To appear in Journal of Statistical Planning and Inference

Convergence of normalized quadratic forms *†

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April 1998

Abstract

The asymptotic behavior of quadratic forms of stationary sequences plays an important role in statistics, for example, in the context of the Whittle approximation to maximum likelihood. The quadratic form, appropriately normalized, may have Gaussian or non-Gaussian limits. Under what circumstances will the limits be of one type or another? And if the limits are non-Gaussian, what are they? The goal of this paper is to describe the historical development of the problem and provide further extensions of recent results.

1 Introduction

Let $\{X_t,\ t = 1, 2, \ldots\}$ be a stationary sequence with mean zero and finite variance. Suppose that its spectral density f is regularly varying with exponent $-\alpha$, that is

$$f(x) = |x|^{-\alpha} L(1/|x|),\ x \in (-\pi, \pi], \tag{1.1}$$

where L is slowly varying of infinity. We shall suppose first that $0 < \alpha < 1$. The assumption $\alpha < 1$ is necessary because the relation $EX_t^2 = \int_{-\pi}^{\pi} f(x)dx$ requires that the spectral density f be integrable.

The sequence (X_t) is said to have strong dependence (long-range dependence or long memory) because f diverges at low frequencies ($x \to 0$). Statistical models with this type of spectral density include fractional Gaussian noise (Mandelbrot and Van Ness [23] and fractional ARMA Granger and Joyeux [19] and Hosking [21]. See also Beran [3] and Samorodnitsky and Taqqu [24]).

When (X_t) has spectral density (1.1), with $0 < \alpha < 1$ then its covariances $r(k) = EX_tX_{t+k}$ decay to zero, typically like $r(k) \sim Ck^{-(1-\alpha)}L(k)$ as $k \to \infty$ where C is a positive constant, so that $\sum_{k=-\infty}^{\infty} r(k) = \infty$. The non-summability of $r(k)$ is a characteristic of strong dependence. Weak dependence occurs when $0 < \sum_{k=-\infty}^{\infty} |r(k)| < \infty$. The special case $\sum_{k=-\infty}^{\infty} r(k) = 0$, which can arise when $\alpha < 0$, corresponds to "negative dependence" because $r(k)$ is then typically negative for large k.

With $A_N \geq 0$ being a suitable normalization factor, we want to describe the limit as $N \to \infty$ of the finite-dimensional distributions of the linear form $A_N^{-1} \sum_{t=1}^{[N\tau]} X_t,\ \tau \geq 0$, and of quadratic forms $A_N^{-1} \sum_{t=1}^{[N\tau]} \sum_{s=1}^{[N\tau]} b(t-s)(X_tX_s - EX_tX_s)$ and $A_N^{-1} \sum_{t=1}^{[N\tau]} \sum_{s=1}^{[N\tau]} b(t-s)P_{m,n}(X_t, X_s)$. Here, $P_{m,n}$ are the *Appell polynomials* (see section 4), which reduce, for example, to Hermite polynomials if (X_t) is Gaussian.

*The first author was supported by ESRC Grant R000235892. The second author was partially supported by the NSF Grants DMS-9404093 and NCR-9404931 at Boston University.

†*AMS Subject classification.* 60F05, 62M10. *Short title:* Functional Non-CLT and CLT. *Keywords and phrases:* Appell polynomials, non-central limit theorem, long-range dependence, quadratic forms, time series.

The limiting processes can be Gaussian or not. We say that we have a *non-central limit theorem* (NCLT) if the limit is non-Gaussian or, if Gaussian, if it is not Brownian motion. We have a *central limit theorem* (CLT) if the limit is Brownian motion. The limiting processes may not have stationary increments but they are self-similar. Recall that a process $\{Z(\tau),\ \tau \geq 0\}$ is *self-similar* with index $H > 0$ if, for any $a > 0$,

$$\{Z(a\tau),\ \tau \geq 0\} \stackrel{d}{=} \{a^H Z(\tau),\ \tau \geq 0\},$$

where the equality is in the sense of the finite-dimensional distributions.

The paper is organized as follows. In order to gain insight into the limiting behavior of the quadratic forms, we first review, in Section 2, what happens to linear forms. We then turn, in Section 3 to quadratic forms and consider both Gaussian variables and Hermite polynomials of such variables. The Appell polynomials are introduced in Section 4. In Section 5, we define the multiple Wiener-Itô integrals with respect to Gaussian complex vector measures because such multiple integrals are used to represent the limiting process. Both non-central and central limit theorems involving quadratic forms in Appell polynomials are presented in Section 6. Multivariate extensions will be found in Section 7. These results are obtained by making spectral domain assumptions, namely assumptions on the spectral density $f(x)$ of $X(t)$ and on the Fourier transform $\widehat{b}(x)$ of $b(t)$. One can also make time-domain assumptions, namely ℓ^p-type assumptions on the covariance $r(t)$ of (X_t) and on $b(t)$. Such assumptions which are used to obtain central limit theorems are presented in Section 8.

Throughout the paper, the symbol "$\Rightarrow$" denotes convergence of the finite-dimensional distributions.

2 Linear forms

Assume for simplicity that the stationary sequence (X_t) is Gaussian. (We will relax this assumption in Section 4). Assume $EX_t = 0$ and $EX_t^2 = 1$ and let $Y_t = G(X_t)$ be a non-linear function of X_t with mean $EY_t = 0$ and finite variance $EY_t^2 < \infty$. We are interested in the limit in distribution of

$$\frac{1}{\sqrt{A_N}} \sum_{t=1}^{N} Y_t \tag{2.1}$$

as $N \to \infty$, where A_N is a suitable normalization factor. Observe that Relation (2.1) involves the *linear form* $\sum_{t=1}^{N} G(X_t)$. Since the Hermite polynomials $H_m(x)$, $m \geq 0$, span the L^2 space generated by the Gaussian measure, it is enough to study what happens when $G = H_m$. The Hermite polynomials H_m can be defined through their generating function

$$\sum_{m=0}^{\infty} \frac{z^m}{m!} H_m(x) = \frac{c^{2x}}{Ee^{zX_0}} = e^{zx - \frac{1}{2}z^2x^2}.$$

One has $H_0(x) = 1$ (but the choice $G(X_t) = H_0(X_t) = 1$ is excluded since $EG(X_t) = 0$), $H_1(x) = x, H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$). For other choices of G, one can use the *reduction principle* (Taqqu [27]), which says that the limit depends only on *Hermite rank* of G, that is on the first non-zero term in the Hermite expansion of $G(X_t)$.

If $m = 1$, then $Y_t = H_1(X_t) = X_t$ and in this case the limit is necessarily Gaussian. We have, however, a non-central limit theorem because after setting $A_N = N^{\gamma} L^*(N)$, with $L^*(N) = L(N)$ and

$$\gamma = (-1 + \alpha) + 2 = 1 + \alpha \in (1, 2),$$

one can show (Taqqu [27]) that

$$\frac{1}{\sqrt{A_N}} \sum_{t=1}^{[N\tau]} X_t, \quad 0 \le \tau < \infty \tag{2.2}$$

converges in the sense of finite-dimensional distributions (in fact, weakly in $D[0,\infty)$) to *fractional Brownian motion*, which is a self-similar process with index

$$H = \gamma/2.$$

Fractional Brownian motion $B_H(\tau)$ is defined as a Gaussian process with stationary increments and variance $EB_H^2(\tau) = C\tau^{2H}$ and can be represented by the integral

$$\int_{\mathbf{R}} \frac{e^{ix\tau} - 1}{ix} |x|^{-\alpha/2} Z_1(dx),$$

where Z_1 is a Gaussian measure defined as in Section 5. Hence, while the limit of (2.2) is Gaussian, the limit is not Brownian motion and $\sqrt{A_N}$ grows like $N^H \sqrt{L(N)}$, $1/2 < H < 1$, that is, it grows faster than the usual $\sqrt{N}$ which characterizes the ordinary central limit theorem.

