Decomposition of self-similar stable mixed moving averages *†‡

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Abstract

Let $\alpha \in (1,2)$ and X_α be a symmetric α-stable ($S\alpha S$) process with stationary increments given by the mixed moving average

$$X_\alpha(t) = \int_X \int_{\mathbb{R}} (G(x,t+u) - G(x,u)) M_\alpha(dx,du),\ t \in \mathbb{R},$$

where $(X, \mathcal{X}, \mu)$ is a standard Lebesgue space, $G : X \times \mathbb{R} \mapsto \mathbb{R}$ is some measurable function and M_α is a $S\alpha S$ random measure on $X \times \mathbb{R}$ with the control measure $m_\alpha(dx,du) = \mu(dx)du$. We show that if X_α is self-similar, then it is determined by a nonsingular flow, a related cocycle and a semi-additive functional. By using the Hopf decomposition of the flow into its dissipative and conservative components, we establish a unique decomposition in distribution of X_α into two independent processes

$$X_\alpha \stackrel{d}{=} X_\alpha^D + X_\alpha^C,$$

where the process X_α^D is determined by a nonsingular dissipative flow and the process X_α^C is determined by a nonsingular conservative flow. In this decomposition, the linear fractional stable motion, for example, is determined by a conservative flow.

1 Introduction

In a fundamental paper, Rosiński (1995) considered general symmetric α-stable, real-valued *stationary* processes $\{X_\alpha(t)\}_{t\in\mathbb{R}}$ (they may be indexed by $t \in \mathbb{Z}$ and can be complex-valued as well) having the representation

$$\{X_\alpha(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{ \int_S f_t(s) M_\alpha(ds) \right\}_{t\in\mathbb{R}}, \tag{1.1}$$

where $\stackrel{d}{=}$ stands for the equality in the sense of the finite-dimensional distributions. Here $(S, \mathcal{S}, \nu)$ is a standard Lebesgue space, $\{f_t\}_{t\in\mathbb{R}} \subset L^\alpha(S,\nu)$ with $\alpha \in (0,2)$ and M_α is a symmetric α-stable ($S\alpha S$) random measure on S with the control measure $\nu(ds)$ (see Chapter 3 Samorodnitsky and Taqqu (1994)). Recall that a random variable ξ is $S\alpha S$ with $\alpha \in (0,2)$ if its characteristic function

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satisfies $E\exp(i\theta\xi) = \exp(-\sigma^\alpha|\theta|^\alpha)$ for some $\sigma > 0$ and all $\theta \in \mathbb{R}$ and that a real-valued stochastic process $\{X_\alpha(t)\}_{t\in\mathbb{R}}$ is $S\alpha S$ with $\alpha \in (0,2)$ if all its linear combinations are $S\alpha S$ random variables. An example of a standard Lebesgue space is the Euclidean space $\mathbb{R}^n$, with a measure consisting of Lebesgue measure and discrete point masses (for a general definition, see Section 3 below).

Assuming some minimality conditions on the spectral representation $\{f_t\}_{t\in\mathbb{R}}$, Rosiński deduced that there exist a unique (modulo ν) flow $\{\phi_t\}_{t\in\mathbb{R}}$ and a cocycle $\{a_t\}_{t\in\mathbb{R}}$ for $\{\phi_t\}_{t\in\mathbb{R}}$ taking values in $\{-1,1\}$ such that, for all $t \in \mathbb{R}$,

$$f_t(s) = a_t(s)\left\{\frac{d(\nu\circ\phi_t)}{d\nu}(s)\right\}^{1/\alpha} f_0(\phi_t(s)) \quad \text{a.e. } \nu(ds). \tag{1.2}$$

(Informally, a flow means that $\phi_{s+t} = \phi_s \circ \phi_t$ and a cocycle means that $a_{s+t} = (a_s)(a_t \circ \phi_s)$ for all $s,t \in \mathbb{R}$. For a precise definition see Section 3 below.) Then, using standard results of ergodic theory, Rosiński decomposed uniquely any stationary $S\alpha S$ process X_α having the representation (1.1) into two independent processes

$$X_\alpha \stackrel{d}{=} X_\alpha^D + X_\alpha^C. \tag{1.3}$$

The process $\{X_\alpha^D(t)\}_{t\in\mathbb{R}}$ is generated by a dissipative flow (see Section 3 for a definition) and has the representation

$$\int_X\int_{\mathbb{R}} g(x,t+u)M_\alpha(dx,du),\ t\in\mathbb{R}, \tag{1.4}$$

where (X,μ) is some standard Lebesgue space, M_α is a $S\alpha S$ random measure on $X\times\mathbb{R}$ with the control measure $\mu(dx)du$ and $g \in L^\alpha(X\times\mathbb{R},\mu(dx)du)$. The process (1.4) is called a (stationary) *mixed moving average*. The process $\{X_\alpha^C(t)\}_{t\in\mathbb{R}}$, on the other hand, is generated by a conservative flow (see Section 3 for a definition). It can be further decomposed into a harmonizable process and, to use the author's language, a "third kind" process.

