le f(x) \le C|x|^{-\alpha}$, $|x| \le \pi$ $(\alpha < 1)$. *Assume* $k \ge 2$, $d_k(\alpha) = 1 - k(1-\alpha) \neq 0$ *and define* $d^+_k(\alpha)$ *as in* (2.9). *Then*

$$f^{(*k)}(x) \le C|x|^{-d^+_k(\alpha)}, \qquad |x| \le \pi. \tag{5.13}$$

Proof. If $\alpha < 0$, then (5.13) holds trivially because f is bounded and so is the convolution $f^{(*k)}$. Suppose $\alpha > 0$ and note that $d_j(\alpha)$ decreases as j increases. If $d_j(\alpha) > 0$ for some $j \le k-1$, then using the inequality (5.19) gives

$$f^{(*k)}(y) \le C \int_{[-\pi,\pi]^{k-j}} |y - x_j - \ldots - x_{k-1}|^{-d_j(\alpha)} |x_j|^{-\alpha} \ldots |x_{k-1}|^{-\alpha} d^{k-j}x. \tag{5.14}$$

Thus $d_k(\alpha) > 0$ implies

$$f^{*k}(y) \le C \int_{-\pi}^{\pi} |y - x_{k-1}|^{-d_{k-1}(\alpha)} |x_{k-1}|^{-\alpha} d^{k-1}x \le C|y|^{-d_k(\alpha)}.$$

If on the other hand, $d_j(\alpha) \ge 0$ and $d_{j+1}(\alpha) < 0$ then $d_j(\alpha) + \alpha < 1$. We can then apply the Hölder inequality on (5.14) to separate variables and conclude that the convolution $f^{(*k)}$ is bounded. Relation (5.13) still holds because $d_k(\alpha) = 0$ and hence $d_k^+(\alpha) = 0$. □

Proof of Theorem 2.4. It is again sufficient to show that (2.5) holds. Note that $f(x) \le C|x|^{-\alpha}$ implies $f \in L^p$ with $p < 1/\alpha$ ($\alpha > 0$). Let $\alpha\beta > 0$. If α and β are both negative then f and $\widehat{b}$ are bounded, $p = q = \infty$, and so both (2.10) and (2.11) hold. If α and β are both positive then (2.12) and (2.13) imply (2.10) and (2.11). Then, in view of assumptions (2.12), (2.13) and Lemma 5.2, the conditions of Theorem 2.3 are satisfied, and hence (2.5) holds. Let $\alpha\beta \le 0$, that is, α and β have different signs. According to the proof of Theorem 2.3, (2.5) follows from the relations (5.3) and (5.4). Since (5.3) is always satisfied, it remains to check the assertion (5.4), or merely (5.7). Lemma 5.2 and (2.12) imply that function F has the bound

$$|F(x_1, x_2, x_3, x_4)| \le C|x_1|^{-d_m^+(\alpha)} |x_2|^{-d_n^+(\alpha)} |x_3|^{-\beta} |x_4|^{-\beta},$$

where, by assumption, $d_m^+(\alpha), d_n^+(\alpha), \beta$ satisfy (2.13). We prove (5.7) for the (more general) function

$$F(x_1, x_2, x_3, x_4) = |x_1|^{-\alpha_1} |x_2|^{-\alpha_2} |x_3|^{-\beta_1} |x_4|^{-\beta_2}$$

with either $0 \le \alpha_1,\ \alpha_2 < 1, \quad -1 \le \beta_1, \beta_2 \le 0$ or $0 \le \beta_1,\ \beta_2 < 1, \quad -1 \le \alpha_1, \alpha_2 \le 0$ and such that $0 < \alpha_1 + \alpha_2 + \beta_1 + \beta_2 < 1$. We suppose below without loss of generality that $0 < \alpha_1,\ \alpha_2 < 1, \quad -1 < \beta_1, \beta_2 \le 0$. Clearly, (5.7) is satisfied for $j = 3, 4$, since the functions $|x_3|^{-\beta_1}$ and $|x_4|^{-\beta_2}$ are bounded and hence $T_K^{c,j} = \emptyset$ for large $K > 0$. It remains to show (5.7) for $j = 1$. (Because of the symmetry the same proof holds for $j = 2$.) Since $|x_j|^{-\alpha_1}$ is large only at the origin, there exits a sequence $\epsilon = \epsilon(K) \to 0$ ($K \to \infty$), such that $T_K^{c,1} \subset B_3 \cup B_4 \cup W$ where $B_i = \{x \in [-\pi, \pi]^4 : |x_1| < |x_i|/2\}$, $i = 3, 4$ and $W = \{x \in [-\pi, \pi]^4 :\ \varepsilon > |x_1| > \max_{i=3,4} |x_i|/2\}$. Thus it remains to check that