The situation is more complicated when $m \ge 2$. In this case, the limit of

$$\frac{1}{\sqrt{A_N}} \sum_{t=1}^{N} H_m(X_t) \tag{2.3}$$

is either Gaussian or not, depending on the value of

$$d_m(\alpha) = 1 - m(1-\alpha). \tag{2.4}$$

$d_m(\alpha)$ may be negative but one always has $d_m(\alpha) < 1$ because $\alpha < 1$. To understand the role of $d_m(\alpha)$, observe that the relation $EH_m(X_t)H_m(X_{t+k}) = Cr^m(k)$ implies

$$A_n = \text{Var}\left(\sum_{t=1}^{N} Y_t\right) = \sum_{t=1}^{N}\sum_{s=1}^{N} EH_m(X_t)H_m(X_s) = C\sum_{t=1}^{N}\sum_{s=1}^{N} r^m(t-s).$$

But $r^m(k)$ behaves asymptotically like $k^{-m(1-\alpha)}L^m(k)$ as $k \to \infty$. Hence $r^m(k)$ is summable if $d_m(\alpha) < 0$ but is not summable if $d_m(\alpha) > 0$.

When $d_m(\alpha) < 0$, $H_m(X_t)$ is weakly dependent and the limit of (2.3) as $N \to \infty$ is Gaussian with $A_N = N$. When $d_m(\alpha) > 0$, $\sum_{k=-\infty}^{\infty} r^m(k)$ diverges and hence $H_m(X_t)$ is strongly dependent. Observe, however, that for $d_m(\alpha)$ to be positive, α has to be large enough, namely

$$\alpha > 1 - 1/m.$$

Such a large value of α compensates for the weakening of the dependence structure which occurs because we consider the non-linear function $Y_t = H_m(X_t)$ instead of X_t. When $m = 2$ and 3 for example, one needs $\alpha > 1/2$ and $\alpha > 2/3$, respectively to ensure $d_m(\alpha) > 0$.

When $d_m(\alpha) > 0$, the limit of (2.3) is non-Gaussian, and requires

$$A_N = N^\gamma L^*(N)$$

with $L^*(N) = L^m(N)$ and

$$\gamma = d_m(\alpha) + 1 \in (1, 2).$$

(See Taqqu [27], Dobrushin and Major [6]). The non-Gaussian limit of (2.3) can be represented as a Wiener-Itô multiple integral of order m (see Section 5)

$$\int_{\mathbf{R}^m} \frac{e^{i(x_1+\ldots+x_m)} - 1}{i(x_1 + \ldots + x_m)} |x_1|^{-\alpha/2} \ldots |x_m|^{-\alpha/2} Z_1(dx_1) \ldots Z_1(dx_m). \tag{2.5}$$

Moreover, $A_N^{-1/2} \sum_{t=1}^{[N\tau]} H_m(X_t)$ converges weakly to a non-Gaussian self-similar process with $H = \gamma/2 \in (1/2, 1)$.

What about $\alpha < 0$? In this case, the spectral density f does not diverge at the origin. This situation typically corresponds to weak dependence, but one can have $\sum_{k=-\infty}^{\infty} r(k) = 0$, that is, negative dependence if $-1 < \alpha < 0$. Thus,

1. If $m = 1$ and $-1 < \alpha < 0$, then the limit of (2.2) is fractional Brownian motion; in this case $A_N = N^\gamma L(N)$, $\gamma = d_1(\alpha) + 1 = \alpha + 1 \in (0, 1)$.
2. If $m = 1$ and $\alpha < -1$, then

$$E(\sum_{t=1}^{N} X_t)^2 = \int_{-\pi}^{\pi} |\sum_{t=1}^{N} e^{itx}|^2 f(x)dx \le C \int_{-\pi}^{\pi} (\frac{N}{1 + N|x|})^2 |x|^{-\alpha} L(1/|x|)dx$$

$$\le C \int_{-\pi}^{\pi} (\frac{N}{1 + N|x|})^2 |x|^{-\alpha+\epsilon} dx \le C \int_{-\pi}^{\pi} |x|^{-(2+\alpha-\epsilon)} dx < \infty$$

   since $2 + \alpha - \epsilon < 1$ for small $\epsilon > 0$ (the constant C may change from expression to expression). Thus the variance does not grow to infinity and hence we do not have any limit in this case.

3. If $m \ge 2$ and $\alpha < 0$, then $A_N = N$ and $N^{-1/2} \sum_{t=1}^{[N\tau]} H_m(X_t)$ converges to Brownian motion.

Section 8 contains further discussion about the central limit theorems (where the limit is Brownian motion).

Observe that the exponent γ in the normalization factor $A_N = N^\gamma L^*(N)$ varies depending on the case considered. One can, however, summarize in one formula, all the NCLT cases corresponding to $-\infty < \alpha < 1$, by setting

$$d_m^+(\alpha) = \begin{cases} \alpha & \text{if } m = 1 \\ \max(d_m(\alpha), 0) & \text{if } m \ge 2 \end{cases} \tag{2.6}$$

and

$$0 < \gamma = d_m^+(\alpha) + 1. \tag{2.7}$$

Observe that the condition $\gamma > 0$ ensures that α must be greater than -1 in the case $m = 1$.

We considered here Gaussian processes and Hermite polynomials. Results similar to those described here were obtained by Surgailis [26], Giraitis [9], Giraitis and Surgailis [11], Avram and Taqqu [2] when X_t is a non-Gaussian linear process and H_m is replaced by an Appell polynomial (see Sections 4 and 6 below).

The empirical process $Y_t = \mathbb{I}(X_t \le x)$ of a strongly dependent Gaussian process (X_t) was considered by Dehling and Taqqu [5]. In the case of a non-Gaussian linear process (X_t), it was considered by Giraitis and Surgailis [14] using a method based on the Appell expansion of Y_t, and also by Ho and Hsing [20] and Koul and Surgailis [22] who applied an interesting new method based on the martingale decomposition of Y_t.

3 Quadratic forms in Gaussian variables

We now turn to the quadratic forms

$$Q_N = \sum_{t=1}^{N} \sum_{s=1}^{N} b(t-s)(X_t X_s - E X_t X_s) \tag{3.1}$$

where (X_t) is Gaussian and defined as in Section 2 with $-\infty < \alpha < 1$. More general results are given in Section 6.

Recall that the spectral density f of (X_t), which is given by (1.1), varies regularly with exponent $-\alpha$. We make an analogous assumption on the Fourier transform $\widehat{b}(x)$ of the coefficients $b(k)$, $-\infty < k < 1$: we shall assume that $\widehat{b}(x)$ is regularly varying with exponent $-\beta$, more precisely,

$$\widehat{b}(x) = |x|^{-\beta} L_1(1/|x|), \quad x \in (-\pi, \pi], \tag{3.2}$$

with $|L_1|$ slowly varying at infinity (L_1 can be negative) and $-\infty < \beta < 1$, so as to assure that the Fourier transform is well-defined. We want to find the limiting distribution of

$$\frac{1}{\sqrt{A_N}} Q_N = \frac{1}{\sqrt{A_N}} \sum_{t=1}^{N} \sum_{s=1}^{N} b(t-s)(X_t X_s - E X_t X_s) \tag{3.3}$$

as $N \to \infty$, and determine what the adequate normalization factor A_N should be. The limiting distribution, it turns out, depends on the sum $\alpha + \beta$ of the exponents.

(i) If $\alpha + \beta < 1/2$, then the limit is Gaussian (Fox and Taqqu [8]) and one has a CLT ($A_N = N$) even though Q_N involves a double sum. The condition "$\alpha + \beta < 1/2$" allows for example a negative β to *compensate* for a positive α. Results that allow "compensation" are quite delicate.