Subsequently, the ideas of Rosiński were adapted by Burnecki, Rosiński and Weron (1998) to decompose *self-similar* stable processes $\{X_\alpha(t)\}_{t>0}$. Recall that the process X_α is self-similar with index $H > 0$ (H-ss), if, for any $c > 0$,

$$\{X_\alpha(ct)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{c^H X_\alpha(t)\}_{t\in\mathbb{R}} \tag{1.5}$$

($\mathbb{R}$ may be replaced by $(0,\infty)$ in (1.5)). To obtain a decomposition of self-similar stable processes, the authors suggested two equivalent approaches: one may proceed directly by using the property of self-similarity instead of that of stationarity, or one may use the well-known Lamperti transformation which is a one-to-one correspondence between self-similar stable processes $\{X_\alpha(t)\}_{t>0}$ and stationary stable processes $\{Y_\alpha(t)\}_{t\in\mathbb{R}}$ (see Samorodnitsky and Taqqu (1994), p. 312). The decomposition of self-similar stable processes obtained in Burnecki et al. (1998) is similar to the decomposition (1.3) of stationary stable processes obtained by Rosiński (1995). For example, the self-similar processes $\{X_\alpha(t)\}_{t>0}$ described by dissipative flows are now called mixed fractional motions and have the representation

$$\int_X\int_0^\infty u^{H-\frac{1}{\alpha}} g\left(x,\frac{u}{t}\right) M(dx,du),\ t>0, \tag{1.6}$$

where (X,μ) is again some standard Lebesgue space, M_α is a $S\alpha S$ random measure on $X\times\mathbb{R}$ with the control measure $\mu(dx)du$ and $g\in L^\alpha(X\times\mathbb{R})$. When Lamperti's transformation is applied, they become stationary mixed moving averages X_α^D in (1.4).

To characterize stable processes with *stationary increments*, Surgailis, Rosiński, Mandrekar and Cambanis (1998) used a transformation which allows one to go from stable processes with stationary increments to stationary stable processes (the transformation used imposes some conditions on the processes involved), obtain their decomposition by using Rosiński's result (1.3) and then go back to the original processes with stationary increments. The stable processes with stationary increments were again characterized by flows and cocycles. The stable processes with stationary increments (zero at $t = 0$) characterized by dissipative flows were shown in Surgailis et al. (1998) to have representation

$$\{X_\alpha(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{\int_X \int_{\mathbb{R}} (G(x, t+u) - G(x, u)) M_\alpha(dx, du)\right\}_{t\in\mathbb{R}}, \tag{1.7}$$

where M_α, $\alpha \in (0, 2)$, is a $S\alpha S$ random measure with the control measure $m_\alpha(dx, du) = \mu(dx)du$, and $G : X \times \mathbb{R} \mapsto \mathbb{R}$ is a measurable function such that, for all $t \in \mathbb{R}$,

$$G_t(x, u) = G(x, t+u) - G(x, u),\ u \in \mathbb{R},\ x \in X, \tag{1.8}$$

is in $L^\alpha(X \times \mathbb{R}, \mu(dx)du)$. We shall call the process X_α in (1.7) a (stationary increments) *mixed moving average* because, if it were differentiable, its derivative would be the (stationary) mixed moving average (1.4).

We shall focus on $S\alpha S$ processes that are *both* self-similar and have stationary increments (sssi processes, in short). These processes are of interest, because in practice, one often wants a model to have these two characteristics. To describe sssi processes, the decompositions obtained by Burnecki et al. (1998) and Surgailis et al. (1998) are not that useful because they rely solely on either the self-similarity or the stationarity of the increments property. It would be desirable to have a decomposition of sssi processes which would incorporate both properties. We will show that such a decomposition exists for a subclass of sssi processes, namely, for those self-similar processes with stationary increments that have the representation (1.7). Such processes appear in a number of applications, for example in the context of renewal reward processes (see Pipiras and Taqqu (2000)).

The basic question of this work is then the following. Suppose that the $S\alpha S$ process X_α, having the representation (1.7) and hence with stationary increments, is also self-similar. We want to know whether there is a way to decompose it, using both properties, the stationarity of the increments and self-similarity. Two approaches are possible. One can try to specialize the Burnecki et al. (1998) flow corresponding to self-similarity in order to take advantage of the special structure of the kernel in (1.7). Alternatively, one can start directly with (1.7) which has built-in stationarity of the increments, and then impose the further condition of self-similarity. We shall follow the second approach.

The Surgailis et al. (1998) flow for the process (1.7) is dissipative and Burnecki et al. (1998) flow for the process (1.6) is also dissipative. But observe that a kernel may be associated with different types of flows. We shall use the additional property of self-similarity to show that there is *another* flow that can be associated with (1.7) which can be either dissipative or conservative. This will allow us to decompose the process X_α in (1.7) in a dissipative component X_α^D and a conservative component X_α^C, and we will have $X_\alpha =_d X_\alpha^D + X_\alpha^C$. This decomposition which looks like (1.3) is, in fact, quite different, because the flows involved are different.

Our flow involves a finer characterization, because, as we shall show, it stems from only one component of Burnecki et al. (1998) flow associated with self-similarity. We will be able, consequently, to distinguish between processes which have both representations (1.6) and (1.7), that is, which are dissipative both in the sense of Burnecki et al. (1998) and Surgailis et al. (1998). Consider for example the $S\alpha S$ process (see (6.1) below) which is obtained as a limit of renewal reward type processes and is of the form (1.7) and consider also the linear fractional stable motion (see (6.2) below) process. These two processes are dissipative in the sense of Surgailis et al. (1998) and, since they are also of the form (1.6), they are both dissipative in the sense of Burnecki et al. (1998) as well. We show in Section 6, however, that while the renewal reward limit process (6.1) is dissipative even in our sense, linear fractional stable motion is conservative in our sense. This allows us, for example, to conclude that these two processes are different, that is, they possess different finite-dimensional distributions.