$$\int_{B_i} F\, d|\mu_N| \to 0, \quad i = 3, 4 \quad (N \to \infty) \tag{5.15}$$

and

$$\limsup_N \int_W F\, d|\mu_N| = \delta(\varepsilon) \to 0 \quad (\varepsilon \to 0) \tag{5.16}$$

which proves (5.7). We first consider (5.15). It is sufficient to prove it for $i = 3$. Set

$$\begin{aligned} v_1 &= \max(\alpha_1 + \beta_2, 2\delta) - \delta, \\ v_2 &= \max(\alpha_2 + \beta_1, 2\delta) - \delta, \\ v_3 &= v_4 = 2\delta, \end{aligned}$$

where $\delta > 0$. In view of the assumptions on $\alpha_1, \alpha_2, \beta_1, \beta_2$, for all δ sufficiently small,

$$1 > v_1 + \ldots + v_4 > \alpha_1 + \alpha_2 + \beta_1 + \beta_2 > 0. \tag{5.17}$$

Therefore using the estimate $|D_N(x)| \leq CN^u|x|^{u-1} \quad (0<u<1)$, which easily follows from (5.8), one can show that

$$\int_{B_3} F\, d|\mu_N| \leq CN^{v_1+\ldots+v_4-1} \int_{B_3} \prod_{i=1}^{2} \prod_{j=3}^{4} |x_i|^{-\alpha_i}|x_j|^{-\beta_j}|x_i - x_j|^{v_i-1}\, d^4x$$

$$\leq CN^{v_1+\ldots+v_4-1} \int_{[-\pi,\pi]^4} |x_1|^{-\alpha_1}|x_2|^{-\alpha_2}|x_3|^{v_2-\beta_1-1}|x_4|^{-\beta_2}$$

$$\cdot|x_1-x_4|^{v_1-1}|x_2-x_3|^{v_3-1}|x_2-x_4|^{v_4-1}\, d^4x \equiv: CN^{v_1+\ldots+v_4-1}j, \tag{5.18}$$

since $|x_1-x_3| > |x_3|/2$ on B_3. Since the exponent of N is negative it is enough to show that $j<\infty$. Applying the inequality

$$\int_{[-\pi,\pi]} |x|^{-p}|x-y|^{-q}\, dx \leq C(p,q)|y|^{1-p-q} \quad (0<p,q<1,\ p+q>1) \tag{5.19}$$

to the right-hand side of (5.18), we obtain

$$\begin{aligned} j &\leq C\int_{[-\pi,\pi]^2} |x_2|^{v_2+v_3-\alpha_2-\beta_1-1}|x_4|^{v_1-\beta_2-\alpha_1}|x_2-x_4|^{v_4-1}dx_2dx_4 \\ &\leq C\int_{[-\pi,\pi]} |x_4|^{v_1+\ldots+v_4-\alpha_1-\alpha_2-\beta_1-\beta_2-1}\, dx_4 < \infty, \end{aligned}$$

in view of (5.17). Now we prove (5.16). Set $W' = \{x \in [-\pi,\pi]^4 : \varepsilon > |x_j| > \max_{i=3,4}|x_i|/2, \quad j = 1,2\} \subset W$. Since $W \setminus W' \subset (x \in [-\pi,\pi]^4 : |x_2| \leq |x_3|/2 \text{ or } |x_2| \leq |x_4|/2\})$ it follows from (5.15) that

$$\int_{W\setminus W'} Fd\mu_N \to 0 \quad (N\to\infty).$$

It is therefore sufficient to prove (5.16) with W replaced by W'. For any $(x_1,...,x_4) \in W'$ we have

$$F(x_1,...,x_4) \leq |x_1|^{-\alpha_1}|x_2|^{-\alpha_2}(|x_3|^{-(\beta_1+\beta_2)} + |x_4|^{-(\beta_1+\beta_2)})$$

$$\leq C(|x_3|^{-(\alpha_1+\alpha_2+\beta_1+\beta_2)} + |x_4|^{-(\alpha_1+\alpha_2+\beta_1+\beta_2)}).$$

Thus, from (5.9) we obtain

$$\int_{W'} Fd|\mu_N| \leq C\int_{[-\pi,\pi]^4} |x_3|^{-(\alpha_1+\alpha_2+\beta_1+\beta_2)}1(\{|x_3| \leq \epsilon\})d|\mu_N|$$

$$\leq C\int_0^\epsilon |x|^{-(\alpha_1+\alpha_2+\beta_1+\beta_2)}dx \to 0 \quad (\epsilon\to 0).$$

□

6 Multivariate extensions

The following result is a multivariate generalization of Theorem 2.1.