(ii) If $\alpha + \beta > 1/2$, then the limit is non-Gaussian (Terrin and Taqqu [28]). The normalization in this case is $A_N = N^{\gamma} L^*(N)$, with $L^*(N) = L(N) L_1^2(N)$ and

$$\gamma = 2(\alpha + \beta) > 1.$$

Observe that γ can be as large as 4, which is natural because Q_N involves a double summation. Observe also that since $\alpha < 1$, the relation $\gamma > 1$ implies $\beta > -1/2$, and hence β cannot be too small.
The non-Gaussian limit is given by the double Wiener-Itô integral (see Section 5):

$$\int_{\mathbf{R}^2} \Phi(x_1, x_2) |x_1|^{-\alpha/2} |x_2|^{-\alpha/2} Z_1(dx_1) Z_1(dx_2), \tag{3.4}$$

where

$$\Phi(x_1, x_2) = \int_{-\infty}^{\infty} \frac{e^{i(x_1+u)} - 1}{i(x_1+u)} \frac{e^{i(x_2-u)} - 1}{i(x_2-u)} |u|^{-\beta} du.$$

Suppose now that we let $b(t) = \delta(t)$, that is equal to 1 if $t = 0$ and equal to 0 otherwise. Then $\widehat{b}(x)$ is constant and $\beta = 0$. According to the result on quadratic forms stated above, the limit of (3.3) will be non-Gaussian with $A_N = N^{\gamma} L^*(N)$ and $\gamma = 2(\alpha + \beta) = 2\alpha$ if $\alpha > 1/2$. It is given by (3.4). Since $\beta = 0$, $\Phi(x_1, x_2)$ becomes $2\pi(e^{i(x_1+x_2)} - 1)/i(x_1 + x_2)$ (see Giraitis and Taqqu [17]). On the other hand, with this choice of b, the quadratic form reduces to

$$Q_N = \sum_{t=1}^{N} (X_t^2 - 1) = \sum_{t=1}^{N} H_2(X_t),$$

that is to a linear form involving the Hermite polynomial H_2 of order 2. We have seen that the normalized limit of such a linear form is non-Gaussian, given by (2.5), with $\gamma = d_2^+(\alpha) + 1 = 2\alpha$ if $\alpha > 1/2$. The two results, therefore, are consistent.

4 Non-Gaussian sequences and associated Appell polynomials

We saw that linear forms can yield non-Gaussian limits when X_t is replaced by $H_m(X_t)$ with $m \geq 2$. What happens if the product $X_t X_s$ in the quadratic form (3.1) is replaced by a bivariate Hermite polynomial? We shall answer this question in the following section in the more general setting, where the sequence (X_t) is not assumed Gaussian anymore. The polynomials associated with a stationary sequence (X_t) with finite variance are the Appell polynomials. The Appell polynomials P_m satisfy the differential equations

$$P_m'(x) = mP_{m-1}(x), \qquad EP_m(X_t) = 0. \tag{4.1}$$

The first relation indicates that these polynomials behave like power functions. The second relation provides the constants of integration and makes P_m depend on the marginal distribution of the stationary process (X_t). The polynomial $P_m(x)$ is defined as long as $E|X_0|^m < \infty$. If $EX_t = 0$, then we have $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = x^2 - EX_0^2, \ldots$ (see Avram and Taqqu [2]). If X_t is Gaussian, then $P_m(X_t)$ is the Hermite polynomial. Among all Appell polynomials, the Hermite polynomials are the only ones that are orthogonal.

The multivariate Appell polynomials $P_{m_1,\ldots,m_k}$ can be defined by the recurrence relations

$$\frac{\partial}{\partial x_j} P_{m_1,\ldots,m_k}(x_1, \ldots, x_k) = m_j P_{m_1,\ldots,m_j-1,\ldots,m_k}(x_1, \ldots, x_k),$$

$$EP_{m_1,\ldots,m_k}(X_{t_1}, \ldots, X_{t_k}) = 0.$$

The second relation provides the constants of integration and relates the polynomial to the joint distribution of the X_t's. The multivariate Appell polynomials can also be defined by the generating function

$$\sum_{m_1,\ldots,m_k=0}^{\infty} \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_k^{m_k}}{m_k!} P_{m_1,\ldots,m_k}(x_1, \ldots, x_k) = \frac{\exp(\sum_1^k z_j x_j)}{E\exp(\sum_1^k z_j X_{t_j})}$$

(see [2], [10] for more details). One gets for example,

$$P_{1,1}(X_1, X_2) =: X_1 X_2 := (X_1 - EX_1)(X_2 - EX_2) - E(X_1 - EX_1)(X_2 - EX_2);$$

and, if $EX_1 = EX_2 = 0$,

$$P_{1,2}(X_1, X_2) =: X_1 X_2^2 := X_1 X_2^2 - X_1 EX_2^2 - 2X_2^2 EX_1 X_2 - EX_1 X_2^2.$$

When applying the Appell polynomials to (X_t), we write

$$P_{m_1,\ldots,m_k}(X_{t_1}, \ldots, X_{t_k}) \equiv: \underbrace{X_{t_1}, \ldots, X_{t_1}}_{m_1}, \ldots, \underbrace{X_{t_k}, \ldots, X_{t_k}}_{m_k} :, \quad m_1, \ldots, m_k = 0, 1, \ldots \tag{4.2}$$

The alternate notation $'' : \ldots :''$, called *Wick product*, is convenient (the indices in P correspond to the number of times that the variables in ": :" are repeated).

In particular,

$$P_m(X_t) =: \underbrace{X_t, \ldots, X_t}_{m} :\equiv: X_t^{(m)} : \tag{4.3}$$

characterize the univariate Appell polynomials. Thus : $X_t^{(m)} := H_m(X_t)$ if (X_t) is Gaussian.

The multivariate Appell polynomials are orthogonal when (X_t) is Gaussian, but not otherwise. In spite of their general lack of orthogonality, the multivariate Appell polynomials (4.2) possess nice properties, for example, the diagram formula for moments and cumulants.

5 Multiple stochastic integrals with respect to vector Gaussian measure

The limiting processes considered in the previous sections are defined as multiple Wiener-Itô stochastic integrals. We shall need in the sequel more general ones, namely multiple Wiener-Itô stochastic integrals with respect to Gaussian complex-valued *vector* measures. These integrals were introduced by Surgailis [25] and we recall here their basic properties.

Let $Z = (Z_1, \ldots, Z_m)$ be a family of complex-valued Gaussian measures on $\mathbf{R}$ with zero mean, and $EZ_i(A)\overline{Z_j(B)} = r_{ij}|A \cap B|$, $\overline{Z_i(A)} = Z_i(-A)$, $i, j = 1, = \ldots, m$, where the covariance matrix $(r_{ij})_{i,j=1,\ldots,m}$ is positive definite. Denote by $L^2(\mathbf{R}^m)$ the Hilbert space of functions $f : \mathbf{R}^m \to \mathbf{C}$ equipped with the norm

$$(f,f)^{1/2} \equiv \left(\int_{\mathbf{R}^m} r_{1,\pi(1)} \ldots r_{m,\pi(m)} f(x_1, \ldots, x_m) \overline{f(x_{\pi(1)}, \ldots, x_{\pi(m)})}/m! \right)^{1/2}.$$

Define the symmetrized function

$$(\mathrm{sym} f)(x_1, \ldots, x_m) = \sum_{\pi(1),\ldots,\pi(m)} f(x_{\pi(1)}, \ldots, x_{\pi(m)})/m!$$

where the sum is taken over all permutations $\pi = (\pi(1), \ldots, \pi(m))$ of the set $(1, \ldots, m)$.

Proposition 5.1 (see Surgailis [25]), Theorem 1.1) *Let $(Z_1, \ldots, Z_m)$ and (r_{ij}) satisfy the assumptions above. Then, for any $f \in L^2(\mathbf{R}^m)$, the multiple stochastic integral of order m,*

$$I_m(f) = I_m(\mathrm{sym} f) = \int_{\mathbf{R}^m} f(x_1, \ldots, x_m) Z_1(dx_1) \ldots Z_m(dx_m),$$

is well-defined and has the following properties:

(1) $E|I_m(f)|^2 < \infty$,

(2) $EI_m(f) = 0$

(3) $EI_m(f)\overline{I_n(g)} = \delta_{nk} m!\ (\mathrm{sym} f, \mathrm{sym} g)_m$

where δ_{nk} is Kronecker's symbol and $(.,.)_m$ denotes the scalar product in $L^2(\mathbf{R})$.

(4) $I_m(f)$ is real if $\overline{f(x_1, \ldots, x_m)} = f(-x_1, \ldots, -x_m)$.

We let

$$\int_A f(x_1, \ldots, x_m) Z_1(dx_1) \ldots Z_m(dx_m) = \int_{\mathbf{R}^m} f(x_1, \ldots, x_m) \mathbb{I}_A Z_1(dx_1) \ldots Z_m(dx_m).$$

and write $||f||_{L^2} = (\int_{\mathbf{R}^m} f^2(x_1, \ldots, x_m) dx_1 \ldots dx_m)^{1/2}$.