The rest of the paper is organized as follows. In Section 2, we state the assumptions and describe our results. In Section 3, we recall some basic definitions on flows and their related functionals and prove some technical results which will be used in the sequel. The reader may want at first to skip Section 3 and return to it when she/he wants to check a concept or result which is being used. In Section 4, we characterize minimal kernels. The results about the decomposition of the process are established in Section 5. Section 6 contains the examples. Finally, in Section 7, we provide a summary and draw some conclusions.

2 Description of the results

We need first to introduce the notion of a minimal spectral representation introduced by Hardin (1982) (and developed by Rosiński (1998)). Let T be an arbitrary index set, $\alpha \in (0,2)$ and $(E,\mathcal{E},m)$ be a measure space. Suppose that $\{f_t\}_{t\in T} \subset L^\alpha(E,\mathcal{E},m)$ is a *spectral representation* of a $S\alpha S$ process X_α, that is,

$$\{X_\alpha(t)\}_{t\in T} \stackrel{d}{=} \left\{ \int_E f_t(x) M_\alpha(dx) \right\}_{t\in T},$$

where M_α is a $S\alpha S$ random measure with the control measure m. Let $F = \{f_t, t \in T\}$, $\mathrm{sp}(F) = \{\sum_{k=1}^n c_k f_{t_k} : c_k \in \mathbb{R}, t_k \in T, n \in \mathbb{N}\}$ and $\overline{\mathrm{sp}}(F)$ be the closure of the span in $L^\alpha(E,m)$. We write $A = B$ m-a.e. if $m(A \triangle B) = 0$ and say that two σ-algebras are equal modulo m if their sets are equal m-a.e. Define $\mathrm{supp}(F)$ (the support of $f_t, t \in T$) as a minimal (m-a.e.) set $A \in \mathcal{E}$ such that $m\{x \in E : f_t(x) \neq 0, x \notin A\} = 0$ for every $t \in T$.

Definition 2.1 The spectral representation $\{f_t\}_{t\in T}$ is called *minimal* for the process X_α if the following two conditions are satisfied:

(M1) $\mathrm{supp}(F) = E$ m–a.e.,

(M2) $\rho(F) = \mathcal{E}$ (modulo m), where $\rho(F)$ is the so-called ratio σ-algebra, that is, the smallest σ–algebra generated by the extended-valued functions f/g, $f, g \in \overline{\mathrm{sp}}(F)$.

Minimality implies richness. For example, the representation $X_\alpha(t) = \int_0^1 1_{[0,t]}(x) M_\alpha(dx)$, $t \in [0,1]$, of the Lévy stable motion is minimal because the functions $\{1_{[0,t]}(x)\}_{t\in[0,1]}$ generate the Borel σ-algebra on $[0,1]$ (one can take $g = 1_{[0,1]}$ here). It is known that every $S\alpha S$ process separable in probability has a minimal spectral representation (Hardin (1982)).

Minimal spectral representations are useful to describe the structures of $S\alpha S$ processes. To understand why, suppose that a $S\alpha S$ process $\{X_\alpha(t)\}_{t\in T}$ ($T = \mathbb{R}$ or $\mathbb{Z}$) has two minimal representations $\{f_t^1\}_{t\in T}$ and $\{f_t^2\}_{t\in T}$ on the standard Lebesgue spaces $(E_1,\mathcal{E}_1,m_1)$ and $(E_2,\mathcal{E}_2,m_2)$, respectively. Then, by the Rigidity Lemma 4.1, (ii), below, there exist unique (modulo m_2) functions $\Phi : E_2 \mapsto E_1$

and $h : E_2 \mapsto \mathbb{R} \setminus \{0\}$ such that Φ is one-to-one and onto and, for each $t \in T$,

$$f_t^2(x) = h(x) f_t^1(\Phi(x)) \quad \text{and} \quad \frac{d(m_1 \circ \Phi)}{dm_2}(x) = |h(x)|^\alpha, \tag{2.1}$$

a.e. $m_2(dx)$. Relation (2.1) allows one to go from one representation to the other. For example, one gets

$$\int_{E_2} \left| \sum_{k=1}^n \theta_k f_{t_k}^2 \right|^\alpha dm_2 = \int_{E_2} \left| \sum_{k=1}^n \theta_k f_{t_k}^1 \circ \Phi \right|^\alpha \frac{d(m_1 \circ \Phi)}{dm_2} dm_2 = \int_{E_1} \left| \sum_{k=1}^n \theta_k f_{t_k}^1 \right|^\alpha dm_1,$$

for $t_1, \ldots, t_n \in T$ and $\theta_1, \ldots, \theta_n \in \mathbb{R}$. This equality expresses the fact that the finite-dimensional distributions of the process can be expressed through either representation.