Let $\{X_t^{(i,l)}\} = \sum_{u\in\mathbf{Z}} a^{(i,l)}(t-u)\xi_u$, $i=1,\dots,k; l=1,2$, be k stationary time series with the same mean 0, finite variance innovations sequence $\{\xi_u, u\in\mathbf{Z}\}$ and consider

$$Q_N^{(i)} := \sum_{t,s=1}^{N} b_i(t-s) P_{m_i,n_i}(X_t^{(i,1)}, X_s^{(i,2)}),$$

$i=1,...,k$. Assume $E|\xi_0|^{2(m_i+n_i)} < \infty$, $i=1,\dots,k$ in order to ensure that $Q_N^{(i)}$ has finite second moments for all i. As in Theorem 2.1 let $f^{(i,l)}$, $f^{(i,l)(*k)}$, $R_{N,(i,l)}^{(k)}$ corresponds to $\{X_t^{(i,l)}\}$ and let $\widehat{b}_i(x)$ denotes the Fourier transform of b_i, $B_{N,i} = (b_i(t-s))_{t,s=1,...,N}$.

Theorem 6.1 *Suppose that for each $i=1,\dots,k$, one has $m_i,\ n_i \ge 1$ and that the quadratic form $Q_N^{(i)}$, satisfies the assumption*

$$N^{-1} Tr(R_{N,(i,1)}^{(m_i)} B_{N,i} R_{N,(i,2)}^{(n_i)} B_{N,i}) \to (2\pi)^3 \int_{[-\pi,\pi]} f^{(i,1)(*m)}(x) f^{(i,2)(*n)}(x) \widehat{b}_i^2(x)\, dx \tag{6.1}$$

as $N\to\infty$.

Let

$$Y_t^{(i)} = \sum_{l_1,l_2=-\infty}^{\infty} \beta_i^{(1)}(l_1)\beta_i^{(2)}(l_2) P_{m_i,n_i}(X_{t-l_1}^{(i,1)}, X_{t-l_2}^{(i,2)}), \quad (i=1,\dots,k)$$

where

$$\beta_i^{(1)}(s) = (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} \exp(isx) |\widehat{b}_i(x)|^{1/2} dx$$

and

$$\beta_i^{(2)}(s) = (2\pi)^{-\frac{1}{2}} \int_{-\pi}^{\pi} \exp(isx) (\widehat{b}_i(x))^{\langle 1/2\rangle} dx, \ s\in\mathbf{Z}.$$

Define the cross-spectral densities $f_{i,j}(x)$ of the vector $(Y_{t_1}^{(1)},...,Y_{t_k}^{(k)})$ by $EY_t^{(i)}Y_s^{(j)} = \int_{-\pi}^{\pi}\exp(i(t-s)x)f_{i,j}(x)dx$, $i,j=1,...,k$. Then as $N\to\infty$

$$N^{-1}\mathrm{Var}\,(Q_N^{(i)} - \sum_{t=1}^{N} Y_t^{(i)}) \to 0 \quad (i=1,\dots,k) \tag{6.2}$$

and

$$(N^{-1/2}Q_N^{(1)},...,N^{-1/2}Q_N^{(k)}) \Rightarrow (Z^{(1)},...,Z^{(k)}), \tag{6.3}$$

where the vector $(Z^{(1)},...,Z^{(k)})$ is Gaussian with zero mean and covariance

$$\sigma_{i,j} \equiv EZ^{(i)}Z^{(j)} = \sum_{\gamma\in\Gamma(m_i+n_i,m_j+n_j)} d(\gamma)(2\pi)^{r+2} \int_{H(\gamma)} \widehat{b}_i(x_{1,1}+\cdots+x_{1,m_i})$$

$$\cdot\widehat{a}^{(i,1)}(x_{1,1})\dots\widehat{a}^{(i,1)}(x_{1,m_i})\widehat{a}^{(i,2)}(x_{1,m_i+1})\dots\widehat{a}^{(i,2)}(x_{1,m_i+n_i})\widehat{b}_j(x_{2,1}+\cdots+x_{2,m_j})$$

$$\cdot\widehat{a}^{(j,1)}(x_{2,1})\dots a^{(j,1)}(x_{2,m_j})\widehat{a}^{(j,2)}(x_{2,m_j+1})\dots\widehat{a}^{(j,2)}(x_{2,m_j+n_j}) d^{(m_i+n_i+m_j+n_j)}x.$$

Moreover, $\sigma_{i,j} = 2\pi f_{i,j}(0)$ under the assumptions of Theorems 2.3 or 2.4.