Corollary 5.1 *If* $||f||_{L^2} < \infty, ||g||_{L^2} < \infty$, *then*

$$|EI_m(f)I_m(g)| \leq m!||f||_{L^2}||g||_{L^2}$$

Special case. If $r_{i,j} = 1$ for $i, j = 1, \ldots m$, then the vector measure $(Z_1, \ldots, Z_m)$ becomes $(Z_1, \ldots, Z_1)$, that is, only Z_1 is involved. The multiple Wiener-Itô integrals in the preceding sections belonged to this special case.

6 Quadratic forms in Appell polynomials

Instead of supposing that X_t is Gaussian we shall now assume that it is given by a linear sequence

$$X_t = \sum_{u=-\infty}^{\infty} a(t-u)\xi_u, \quad t \in \mathbf{Z}. \tag{6.1}$$

The innovations ξ_u are i.i.d. random variables with zero mean and variance 1 and the weights $a(u)$ satisfy $\sum_u a^2(u) < \infty$ and are such that the spectral density $f(x)$ of X_t is given by (1.1), namely

$$f(x) = 2\pi|\widehat{a}(x)|^2 = |x|^{-\alpha}L(1/|x|), \quad -\infty < \alpha < 1. \tag{6.2}$$

We will use the Appell polynomials $P_{m,n}(X_t, X_s)$ where $m \geq 0, n \geq 0$ and $m + n \geq 1$. To ensure that these have finite variance, we suppose in the sequel that the innovations ξ_u satisfy $E|\xi_u|^{2(m+n)} < \infty$.

Set

$$Q_{[N\tau]} = \sum_{t=1}^{[N\tau]}\sum_{s=1}^{[N\tau]} b(t-s)P_{m,n}(X_t, X_s) \tag{6.3}$$

where $\tau \geq 0$ is the continuous time parameter. The weights $b(j) = \int_{-\pi}^{\pi} e^{ijx}\widehat{b}(x)dx$ are assumed to have the Fourier transform

$$\widehat{b}(x) = |x|^{-\beta}L_1(1/|x|), \ \ |x| \leq \pi, \quad (\beta < 1) \tag{6.4}$$

where L_1 is slowly varying function at infinity. When $n = 0$, $Q_{[N\tau]}$ reduces to

$$Q_{[N\tau]} = \sum_{t=1}^{[N\tau]}\sum_{s=1}^{[N\tau]} b(t-s)P_m(X_t). \tag{6.5}$$

To obtain other interesting special cases, set $\beta = 0$ and $L_1 = 1/2\pi$. Then $b(t) = \delta(t)$ and $Q_{[N\tau]}$ reduces to the linear form

$$Q_{[N\tau]} = \sum_{t=1}^{[N\tau]} P_{m,n}(X_t, X_t) = \sum_{t=1}^{[N\tau]} P_{m+n}(X_t), \tag{6.6}$$

and becomes

$$Q_{[N\tau]} = \sum_{t=1}^{[N\tau]} P_m(X_t) \tag{6.7}$$

if $n = 0$. Relation (6.7) is of the type considered in Section 2.

Limits for $Q_{[N\tau]}$ adequately normalized were obtained in special cases by Avram and Fox [1], Giraitis and Surgailis [12], [13], Terrin and Taqqu [28], [29]. To describe the general situation, define as before

$$d_m(\alpha) = 1 - m(1-\alpha), \tag{6.8}$$

but this time for $m = 0, 1, 2, \ldots$. The number $d_m(\alpha)$ is important because of the following fact: the spectral density $\psi_{m,m}(x)$ of $P_m(X_t) =: X_t^{(m)}:$, $m \geq 1$, is a continuous function if $d_m(\alpha) < 0$ and behaves $C|x|^{-d_m(\alpha)} L^m(1/|x|)$ if $d_m(\alpha) > 0$. For this reason, we say that $: X_t^{(m)} :$ is weakly dependent if $d_m(\alpha) < 0$ and strongly dependent if $d_m(\alpha) > 0$.

As in Giraitis, Taqqu and Terrin [18], we suppose for convenience $d_m(\alpha) \neq 0, d_n(\alpha) \neq 0$ when $m, n \geq 2$. We will consider the boundary case $d_m(\alpha) = 0$ only when $m = 1$ (then $0 = d_1(\alpha) = \alpha$, that is, $f(x) = L(1/|x|)$ in (6.2).)

It will be important to differentiate between "components" $:X_t^{(m)}:$ that are linear ($m = 1, : X_t^{(m)} := X_t$) and non-linear ($m \geq 2$) and also between components that are weakly dependent ($d_m(\alpha) < 0$) and strongly dependent ($d_m(\alpha) > 0$).

We have to distinguish five cases:

$$\begin{array}{lll}
(\mathbf{A_1}) & \text{either} & m = n = 1; \\
& \text{or} & 1 = m < n \text{ and } d_n(\alpha) > 0; \\
& \text{or} & 2 \leq m \leq n, \text{ and } d_m(\alpha) > 0,\ d_n(\alpha) > 0; \\[1em]
(\mathbf{A_2}) & \text{either} & 1 = m < n \text{ and } d_n(\alpha) < 0 \text{ (case } (\mathbf{A_2'})); \\
& \text{or} & 2 \leq m \leq n \text{ and } d_m(\alpha) > 0, d_n(\alpha) < 0; \\[1em]
(\mathbf{A_3}) & & 2 \leq m \leq n \text{ and } d_m(\alpha) < 0, d_n(\alpha) < 0; \\[1em]
(\mathbf{A_4}) & & m \geq 2,\ n = 0,\ d_m(\alpha) < 0; \\[1em]
(\mathbf{A_5}) & & \text{either } m = 1, n = 0; \\
& & \text{or } m \geq 2,\ n = 0,\ d_m(\alpha) > 0.
\end{array}$$

In the cases $(A_1), (A_2)$ and (A_3), we set for convenience $1 \leq m \leq n$.

The cases (A_1) to (A_5) cover a wide range of situations and finding the corresponding limits may involve different considerations. The linear form problem of Section 2 is covered by the case (A_5) and the quadratic form problem of Section 3 involving the product $X_t X_s$ is covered by the case (A_1). The case (A_1) also covers the quadratic form problem with $P_{m,n}(X_t, X_s)$, $m, n \geq 2$, but where both $: X_t^{(m)} :$ and $: X_t^{(n)} :$ are strongly dependent. The cases (A_2) and (A_3) are particularly delicate. The case (A_2) involves the mixed situation where one component is weakly dependent and the other is strongly dependent. The case (A_3) involves weak dependence but a large positive value of β may "compensate" for it and yield a non-Gaussian limit.

We have classified the cases according to the type of limit process one obtains. To summarize:

In the case (A_1), each component $:X_t^{(n)}:$ and $:X_t^{(m)}:$ is either linear or it is non-linear and strongly dependent. In case (A_2), $:X_t^{(m)}:$ satisfies the same condition as in (A_1) but $:X_t^{(n)}:$ is non-linear and weakly dependent. In case (A_3), both components are non-linear and weakly dependent. In case (A_4), $P_m(X_t)$ is weakly dependent ($d_m(\alpha) < 0$), and, in case (A_5), it is either linear or strongly dependent.