To see how (2.1) can be used to characterize *stationary* $S\alpha S$ processes, suppose that $\{X_\alpha(t)\}_{t\in\mathbb{R}}$ is a $S\alpha S$ stationary process having a minimal spectral representation $\{f_t\}_{t\in\mathbb{R}}$ as in (1.1). Then, by stationarity, for any $\tau \in \mathbb{R}$, $\{f_{t+\tau}\}_{t\in\mathbb{R}}$ is also a minimal spectral representation of the process X_α. By using the result (2.1), for each $\tau \in \mathbb{R}$, there exist unique (modulo ν) functions $\Phi_\tau : S \mapsto S$ and $h_\tau : S \mapsto \mathbb{R} \setminus \{0\}$ such that $f_\tau(s) = h_\tau(s) f_0(\Phi_\tau(s))$ a.e. $\nu(ds)$ (plus the condition on h_τ). Then, for any $\tau_1, \tau_2 \in \mathbb{R}$, by iterating twice, $f_{\tau_1+\tau_2}(s) = h_{\tau_1}(s) h_{\tau_2}(\Phi_{\tau_1}(s)) f_0(\Phi_{\tau_2}(\Phi_{\tau_1}(s)))$ a.e. $\nu(ds)$ and, hence, by uniqueness, $\Phi_{\tau_1+\tau_2} = \Phi_{\tau_1} \circ \Phi_{\tau_2}$ a.e. $\nu(ds)$. After some technical (but by no means trivial) work, one can modify $\{\Phi_\tau\}_{\tau\in\mathbb{R}}$ to a flow $\{\phi_\tau\}_{\tau\in\mathbb{R}}$ to get the representation (1.2) and then perform the decomposition of the process X_α based on the ergodic properties of the flow in (1.2) (see Rosiński (1995)). The flows satisfying the relation $\phi_{\tau_1+\tau_2} = \phi_{\tau_1} \circ \phi_{\tau_2}$, $\tau_1, \tau_2 \in \mathbb{R}$, are (*additive*) flows. The flows that we will consider are *multiplicative*, namely such that $\phi_{\tau_1\tau_2} = \phi_{\tau_1} \circ \phi_{\tau_2}$, $\tau_1, \tau_2 > 0$.

Suppose now that the spectral representation $\{G_t\}_{t\in\mathbb{R}}$, as defined in (1.8), is minimal for the process X_α in (1.7) which has stationary increments and is assumed to be self-similar. By using the methodology described above, we show in Section 4 below that there exist a multiplicative flow $\{\psi_c\}_{c>0}$ on (X, μ), a cocycle $\{b_c\}_{c>0}$ taking values in $\{-1, 1\}$ and a semi-additive functional $\{g_c\}_{c>0}$ for the flow $\{\psi_c\}_{c>0}$ (see Section 3 for definitions) such that, for any $c > 0$ and $t \in \mathbb{R}$,

$$\begin{aligned} &c^{-\kappa}(G(x, c(t+u)) - G(x, cu)) \\ &= b_c(x) \left\{ \frac{d(\mu \circ \psi_c)}{d\mu}(x) \right\}^{1/\alpha} (G(\psi_c(x), t + u + g_c(x)) - G(\psi_c(x), u + g_c(x))) \end{aligned} \tag{2.2}$$

a.e. $\mu(dx)du$, where

$$\kappa = H - 1/\alpha. \tag{2.3}$$

Note that (2.2) is consistent with the fact that (1.7) is assumed to be self-similar since, for $t_1, \ldots, t_n \in \mathbb{R}$, $\theta_1, \ldots, \theta_n \in \mathbb{R}$ and $c > 0$, (2.2) yields

$$\begin{aligned} &\int_X \int_{\mathbb{R}} \left| \sum_{k=1}^n \theta_k c^{-\kappa} (G(x, c(t_k + u)) - G(x, cu)) \right|^\alpha \mu(dx) du \\ &= \int_X \int_{\mathbb{R}} \left| \sum_{k=1}^n \theta_k \left(G(\psi_c(x), t_k + u + g_c(x)) - G(\psi_c(x), u + g_c(x)) \right) \right|^\alpha \frac{d(\mu \circ \psi_c)}{d\mu}(x) \; \mu(dx) du \\ &= \int_X \int_{\mathbb{R}} \left| \sum_{k=1}^n \theta_k \left(G(\psi_c(x), t_k + u) - G(\psi_c(x), u) \right) \right|^\alpha (\mu \circ \psi_c)(dx) du \end{aligned}$$

$$= \int_X \int_{\mathbb{R}} \left| \sum_{k=1}^n \theta_k \left(G(x, t_k + u) - G(x, u) \right) \right|^\alpha \mu(dx) du. \tag{2.4}$$

Another way of expressing (2.2) is to set

$$\varphi_c(x, u) = \left(\psi_c(x), \frac{u}{c} + g_c(x) \right). \tag{2.5}$$

We will verify below (see Example 3.4) that φ_c, $c > 0$, is a multiplicative flow on $X \times \mathbb{R}$. Moreover, for all $t > 0$,

$$\begin{aligned} G_t(x,u) &= G(x, t(1 + t^{-1}u)) - G(x, tt^{-1}u) \\ &= t^\kappa \, b_t(x) \left\{ \frac{d(\mu \circ \psi_t)}{d\mu}(x) \right\}^{1/\alpha} \Big(G(\psi_t(x), 1 + t^{-1}u + g_t(x)) - G(\psi_t(x), t^{-1}u + g_t(x)) \Big) \\ &= t^{H - \frac{1}{\alpha}} \, b_t(x) \left\{ \frac{d(\mu \circ \psi_t)}{d\mu}(x) \right\}^{1/\alpha} G_1(\psi_t(x), t^{-1}u + g_t(x)) \\ &= t^H \, b_t(x,u) \left\{ \frac{d((\mu \otimes \mathbb{L}) \circ \varphi_t)}{d(\mu \otimes \mathbb{L})}(x) \right\}^{1/\alpha} G_1(\varphi_t(x,u)) \end{aligned} \tag{2.6}$$