Proof of Theorem 6.1. Observe that $Y_t^{(i)} \equiv Y_\infty^{(i)}(t)$ where $Y_\infty^{(i)}(t)$ is defined by (4.4) with $K = \infty$, and the index i means that the function $\widehat{b}$ in the definition of $Y_\infty(t)$ is replaced by $\widehat{b}_i$. Therefore, the relations (6.2) and (6.3) follow as in the proof of the Theorem 2.1. Using the diagram formula (Giraitis and Surgailis (1986), Theorem 4 (iv)) one can easily obtain the following expression for the spectral density:

$$f_{i,j}(y) = \sum_{\gamma \in \Gamma(m_i+n_i, m_j+n_j)} d(\gamma)(2\pi)^{r+2}$$

$$\int_{H(V_1)} d^{|V_1|}x \int_{H(V_2)} d^{|V_2|}x \ldots \int_{H(V_r)} d^{|V_r|}x \, \mathbb{1}(x_{1,1} + \ldots + x_{1,m_i+n_i} = y \bmod 2\pi)$$

$$\cdot |\widehat{b}_i(x_{1,1} + \ldots + x_{1,m_i})|^{1/2} (\widehat{b}_i(x_{1,m_i+1} + \ldots + x_{1,m_i+n_i}))^{\langle 1/2 \rangle}$$

$$\cdot \widehat{a}^{(i,1)}(x_{1,1}) \ldots \widehat{a}^{(i,1)}(x_{1,m_i}) \widehat{a}^{(i,2)}(x_{1,m_i+1}) \ldots \widehat{a}^{(i,2)}(x_{1,m_i+n_i})$$

$$\cdot |\widehat{b}_j(x_{2,1} + \ldots + x_{2,m_j})|^{1/2} (\widehat{b}_j(x_{2,m_j+1} + \ldots + x_{2,m_j+n_j}))^{\langle 1/2 \rangle}$$

$$\cdot \widehat{a}^{(j,1)}(x_{2,1}) \ldots \widehat{a}^{(j,1)}(x_{2,m_j}) \widehat{a}^{(j,2)}(x_{2,m_j+1}) \ldots \widehat{a}^{(j,2)}(x_{2,m_j+n_j}).$$

Using the Hölder inequality, it is possible to show that this is a continuous function under the assumptions of Theorem 2.3 or 2.4. The equality $\sigma_{i,j} = 2\pi f_{i,j}(0)$ is well-known. □

Theorem 6.2 *The assumption* (6.1) *of the Theorem 6.1 holds and so does the CLT* (6.3) *if either*

A1)

$$f^{(i,1)} \in L^{p_{i,1}}, \quad f^{(i,2)} \in L^{p_{i,2}}, \quad \widehat{b}_i \in L^{q_i} \quad (1 \le p_{i,1}, p_{i,2}, q_i \le \infty),$$

and

$$d^+_{m_i}(p_{i,1}^{-1}) + d^+_{n_i}(p_{i,2}^{-1}) + 2q_i^{-1} \le 1, \; i = 1, \ldots, k;$$

or

A2)

$$f^{(i,1)}(x) \le C|x|^{-\alpha_{i,1}}, \quad f^{(i,2)}(x) \le C|x|^{-\alpha_{i,2}}, \quad |\widehat{b}_i(x)| \le C|x|^{-\beta_i}, \quad x \in [-\pi, \pi], \; i = 1, \ldots, k$$

where $\alpha_{i,1}, \alpha_{i,2} < 1, \; \beta_i < 1,$ *and*

$$d^+_{m_i}(\alpha_{i,1}) + d^+_{n_i}(\alpha_{i,2}) + 2\beta_i < 1, \; i = 1, \ldots, k.$$

The validity of (6.1) under assumptions A1) and A2) can be established as in the proofs of Theorems 2.3 and 2.4, respectively.

Theorem 6.3 *The CLT* (6.3) *holds under the assumption A1) of Theorem 6.2 for all quadratic forms* $Q_N^{(i)}$ *such that* $m_i \ge 0, \; n_i \ge 0, \; m_i + n_i \ge 1$, *provided that the following additional conditions hold as well:*

(a) $\widehat{b}_i(x)$ *is continuous at* $x = 0$ *if* $m_i \ge 1, \; n_i = 0$ *or* $n_i \ge 1, \; m_i = 0$.
(b) $f^{(i,1)}(x)$ *is continuous at* $x = 0$ *if* $m_i = 1, n_i = 0$.
(c) $f^{(i,2)}(x)$ *is continuous at* $x = 0$ *if* $m_i = 0, \; n_i = 1$.

The proof is similar to that of Theorem 2.3.

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