To characterize the limit of $A_N^{-1/2} Q_{[N\tau]}$ as $N \to \infty$, introduce a random noise vector

$$(Z^{(1)}(dx), Z^{(m)}(dx), Z^{(n)}(dx)) \tag{6.9}$$

whose components are complex-valued Gaussian measures $Z^{(1)}(dx)$, $Z^{(m)}(dx)$ and $Z^{(n)}(dx)$ with zero mean such that

$$EZ^{(l)}(dx)\overline{Z^{(l')}(dy)} = 0 \quad \text{if} \quad x \neq y; \quad l, l' = 1, m, n$$
$$EZ^{(l)}(dx)\overline{Z^{(l')}(dx)} = v(l, l')dx;$$

where

$$v(1,1) = 1;$$
$$v(1,l) = \text{Cov } (\xi_0, : X_0^{(l)} :) \quad \text{if} \quad 2 \leq l,\ l = m, n;$$
$$v(l,l') = \psi_{l,l'}(0) \quad \text{if} \quad l, l' \geq 2 \quad \text{and either equal to } m \text{ or } \ n.$$

The function $\psi_{l,l'}(x)$ is the cross-spectral density of $: X_t^{(l)} :$ and $: X_t^{(l')} :$ and it is defined by

$$E : X_t^{(l)} :: X_s^{(l')} := \int_{-\pi}^{\pi} e^{(t-s)x} \psi_{l,l'}(x)dx$$

and appears in the case (A_3). In the special case

$$\hat{a}(x) \sim [f(x)/2\pi]^{1/2} \sim [|x|^{-\alpha/2}\ L(1/|x|)]^{1/2} \quad (x \to 0) \tag{6.10}$$

considered in Terrin and Taqqu [29], one can show that

$$\psi_{m,n}(0) = (1/2\pi) \sum_t \text{Cov } (: X^{(m)}(t) :, : X^{(n)}(0) :). \tag{6.11}$$

Each case (A_1), (A_2), ..., (A_5), yields a different limiting process $Z_{m,n}^{(1)}(\tau), Z_{m,n}^{(2)}(\tau), \ldots, Z_{m,n}^{(5)}(\tau)$. These processes are characterized by an integral representation as follows:

$$\begin{aligned}
Z_{m,n}^{(1)}(\tau) &= \int_{\mathbf{R}^{m+n}} \Phi_\tau(\sum_{i=1}^m x_i, \sum_{i=m+1}^{m+n} x_i)|x_1|^{-\alpha/2} \ldots |x_{m+n}|^{-\alpha/2} Z^{(1)}(dx_1) \ldots Z^{(1)}(dx_{m+n}),\\
Z_{m,n}^{(2)}(\tau) &= \int_{\mathbf{R}^{m+1}} \Phi_\tau(\sum_{i=1}^m x_i, x_{m+1})|x_1|^{-\alpha/2} \ldots |x_m|^{-\alpha/2} Z^{(1)}(dx_1) \ldots Z^{(1)}(dx_m) Z^{(n)}(dx_{m+1}),\\
Z_{m,n}^{(3)}(\tau) &= \int_{\mathbf{R}^2} \Phi_\tau(x_1, x_2) Z^{(m)}(dx_1) Z^{(n)}(dx_2),\\
Z_{m,0}^{(4)}(\tau) &= \psi_m(0) \int_{\mathbf{R}} \Phi_\tau(x, 0) Z^{(1)}(dx),\\
Z_{m,0}^{(5)}(\tau) &= \int_{\mathbf{R}^m} \Phi_\tau(x_1 + \ldots + x_m, 0)|x_1|^{-\alpha/2} \ldots |x_m|^{-\alpha/2} Z^{(1)}(dx_1) \ldots Z^{(1)}(dx_m).
\end{aligned}$$

where

$$\Phi_\tau(x_1, x_2) = \int_{\mathbf{R}} K_\tau(x_1 + u) K_\tau(x_2 - u)|u|^{-\beta} du, \quad K_\tau(x) = \frac{e^{i\tau x} - 1}{ix}.$$

The limit processes $(Z_{m,n}^{(j)}(\tau)), j = 1, \ldots, 5$ are *self-similar* with exponent $H = \gamma/2$. They typically do not have stationary increments. Observe, however, that if $\beta = 0$, then $Z_{m,0}^{(4)}$ is Brownian motion and $Z_{m,0}^{(5)}(1)$ is the random variable (2.5).

When the processes $(Z^{(j)}_{m,n}(\tau)), j = 1, \ldots, 5$ are the limit of $A_N^{-1/2} Q_{[N\tau]}$, the normalization factor of A_N is given by

(6.12) $$A_N = N^\gamma L^*(N),$$

where

(6.13) $$L^*(N) = \begin{cases} L(N)^{m+n} L_1^2(N) & \text{in cases } A_1 \text{ and } A_5 \\ L(N)^m L_1^2(N) & \text{in case } A_2 \\ L_1^2(N) & \text{in cases } A_3 \text{ and } A_4. \end{cases}$$

The value of γ is given, in all cases, by

(6.14) $$\gamma = d_m^+(\alpha) + d_n^+(\alpha) + 2\beta$$

where

$$d_m^+(\alpha) = \begin{cases} \alpha & \text{if } m = 1 \\ \max(d_m(\alpha), 0) & \text{if } m = 0 \text{ or } m \geq 2. \end{cases}$$

This definition of $d_m^+(\alpha)$ is similar to the one given in (2.6) but includes also the case $m = 0$. When $m = 0, d_m(\alpha) = 1 - m(1-\alpha)$ (see (6.8)) gives $d_0(\alpha) = 1$ and hence one has $d_0^+(\alpha) = 1$.

Remarks

- When the normalization factor A_N is given by $N^\gamma L^*(N)$, the exponent cannot be negative. Since $d_m(\alpha) < 1$, the condition $\gamma > 0$ implies $\beta > -1$, and in the case where either $m = 1$ or $n = 1$, it implies $\alpha > -1$ as well.
- Suppose (X_t) is Gaussian. Then $v(l, l') = 0$ if $(l \neq l')$, since Cov $(\xi_0 : X_0^{(l)} :) = 0$ if $l \geq 2$. Also $\psi_{m,n}(\lambda) \equiv 0$ if $m \neq n$.

Theorem 6.1 (see Giraitis, Terrin and Taqqu [18], Theorems 2.1-2.3). *If $m, n \geq 1$, suppose*

$$1 < \gamma = d_m^+(\alpha) + d_n^+(\alpha) + 2\beta.$$

If $n = 0, m \geq 1$, suppose

$$0 < \gamma = d_m^+(\alpha) + 1 + 2\beta.$$

Then

(6.15) $$\left(A_N^{-1/2} Q_{[N\tau]}\right)_{0 \leq \tau < \infty} \Rightarrow \left(Z^{(j)}_{m,n}(\tau)\right)_{0 \leq \tau < \infty}, \quad (N \to \infty)$$

holds in the case (A_j) for each $j = 1, 2, 3, 4, 5$, where $\Rightarrow$ denote the convergence of finite dimensional distributions. (In the case (A_2'), also assume (6.10).)

Recall that the cases $(A_1), (A_2)$ and (A_3) involve $m, n \geq 1$ and the cases (A_4) and (A_5) involve $m = 0, m \geq 1$. The result in this second situation is remarkable because γ is allowed to be not only greater than 1 but also less or equal to 1. These situations occur not only in the Gaussian case $m = 1$ but also in the non-Gaussian case $m \geq 2$.

Example 6.1 In the case (A_3) where $2 \le m \le n$ and $d_m(\alpha) < 0,\ d_n(\alpha) < 0$ we have $d_m^+(\alpha) = 0,\ d_n^+(\alpha) = 0$ and hence the condition $\gamma > 1$ reduces to $\beta > 1/2$. We then get

$$\frac{1}{N^\beta |L_1(N)|} \sum_{t=0}^{[N\tau]} \sum_{s=0}^{[N\tau]} b(t-s) P_{m,n}(X_t, X_s) \Rightarrow Z_{m,n}^{(3)}(\tau),$$

which is a non-Gaussian process.

Example 6.2 In the case (A_4) where $n = 0,\ m \ge 2$ and $d_m(\alpha) < 0$, we have $d_m^+(\alpha) = 0$. Hence the condition $\gamma > 0$ reduces again to $\beta > 1/2$ and we get

$$\frac{1}{N^\beta |L_1(N)|} \sum_{t=1}^{[N\tau]} \sum_{s=1}^{[N\tau]} b(t-s) P_m(X_t) \Rightarrow Z_{m,0}^{(4)}(\tau)$$

which is a Gaussian process.

Example 6.3 Suppose that all slowly varying functions equal 1, $n = 0$, $m \ge 2$, $1 - 1/m < \alpha < 1$, so that $d_m(\alpha) = 1 - m(1-\alpha) > 0$. Choose $\beta = -d_m(\alpha)/2$. Then $\gamma = 1$ and

$$\frac{1}{\sqrt{N}} Q_{[N\tau]} \overset{d}{\Rightarrow} C Z_{m,0}^{(5)}(\tau)$$

where C is a constant and $Z_{m,0}^{(5)}$ is a non-Gaussian process. Observe that the convergence takes place with the classical normalization $\sqrt{N}$.