a.e. $\mu(dx)du$, where $\mathbb{L}$ denotes the Lebesgue measure and $b_t(x,u) = b_t(x)$. While (2.6) and (1.2) look similar, we shall base our decomposition of the process X_α into two parts, a dissipative and conservative one, *only* on the flow $\psi_t(x)$, that is, on the first component of the flow $\varphi_t(x,u) = (\psi_t(x), \frac{u}{t} + g_t(x))$. We will be able to proceed in the spirit of Rosiński (1995). We say that an H-ss process X_α with stationary increments, having the representation (1.7), is generated by a multiplicative flow $\{\psi_c\}_{c>0}$ if the representation (2.2) and some condition on the support of functions $G_t, t \in \mathbb{R}$, hold (see Section 5). We then show that, for $\alpha \in (1,2)$, any H-ss process X_α with stationary increments, having the representation (1.7), is generated by a multiplicative flow. (When $\alpha \in (0,1]$, the arguments which we use work only for a subclass of such processes X_α, as noted in the remark following Theorem 4.2. In order not to obscure the arguments, we decided to provide a decomposition in the case $\alpha \in (1,2)$ only.) We also show that, if the process X_α is generated by a dissipative (conservative, resp.) flow in one spectral representation, then, in any other spectral representation, the flow has to be dissipative (conservative, resp.) as well. This allows us to decompose the process X_α uniquely into two independent parts:

$$X_\alpha \stackrel{d}{=} X_\alpha^D + X_\alpha^C, \tag{2.7}$$

where the process X_α^D is generated by a dissipative flow and the process X_α^C is generated by a conservative flow. Finally, in Section 6 we show that the limit of the renewal reward processes is a process generated by a dissipative flow and that the usual linear fractional stable motion, the log-fractional stable motion and the Lévy stable motion are processes generated by conservative flows.

In a subsequent paper, Pipiras and Taqqu (2001), we examine processes generated by dissipative and conservative flows in greater detail. In particular, we show that processes generated by dissipative flows can be represented in distribution as

$$\int_Y \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\kappa s} (F(y, e^s(t+u)) - F(y, e^s u)) M_\alpha(dy, ds, du), \; t \in \mathbb{R}, \tag{2.8}$$

where $(Y, \mathcal{Y}, \nu)$ is some standard Lebesgue space, $F : Y \times \mathbb{R} \mapsto \mathbb{R}$ is a measurable function, M_α is a $S\alpha S$ random measure on $Y \times \mathbb{R} \times \mathbb{R}$ with the control measure $\nu(dy)dsdu$ and $\kappa = H - 1/\alpha$. The case of processes generated by conservative flows, which is also considered, is more involved.

3 Flows, cocycles and other functionals

A standard Lebesgue space $(X, \mathcal{X}, \mu)$ is made up of a standard Borel space $(X, \mathcal{X})$ and a σ-finite measure μ. A standard Borel space is a measurable space measurably isomorphic (i.e. there is a one-to-one, onto and bimeasurable map) to a Borel subset of a complete separable metric space[1]. A standard Borel space, when equipped with a finite measure μ, is called a Lebesgue space in ergodic theory[2]. The term "standard Lebesgue", found in Rosiński (1995), extends this use to the case of a σ-finite, possibly infinite, measure μ. As indicated in the introduction, a typical example is $\mathbb{R}^n$ with a measure consisting of Lebesgue measure and discrete point masses.

Let $(X, \mathcal{X}, \mu)$ be a standard Lebesgue space. A family $\{\phi_t\}_{t\in\mathbb{R}}$ of measurable maps from X onto X is called an *(additive) flow* if $\phi_0(x) = x$ and

$$\phi_{t_1}(\phi_{t_2}(x)) = \phi_{t_1+t_2}(x), \tag{3.1}$$

for all $t_1, t_2 \in \mathbb{R}$ and $x \in X$. The flow $\{\phi_t\}_{t\in\mathbb{R}}$ is said to be *nonsingular* if $\mu(\phi_t^{-1}(B)) = 0$ if and only if $\mu(B) = 0$ for every $t \in \mathbb{R}$ and $B \in \mathcal{X}$. It is said to be *measurable* if the map $\phi_t(x) : \mathbb{R} \times X \mapsto X$ is measurable. Similarly, a family $\{\psi_c\}_{c>0}$ of measurable maps from X onto X is called a (*nonsingular, measurable*) *multiplicative flow* if $\{\phi_t\}_{t\in\mathbb{R}} = \{\psi_{e^t}\}_{t\in\mathbb{R}}$ is a (nonsingular, measurable) flow. Hence, $\{\psi_c\}_{c>0}$ is a multiplicative flow if $\psi_1(x) = x$ and, for all $c_1, c_2 > 0$ and $x \in X$,

$$\psi_{c_1}(\psi_{c_2}(x)) = \psi_{c_1c_2}(x). \tag{3.2}$$

Example 3.1 The flows $\phi_t(x) = x$, $t \in \mathbb{R}, x \in X$ (identity flow), $\phi_t(x) = \{x + t\}$, $t \in \mathbb{R}, x \in [0, 1)$ (cyclic flow), where $\{\cdot\}$ denotes the fractional part function, $\phi_t(y, u) = (y, u + t)$, $t \in \mathbb{R}, y \in Y, u \in \mathbb{R}$, are examples of additive flows. The corresponding multiplicative flows are

$$\psi_c(x) = x, \quad \psi_c(x) = \{x + \ln c\}, \quad \psi_c(y, u) = (y, u + \ln c),$$

where $c > 0$.

Let $A = \{-1, 1\}$. A measurable map $a_t(x) : \mathbb{R} \times X \mapsto A$ is said to be a *cocycle* for a measurable flow $\{\phi_t\}_{t\in\mathbb{R}}$ if

$$a_{t_1+t_2}(x) = a_{t_2}(x)a_{t_1}(\phi_{t_2}(x)), \tag{3.3}$$

for all $t_1, t_2 \in \mathbb{R}$ and $x \in X$. Similarly, a measurable map $b_c(x) : \mathbb{R}_+ \times X \mapsto A$ is said to be a *cocycle* for a measurable *multiplicative* flow $\{\psi_c\}_{c>0}$ if $\{a_t\}_{t\in\mathbb{R}} = \{b_{e^t}\}_{t\in\mathbb{R}}$ is a cocycle for a measurable flow $\{\phi_t\}_{t\in\mathbb{R}} = \{\psi_{e^t}\}_{t\in\mathbb{R}}$. Hence, $\{b_c\}_{c>0}$ is a cocycle for for the measurable multiplicative flow $\{\psi_c\}_{c>0}$ if, for all $c_1, c_2 > 0$ and $x \in X$,

$$b_{c_1c_2}(x) = b_{c_2}(x)b_{c_1}(\psi_{c_2}(x)). \tag{3.4}$$

Example 3.2 Let $\{\phi_t\}_{t\in\mathbb{R}}$ be a flow on a space X and $a : X \mapsto \{-1, 1\}$ be a measurable function. Then, the collection of maps

$$a_t(x) = \frac{a(\phi_t(x))}{a(\phi_0(x))} = \frac{a(\phi_t(x))}{a(x)}, \quad t \in \mathbb{R}, x \in X,$$

[1] See, for example, Appendix A in Zimmer (1984) or Arveson (1976) and Mackey (1957); these authors work in the context of algebraic groups.

[2] See, for example, Walters (1982), Definition 2.3, or Petersen (1983), Definition 4.5.

is a cocycle for the flow $\{\phi_t\}_{t\in\mathbb{R}}$. Indeed, for $t_1, t_2 \in \mathbb{R}$ and $x \in X$,

$$a_{t_2}(x)a_{t_1}(\phi_{t_2}(x)) = \frac{a(\phi_{t_2}(x))}{a(x)}\frac{a(\phi_{t_1}(\phi_{t_2}(x)))}{a(\phi_{t_2}(x))} = \frac{a(\phi_{t_1+t_2}(x))}{a(x)} = a_{t_1+t_2}(x).$$

Similarly, if $\{\psi_c\}_{c>0}$ is a multiplicative flow on X and $b : X \mapsto \{-1, 1\}$ is a measurable function, then

$$b_c(x) = \frac{b(\psi_c(x))}{b(x)}, \quad c > 0, x \in X,$$

is a cocycle for the flow $\{\psi_c\}_{c>0}$.

A measurable map $f_t(x) : \mathbb{R} \times X \mapsto \mathbb{R}$ is said to be a *semi-additive functional* for a measurable flow $\{\phi_t\}_{t\in\mathbb{R}}$ if

$$f_{t_1+t_2}(x) = e^{-t_2} f_{t_1}(x) + f_{t_2}(\phi_{t_1}(x)) \tag{3.5}$$

for all $t_1, t_2 \in \mathbb{R}$ and $x \in X$. (Relation (3.5) without the multiplicative term e^{-t_2} defines additive functionals for measurable flows - see Kubo (1969, 1970).) Similarly, a measurable map $g_c(x) : \mathbb{R}_+ \times X \mapsto \mathbb{R}$ is said to be a *semi-additive functional* for a measurable *multiplicative* flow $\{\psi_c\}_{c>0}$ if $\{f_t\}_{t\in\mathbb{R}} = \{g_{e^t}\}_{t\in\mathbb{R}}$ is a semi-additive functional for a measurable flow $\{\phi_t\}_{t\in\mathbb{R}} = \{\psi_{e^t}\}_{t\in\mathbb{R}}$. Hence, $\{g_c\}_{c>0}$ is a semi-additive functional for $\{\psi_c\}_{c>0}$ if

$$g_{c_1c_2}(x) = c_2^{-1} g_{c_1}(x) + g_{c_2}(\psi_{c_1}(x)), \tag{3.6}$$

for all $c_1, c_2 > 0$ and $x \in X$.