What happens if $-\infty < \gamma < 1$ when $m, n \ge 1$? In this situation, the normalization is not $A_N = N^\gamma L^*(N)$ anymore, but it is $A_N = N$ and the limit is Brownian motion, even though Q_N involves a double summation:

Theorem 6.2 (see Theorems 2.4, 6.2 in Giraitis and Taqqu [15]). Let $m, n \ge 1$. Suppose (6.2), (6.4) *and* $-\infty < \gamma < 1$. *Then*

$$\left(N^{-1/2} Q_{[N\tau]}\right)_{0 \le \tau < \infty} \Rightarrow \left(\sigma B(\tau)\right)_{0 \le \tau < \infty}, \quad (N \to \infty)$$

where $(B(\tau))$ *is Brownian motion, and* σ^2 *is given by Relation (2.6) in Giraitis and Taqqu [15].*

Theorem 6.1 is a non-central limit theorem (NCLT) whereas Theorem 6.2 is a central limit theorem (CLT).

Example 6.4 Case (A_1) with $m = n = 1$. Then $d_m(\alpha) = d_n(\alpha) = \alpha$ and $\gamma = 2\alpha + 2\beta$. We get the NCLT if $\alpha + \beta > 1/2$ (because $\gamma > 1$) and the CLT if $\alpha + \beta < 1/2$ (because $\gamma < 1$). This is the case discussed in Section 3 with (X_t) Gaussian.

7 Multivariate limit theorems

We provide in this section some multivariate generalizations of the non-central and central limit theorems given in the previous sections.

7.1 Different weight sequences b

We now consider the situation where there are $k \geq 1$ different quadratic forms $Q^{(i)}_{[N\tau]}$, $i = 1, \dots, k$, each with its own weight sequence $b_i(.)$. Thus suppose

$$Q^{(i)}_{[N\tau]} := \sum_{t=1}^{[N\tau]} \sum_{s=1}^{[N\tau]} b_i(t-s) P_{m_i,n_i}(X_t, X_s), \quad (m_i + n_i \geq 1), \quad i = 1, ..., k, \tag{7.1}$$

where (X_t) is the linear sequence (6.1) whose spectral density satisfies the Relation (6.2) and where the weights $b_i(.)$ are Fourier coefficients of the functions $\widehat{b}_i(x)$, defined by

$$\widehat{b}_i(x) = |x|^{-\beta_i} L_i(1/|x|), \;\; |x| \leq \pi, \quad (\beta_i < 1) \quad i = 1, \dots, k, \tag{7.2}$$

where $L_i, i = 1, \dots, k$ are slowly varying functions at infinity. Assume $E|\xi_0|^{2(m_i+n_i)} < \infty$, in order to ensure that $Q^{(i)}_N$ has finite second moments for all i. For each $i = 1, \dots, k$:

γ_i is defined by (6.14) using β_i instead of β.

$(Z^{(j)}_{m_i,n_i}(\tau))$ is defined as $(Z^{(j)}_{m,n}(\tau))$ (see Section 6) using β_i instead of β, and where $j = 1, \dots, 5$ labels the type of process. These processes correspond to the cases (A_j), $j = 1, \dots, 5$, respectively, considered in Section 6.

$L^*_i(N)$ is defined as $L^*(N)$ by replacing $L_1(N)$ by $L_i(N)$.

$A_{N,i} = N^{\gamma_i} L^*_i(N)$.

The following theorem extends the NCLT Theorem 6.1.

Theorem 7.1 *Suppose that each of the quadratic form $Q^{(i)}_N$, $i = 1, ..., k$ satisfies the assumptions of Theorem 6.1. Then, for each of the cases (A_j), $j = 1, \dots, 5$,*

$$(A^{-1/2}_{N,1} Q^{(1)}_{[N\tau]}, \dots, A^{-1/2}_{N,k} Q^{(k)}_{[N\tau]})_{0 \leq \tau < \infty} \Rightarrow (Z^{(j)}_{m_1,n_1}(\tau), \dots, Z^{(j)}_{m_k,n_k}(\tau))_{0 \leq \tau < \infty} \quad (N \to \infty). \tag{7.3}$$

Proof: The proof is similar to that of Theorems 2.1- 2.3 in Giraitis, Taqqu and Terrin [18].

The following is an extension of the CLT Theorem 6.2.

Theorem 7.2 *Let $m, n \geq 1$. Suppose that each of the quadratic form $Q^{(i)}_N$, $i = 1, ..., k$ satisfies the assumptions of the Theorem 6.2. Then*

$$(N^{-1/2} Q^{(1)}_{[N\tau]}, \dots, N^{-1/2} Q^{(k)}_{[N\tau]})_{0 \leq \tau < \infty} \Rightarrow (B_1(\tau), \dots, B_k(\tau))_{0 \leq \tau < \infty} \quad (N \to \infty) \tag{7.4}$$

where $(B_1(\tau), \dots, B_k(\tau))$ is a zero mean Gaussian vector with covariance matrix $EB_i(t)B_j(s) = \sigma_{i,j} \min(t, s)$. The $\sigma_{i,j}$, $i, j = 1, \dots, k$ are the same as in Theorem 6.1 in Giraitis and Taqqu [15].

Proof: This is a simple generalization of the proof of Theorems 2.4, 6.1-6.2 in Giraitis and Taqqu [15]. □

7.2 Different weight sequences b and a

In the preceding theorems, each $Q^{(i)}_{[N\tau]}$ involves its own weight sequence $b_i(t)$ but the process (X_t) is the same for each i. We now allow the weights of the linear sequence (X_t) to depend also on i. Let

$$X_t^{(i,l)} = \sum_{s=-\infty}^{\infty} a^{(i,l)}(t-s)\xi_s,\ t \in \mathbf{Z}, \quad i = 1, \ldots, k;\ l = 1, 2$$

be $2k$ stationary time series with the same i.i.d. innovation sequence $\{\xi_u, u \in \mathbf{Z}\}$ and consider

$$Q^{(i)}_{[N\tau]} := \sum_{t=1}^{[N\tau]} \sum_{s=1}^{[N\tau]} b_i(t-s) P_{m_i,n_i}(X_t^{(i,1)}, X_s^{(i,2)}), \tag{7.5}$$

$i = 1, ..., k$. Assume again $E\xi_0 = 0$ and also $E|\xi_0|^{2(m_i+n_i)} < \infty$ in order to ensure that $Q^{(i)}_N$ has finite second moments for all i.

We assume that the spectral densities $f^{(i,l)}$ of the sequences $(X_t^{(i,l)})$ satisfy, as in (6.2) the condition

$$f^{(i,l)}(\lambda) = L_{(i,l)}(1/|\lambda|)|\lambda|^{-\alpha_{(i,l)}}, \quad |\lambda| \le \pi \quad (\alpha_{(i,l)} < 1) \tag{7.6}$$

and that the Fourier transforms of the weights $b_i(t)$ satisfy the assumption (7.2); throughout, $i = 1, \ldots, k$ and $l = 1, 2$.

The corresponding CLT result is as follows:

Theorem 7.3 *Let $m_i, n_i \ge 1$ and*

$$-\infty < \gamma_i := d^+_{m_i}(\alpha_{i,1}) + d^+_{n_i}(\alpha_{i,2}) + 2\beta_i < 1, \quad i = 1, \ldots, k.$$

Then the CLT holds:

$$(N^{-1/2}Q^{(1)}_{[N\tau]}, \ldots, N^{-1/2}Q^{(k)}_{[N\tau]})_{0\le\tau<\infty} \Rightarrow (B_1(\tau), \ldots, B_k(\tau))_{0\le\tau<\infty} \quad (N \to \infty), \tag{7.7}$$

where $(B_1(\tau), \ldots, B_k(\tau))$ is zero mean Gaussian vector with covariance matrix $EB_i(t)B_j(s) = \sigma_{i,j}\min(t,s)$. The $\sigma_{i,j}$ are the same as in Theorem 6.1 in Giraitis and Taqqu [15].

PROOF: The proof of Theorem 7.2 is similar to that of Theorems 2.4, 6.1-6.2 in Giraitis and Taqqu [15]. □

We now turn to NCLTs. We must again distinguish between the cases (A_j), $j = 1, \ldots, 5$, but some additional notation is needed.