Example 3.3 Let $\{\phi_t\}_{t\in\mathbb{R}}$ be a flow on a space X and $F : X \mapsto \mathbb{R}$ be a measurable function. Then, the collection of maps

$$f_t(x) = F(\phi_t(x)) - e^{-t}F(x), \quad t \in \mathbb{R}, x \in X,$$

is a semi-additive functional for the flow $\{\phi_t\}_{t\in\mathbb{R}}$ because, for $t_1, t_2 \in \mathbb{R}$ and $x \in X$,

$$\begin{aligned} e^{-t_2} f_{t_1}(x) + f_{t_2}(\phi_{t_1}(x)) &= e^{-t_2}(F(\phi_{t_1}(x)) - e^{-t_1}F(x)) + F(\phi_{t_2}(\phi_{t_1}(x))) - e^{-t_2}F(\phi_{t_1}(x)) \\ &= F(\phi_{t_1+t_2}(x)) - e^{-(t_1+t_2)}F(x) = f_{t_1+t_2}(x). \end{aligned}$$

Similarly, if $\{\psi_c\}_{c>0}$ is a multiplicative flow on X and $G : X \mapsto \mathbb{R}$ is a measurable function, then

$$g_c(x) = G(\psi_c(x)) - c^{-1}G(x), \quad c > 0, x \in X,$$

is a semi-additive functional for the flow $\{\psi_c\}_{c>0}$. This example is used in the proof of Proposition 3.1 below.

Example 3.4 Consider the collection of maps $\varphi_c(x, u) = (\psi_c(x), \frac{u}{c} + g_c(x))$ with $c > 0$ and $x \in X$, introduced in (2.5), where $\{\psi_c\}_{c>0}$ is a multiplicative flow on a space X and $\{g_c\}_{c>0}$ is a semi-additive functional for $\{\psi_c\}_{c>0}$. Since

$$\begin{aligned} \varphi_{c_1}(\varphi_{c_2}(x, u)) &= \varphi_{c_1}\left(\psi_{c_2}(x), \frac{u}{c_2} + g_{c_2}(x)\right) \\ &= \left(\psi_{c_1}(\psi_{c_2}(x)), \frac{1}{c_1}\left(\frac{u}{c_2} + g_{c_2}(x)\right) + g_{c_1}(\psi_{c_2}(x))\right) \\ &= \left(\psi_{c_1c_2}(x), \frac{u}{c_1c_2} + g_{c_1c_2}(x)\right) \\ &= \varphi_{c_1c_2}(x, u), \end{aligned}$$

for all $c_1, c_2 > 0$ and $x \in X$, the collection $\{\varphi_c\}_{c>0}$ is a multiplicative flow on $X \times \mathbb{R}$.

The preceding quantities can also be defined almost everywhere $\mu(dx)$. Thus, a measurable map $a_t(x) : \mathbb{R} \times X \mapsto A$ is said to be an *almost cocycle* for a measurable flow $\{\phi_t\}_{t\in\mathbb{R}}$ if the relation (3.3) holds a.e. $\mu(dx)$ for all $t_1, t_2 \in \mathbb{R}$. An *almost cocycle* for a measurable *multiplicative* flow is defined similarly. A measurable map $f_t(x) : \mathbb{R} \times X \mapsto \mathbb{R}$ is said to be an *almost semi-additive functional* for a measurable flow $\{\phi_t\}_{t\in\mathbb{R}}$ if the relation (3.5) holds a.e. $\mu(dx)$ for all $t_1, t_2 \in \mathbb{R}$. An *almost semi-additive functional* for a measurable *multiplicative* flow is defined similarly.

The following elementary lemma is used many times throughout the paper.

Lemma 3.1 *Suppose that $(X_1, \mathcal{X}_1, \mu_1)$ and $(X_2, \mathcal{X}_2, \mu_2)$ are two measure spaces. Let $F : X_1 \times X_2 \mapsto \mathbb{R}$ be a measurable function on $(X_1 \times X_2, \mathcal{X}_1 \otimes \mathcal{X}_2, \mu_1 \times \mu_2)$. Then, (i) if for almost every $x_1 \in X_1$, $F(x_1, x_2) = 0$ a.e. $\mu_2(dx_2)$, we have $F(x_1, x_2) = 0$ a.e. $(\mu_1 \times \mu_2)(dx_1, dx_2)$, and, conversely, (ii) if $F(x_1, x_2) = 0$ a.e. $(\mu_1 \times \mu_2)(dx_1, dx_2)$, then for almost every $x_1 \in X_1$, $F(x_1, x_2) = 0$ a.e. $\mu_2(dx_2)$.*

Let T be an arbitrary index set and let also $\{f_t^1\}_{t\in T}$ and $\{f_t^2\}_{t\in T}$ be collections of measurable maps from X into X. Then $\{f_t^2\}_{t\in T}$ is said to be a *version* of $\{f_t^1\}_{t\in T}$ or that they are equal modulo μ if $\mu(x : f_t^1(x) \neq f_t^2(x)) = 0$, for all $t \in T$. The following two propositions will be useful in the sequel. The first, which extends Proposition 1.2 of Kubo (1970), states that we can replace an almost semi-additive functional by one that is semi-additive everywhere.