For each $i = 1, \ldots, k$, we let the parameters $(\alpha_i, \beta_i, m_i, n_i)$ correspond to one of the cases $(A_1) - (A_5)$ of Section 6.

For example, in the case (A_1), we suppose for all $i = 1, \ldots, k$, either

(i) $m_i = n_i = 1$; or

(ii) $1 = m_i < n_i, d_{n_i}(\alpha_{i,2}) > 0$; or

(iii) $2 \le m_i \le n_i, d_{m_i}(\alpha_{i,1}) > 0,\ d_{n_i}(\alpha_{i,2}) > 0$.

Similar extensions are made to the cases $(A_2) - (A_5)$. Then:

Case ($\mathbf{A_1}$):
Define $A_{N,i} = N^{\gamma_i} L_{(i,1)}^{m_i}(N) L_{(i,2)}^{n_i}(N) L_i^2(N)$, and set

$$Z_{i;m_i,n_i}^{(1)}(\tau) = \int_{\mathbf{R}^{m_i+n_i}} \Phi_\tau^{(i)}(\sum_{j=1}^{m_i} x_j, \sum_{j=m_i+1}^{m_i+n_i} x_j)|x_1|^{-\alpha_{i,1}/2} \dots |x_{m_i}|^{-\alpha_{i,1}/2}$$

$$\times |x_{m_i+1}|^{-\alpha_{i,2}/2} \dots |x_{m_i+n_i}|^{-\alpha_{i,2}/2} Z^{(1)}(dx_1) \dots Z^{(1)}(dx_{m_i}) Z^{(1)}(dx_{m_i+1}) \dots Z^{(1)}(dx_{m_i+n_i}).$$

Case ($\mathbf{A_2}$): Define $A_{N,i} = N^{\gamma_i} L_{(i,1)}^{m_i}(N) L_i^2(N)$,

$$Z_{i;m_i,n_i}^{(2)}(\tau) = \int_{\mathbf{R}^{m_i+1}} \Phi_\tau^{(i)}(\sum_{j=1}^{m_i} x_j, x_{m_i+1})|x_1|^{-\alpha_{i,1}/2} \dots |x_{m_i}|^{-\alpha_{i,1}/2} Z^{(1)}(dx_1) \dots Z^{(1)}(dx_{m_i}) Z_{(i,2)}^{(n_i)}(dx_{m_i+1}).$$

Case ($\mathbf{A_3}$): Set $A_{N,i} = N^{\gamma_i} L_i^2(N)$,

$$Z_{i;m_i,n_i}^{(3)}(\tau) = \int_{\mathbf{R}^2} \Phi_\tau^{(i)}(x_1, x_2) Z_{(i,1)}^{(m_i)}(dx_1) Z_{(i,2)}^{(n_i)}(dx_2).$$

Case ($\mathbf{A_4}$): Set $A_{N,i} = N^{\gamma_i} L_i^2(N)$,

$$Z_{i;m_i,n_i}^{(4)}(\tau) = \int_{\mathbf{R}^1} \Phi_\tau^{(i)}(x_1, 0) Z_{(i,1)}^{(m_i)}(dx_1).$$

Case ($\mathbf{A_5}$): Set $A_{N,i} = N^{\gamma_i} L_{(i,1)}^{m_i} L_i^2(N)$,

$$Z_{i;m_i,n_i}^{(5)}(\tau) = \int_{\mathbf{R}^{m_i}} \Phi_\tau^{(i)}(\sum_{j=1}^{m_i} x_j, 0)|x_1|^{-\alpha_{i,1}/2} \dots |x_{m_i}|^{-\alpha_{i,1}/2} Z^{(1)}(dx_1) \dots Z^{(1)}(dx_{m_i}).$$

The complex-valued Gaussian measures $Z^{(1)}(dx), Z_{(i,l)}^{(m_i)}(dx)$, $i = 1, \dots, k, l = 1, 2$, that appear in the above integrals have the property $Z^{(1)}(dx) = \overline{Z^{(1)}(-dx)}$; $Z_{(i,l)}^{(m_i)}(dx) = \overline{Z_{(i,l)}^{(m_i)}(-dx)}$, have zero mean and are such that

$EZ^{(1)}(A)\overline{Z^{(1)}(B)} = |A \cup B| = \int_{A \cap B} dx$,

$EZ^{(1)}(A)\overline{Z_{(i,l)}^{(m)}(B)} = \text{Cov}\,(\xi_0, P_m(X_0^{(i,l)}))|A \cup B|$, where $m = m_i$ or n_i $(m \geq 2)$;

$EZ_{(i,l)}^{(m)}(A)\overline{Z_{(i',l')}^{(m')}(B)} = \psi_{m,m'}^{(i,l),(i',l')}(0)|A \cup B|$ where $\psi_{m,m'}^{(i,l),(i',l')}(\cdot)$ is the cross-spectral density of the sequences $(P_m(X_t^{(i,l)}))$ and $(P_{m'}(X_t^{(i',l')}))$. This last covariance is relevant in the case (A_3).

Theorem 7.4 *Consider the sequences $(X_t^{(i,l)})$, $i = 1, \dots, k$, $l = 1, 2$ and the weights $(b_i(t))$, $i = 1, \dots, k$ defined above. If $m_i, n_i \geq 1$, then suppose*

(7.8) $$\gamma_i := d_{m_i}^+(\alpha_{i,1}) + d_{n_i}^+(\alpha_{i,2}) + 2\beta_i > 1;$$

if $n_i = 0$, $m_i \geq 1$, then suppose

(7.9) $$0 < \gamma_i := d_{m_i}^+(\alpha_{i,1}) + 1 + 2\beta_i.$$

Under these assumptions, for each of the cases (A_j), $j = 1, \dots, 5$,

(7.10) $$(A_{N,1}^{-1/2} Q_{[N\tau]}^{(1)}, \dots, A_{N,k}^{-1/2} Q_{[N\tau]}^{(k)})_{0 \leq \tau < \infty} \Rightarrow (Z_{1;m_1,n_1}^{(j)}(\tau), \dots, Z_{k;m_k,n_k}^{(j)}(\tau))_{0 \leq \tau < \infty} \quad (N \to \infty).$$

PROOF: The proof is similar to that of Theorems 2.1-2.3 in Giraitis, Taqqu, and Terrin [18], Theorem 2.3. □

8 Central limit theorems with time domain conditions

We consider first the univariate case where $(X(t))$ is given by (6.1) and where $Q_{[N\tau]}$ is defined by (6.3). Here, however, instead of making assumptions on the spectral density f of $(X(t))$ and on the Fourier transform $\widehat{b}$ of b, we shall make ℓ^p-type assumptions on the covariance $r(t) = EX_0X_t$ and on the kernel b. Note first that

$$\sum_{t\in\mathbf{Z}} \Big| \mathrm{Cov}\,(P_{m,n}(X_t, X_{t+h_1}), P_{m,n}(X_0, X_{h_2})) \Big| < \infty, \quad h_1, h_2 \in \mathbf{Z}, \tag{8.1}$$

is equivalent to the condition

$$\sum_t |r(t)|^{m+n} < \infty, \tag{8.2}$$

(Giraitis and Taqqu [16], Lemma 4.1). Because we are looking at quadratic forms (6.3), the condition that will ensure the CLT is a modification of (8.1), namely

$$\sum_{h_1,h_2,t\in\mathbf{Z}} \Big| b(h_1)b(h_2)\,\mathrm{Cov}\,(P_{m,n}(X_t, X_{t+h_1}), P_{m,n}(X_0, X_{h_2})) \Big| < \infty. \tag{8.3}$$

Theorem 8.1 (c.f. Giraitis and Taqqu [16], Theorem 2.1) *Suppose that* (8.3) *holds and, if* $b(0) = 0$, *suppose in addition* $\sum_{t\in\mathbf{Z}} |r(t)|^{m+n} < \infty$. *Then the CLT holds:*

$$N^{-1/2} Q_N \Rightarrow \sigma B(\tau), \tag{8.4}$$

where $B(\tau)$ *is standard Brownian motion and where the limiting variance* σ^2 *is given by*

$$\sigma^2 = \sum_{h_1,h_2,t\in\mathbf{Z}} b(h_1)b(h_2)\,\mathrm{Cov}\,(P_{m,n}(X_t, X_{t+h_1}), P_{m,n}(X_0, X_{h_2})). \tag{8.5}$$

PROOF: The proof is an easy generalization of that of Theorem 2.1 in Giraitis and Taqqu [16], which only involved the case $\tau = 1$. (The other theorems in this section involve similar extensions of results in Giraitis and Taqqu [16].) □

The assumption (8.3) underlines the similarities between the central limit theorem for quadratic forms in linear or Gaussian variables with long-range dependence and the central limit theorem for univariate sums $S_N = N^{-1/2}\sum_{t=1}^N G(X_t)$. Breuer and Major [4] proved that if the covariance of the Gaussian stationary process (X_t) satisfies the condition $\sum_{t\in\mathbf{Z}} |r(t)|^m < \infty$ where m is the "Hermite rank" of the function G (see Section 2), and if $EG(X_t) = 0$ and $EG(X_t)^2 < \infty$, then the CLT for S_N holds. Giraitis and Surgailis [10] showed that the asymptotic normality of $N^{-1/2}\sum_{t=1}^N G(X_t)$ can be established in terms of the correlation function $r_G(t) = EG(X_t)G(X_0)$, without referring specifically to the concept of Hermite or Appell rank (it plays, however, an important role in the proof). They showed that the CLT holds if $\sum_{t\in\mathbf{Z}} |r_G(t)| < \infty$ when X_t is Gaussian. In addition, Giraitis [9] proved that if $\sum_{t\in\mathbf{Z}} |r_G(t)| < \infty$, then the CLT holds also for the linear process X_t (6.1) if the function G is sufficiently smooth, for example, if G is a polynomial. The condition (8.3) thus extends the univariate condition $\sum_{t\in\mathbf{Z}} |r_G(t)| < \infty$ to quadratic forms. However, in contrast to the CLT for univariate sums, it does not cover certain additional cases, first discovered by Fox and Taqqu [7], where the CLT for quadratic form also holds. Specifically, it excludes the possible compensation of the long-range dependence of (X_t) by a fast decay of the weights $b(t)$. These cases, which do not have a simple formulation in the time domain, are best characterized in the spectral domain. Conditions for the CLT in these situations were stated in Sections 6 and 7, under assumptions involving m and n and the spectral density of (X_t). The conditions of this section are stated in the time domain and are, in general, not equivalent to those of Sections 6 and 7.

The method of proof for establishing the CLT is also different in the time domain. Whereas in the spectral domain, one can use approximation methods, here one resorts to the method of moments. This is because, in the spectral domain, one can approximate the possibly unbounded spectrum by a bounded one. In the time domain, however, one has to deal directly with the covariances, which decrease slowly.

The conditions of the next theorem are formulated in terms of the ℓ^p norms of the covariance $r(t)$ (namely, $(\sum_{t\in\mathbf{Z}}|r(t)|^p)^{1/p}$) and also of the weights $b(t)$. These conditions are of a global nature and do not require the power (or regular variation) decay of $r(t)$ and $b(t)$ as $t\to\infty$. Therefore, they do not imply the regular variation of the spectral density $f(x)$ (Fourier transform of the covariance $r(t)$), nor of the Fourier transform $\widehat{b}(x)$ of the weights $b(t)$ as the frequency $x\to 0$.

Theorem 8.2 (c.f. Giraitis and Taqqu [16], Theorem 2.2) *The conclusion of Theorem 8.1 remains true if*

$$r\in\ell^p,\quad b\in\ell^q\quad (p,q\geq 1) \tag{8.6}$$

and

$$\min(mp^{-1},1)+\min(np^{-1},1)+2q^{-1}\geq 3. \tag{8.7}$$

To understand the Condition (8.7) suppose that $r(t)=Ct^{-(1-\alpha)}L(t)$ and $b(t)=C't^{-(1-\beta)}L_1(t)$, $t\in\mathbf{Z}$, which corresponds to the conditions on f and $\widehat{b}$ given in (6.2) and (6.4) respectively when the slowly varying functions are nicely behaved. As usual, we suppose $\alpha<1$ and $\beta<1$. Condition (8.7), however, can only be used effectively if $\alpha>0$ and $\beta>0$, that is, when there is no compensation. Since $r\in\ell^p$ if $p^{-1}<1-\alpha$ and $b\in\ell^q$ if $q^{-1}<1-\beta$, it is easy to see that (8.7) corresponds to $\gamma=d_m^+(\alpha)+d_n^+(\alpha)+2\beta<1$ (we ignore boundary cases). In view of Theorem 6.2, this is when we expect the CLT to hold if $m,n\geq 1$. (If $m\geq 1, n=0$, the condition $\gamma<1$ becomes vacuous.) Thus, we get nothing more when r and b are as above. On the other hand, Theorem 8.2 can be used in cases where r and b or f and $\widehat{b}$ are not regularly varying, for example, when f or $\widehat{b}$ diverge at frequencies other than the origin, or as in the following example.

Example 8.1 The CLT holds if both r and b decrease exponentially fast (in this case $p=q=1$).

We now turn to multivariate versions of the preceding theorems. The processes are defined as in Section 7.

Theorem 8.3 (c.f. Giraitis and Taqqu [16], Theorem 5.1) *Suppose that each quadratic form*

$$Q_N^{(i)}:=\sum_{t,s=1}^{N} b_i(t-s)P_{m_i,n_i}(X_t^{(i,1)},X_s^{(i,2)}),\quad i=1,...,k$$

satisfies the assumptions

$$\sum_{h_1,h_2,t\in\mathbf{Z}}\left|b_i(h_1)b_j(h_2)\operatorname{Cov}\left(P_{m_i,n_i}(X_t^{(i,1)},X_{t+h_1}^{(i,2)}),P_{m_j,n_j}(X_0^{(j,1)},X_{h_2}^{(j,2)})\right)\right|<\infty,\quad 1\leq i,j\leq k \tag{8.8}$$

and

$$\sum_{t\in\mathbf{Z}}|r_{(i,l),(i',l')}(t)|^{m_i+n_i}<\infty,\quad 1\leq i,i'\leq k, 1\leq l,l'\leq 2. \tag{8.9}$$

Then as $N \to \infty$

$$N^{-1/2}(Q^{(1)}_{[N\tau]}, ..., Q^{(k)}_{[N\tau]}) \Rightarrow (B^{(1)}(\tau), ..., B^{(k)}(\tau)), \tag{8.10}$$

where $(B^{(1)}(\tau), ..., B^{(k)}(\tau))$ *is the Gaussian vector process with zero mean and cross-covariances*

$$EB^{(i)}(t)B^{(j)}(s) = \min(t,s)\sigma_{i,j} \tag{8.11}$$

where

$$\sigma_{i,j} := \sum_{h_1,h_2,t\in\mathbf{Z}} b_i(h_1)b_j(h_2)\,\mathrm{Cov}\,(P_{m_i,n_i}(X_t^{(i,1)}, X_{t+h_1}^{(i,2)}), P_{m_j,n_j}(X_0^{(j,1)}, X_{h_2}^{(j,2)})).$$

In view of the equivalence between (8.1) and (8.2), Condition (8.9) is automatically satisfied for any $i = i'$ if $X^{(i,1)} = X^{(i,2)}$ and $b_i(0) \neq 0$.

Theorem 8.4 (c.f. Giraitis and Taqqu [16], Theorem 5.2) *The conclusion of Theorem 8.3 remains true if Condition* (8.8) *is replaced by:*

$$r_{(i,1),(i,1)} \in L^{p_{(i,1)}}, \quad r_{(i,2),(i,2)} \in L^{p_{(i,2)}}, \quad b_i \in L^{q_i} \quad (p_{(i,1)}, p_{(i,2)}, q_i \geq 1), \tag{8.12}$$

for $1 \leq i \leq k$, *and*

$$\min(m_i p_{(i,1)}^{-1}, 1) + \min(n_i p_{(i,2)}^{-1}, 1) + 2q_i^{-1} \geq 3 \quad (1 \leq i \leq k). \tag{8.13}$$

In addition, $a^{(i,1)} \in L^1$ *if* $k > 1$, $m_i = 1, n_i = 0$, *and* $a^{(i,2)} \in L^1$ *if* $k > 1$, $m_i = 0, n_i = 1$.

Remark. This extra condition was not required in the "univariate" version, Theorem 8.2, where only one form was considered.

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