Proposition 3.1 *Let $\{f_t\}_{t\in\mathbb{R}}$ be an almost semi-additive functional for a measurable flow $\{\phi_t\}_{t\in\mathbb{R}}$. Then $\{f_t\}_{t\in\mathbb{R}}$ has a version which is a semi-additive functional for $\{\phi_t\}_{t\in\mathbb{R}}$.*

Proof: We suppose first that $\mu(X) = 1$. (As noted above, this is a typical assumption in ergodic theory where standard Lebesgue spaces with $\mu(X) = 1$ are called *Lebesgue spaces*). To prove the proposition in this case, we will use some ideas of Kubo (1969, 1970). By Remark 3.1 in Kubo (1969), it is sufficient to prove the proposition in the following two cases:

Case 1: The flow $\{\phi_t\}_{t\in\mathbb{R}}$ is an identity flow, that is, $\phi_t(x) = x$ for all $t \in \mathbb{R}$ and $x \in X$. In this case, $f_{t_1+t_2}(x) = e^{-t_2} f_{t_1}(x) + f_{t_2}(x)$ a.e. $\mu(dx)$ for all $t_1, t_2 \in \mathbb{R}$, and hence we also have that $f_{t_1+t_2}(x) = e^{-t_1} f_{t_2}(x) + f_{t_1}(x)$ a.e. $\mu(dx)$. It follows that

$$e^{-t_2} f_{t_1}(x) + f_{t_2}(x) = e^{-t_1} f_{t_2}(x) + f_{t_1}(x)$$

a.e. $\mu(dx)$ and that

$$(e^{-t_2} - 1) f_{t_1}(x) = (e^{-t_1} - 1) f_{t_2}(x)$$

a.e. $\mu(dx)$. Thus, by fixing t_2 and setting $t = t_1$,

$$f_t(x) = (e^{-t} - 1) g(x)$$

a.e. $\mu(dx)$ for some measurable function g. Therefore, $(e^{-t} - 1)g(x)$, $t \in \mathbb{R}$, $x \in X$, is a version of $f_t(x)$, $t \in \mathbb{R}$, $x \in X$, and is a semi-additive functional. This completes the proof for Case 1.

Case 2: This case is more delicate. It assumes that the flow $\{\phi_t\}_{t\in\mathbb{R}}$ is a special flow defined on a space $(\Omega, \mathcal{E}, P)$. (The "almost" part in the proposition then refers to P.) The space and the special flow are defined in the following way[3].

[3]The concept of "special flow", also known as "flow under a function", is well-known in ergodic theory (see, for example, Cornfeld, Fomin and Sinai (1982), Chapter 11).

Definition 3.1 Let $(Y, \mathcal{Y}, \nu)$ be a Lebesgue space, S be a nonsingular, one-to-one and bimeasurable (that is, both S and S^{-1} are measurable) map of the space Y onto itself, and h be a positive measurable function defined on Y such that $h > \theta$ for some $\theta > 0$. Set $\Omega = \{(y,u) : 0 \le u < h(y), y \in Y\}$ and let $\mathcal{E}$ be a restriction of $\mathcal{Y} \otimes \mathcal{B}$ to Ω, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}$. Take a measurable function $p(y,u) > 0$ such that $\int\int_\Omega p(y,u)\nu(dy)du = 1$ and let $dP(y,u) = p(y,u)\nu(dy)du$. Then $(\Omega, \mathcal{E}, P)$ becomes a Lebesgue space.

Now consider the sequence of points $\{h_n(y)\}_{n\in\mathbb{Z}}$ on $\mathbb{R}$ defined by $h_n(y) = \sum_{k=0}^{n-1} h(S^k y)$, if $n \ge 1$, $h_n(y) = 0$, if $n = 0$, and $h_n(y) = -\sum_{k=n}^{-1} h(S^k y)$, if $n \le -1$. Then

$$h_{n+m}(y) = h_n(y) + h_m(S^n y). \tag{3.7}$$

Since $h > \theta$, $h_n(y) \to \mp\infty$ as $n \to \mp\infty$. The special flow $\{\phi_t\}_{t\in\mathbb{R}}$ on $(\Omega, \mathcal{E}, P)$ is defined as follows. First $\phi_0(y,0) = (y,0)$. Then for $t > 0$, $\phi_t(y,0) = (y, 0+t)$ but only if $t < h_1(y)$. For $h_1(y) \le t < h_2(y)$, $\phi_t(y,0) = (S(y), 0 + t - h_1(y))$, ... In general,

$$\phi_t(y,u) = (S^n y, u + t - h_n(y)), \quad \text{for} \quad 0 \le u + t - h_n(y) < h(S^n y), \tag{3.8}$$

or equivalently,

$$\phi_t(y,u) = (S^n y, u + t - h_n(y)), \quad \text{for} \quad h_n(y) \le u + t < h_{n+1}(y). \tag{3.9}$$

It is called a *special flow* built up by a function h such that $h(y) > \theta$ for all y and some $\theta > 0$.

Remark. To verify that $\{\phi_t\}_{t\in\mathbb{R}}$ is indeed a flow on Ω, note first that (3.8) implies $(S^n y, u+t-h_n(y)) \in \Omega$. Fix now $s, t \in \mathbb{R}$ and $(y,u) \in \Omega$. Let $n \in \mathbb{Z}$ be such that $0 \le u + t - h_n(y) < h(S^n y)$ and $m \in \mathbb{Z}$ be such that $0 \le u + s + t - h_n(y) - h_m(S^n y) < h(S^{n+m} y)$. Then, by applying (3.8) twice,

$$\phi_s(\phi_t(y,u)) = \phi_s(S^n y, u + t - h_n(y)) = (S^{n+m} y, u + s + t - h_n(y) - h_m(S^n y)).$$

By using (3.7), we have $\phi_s(\phi_t(y,u)) = (S^{n+m} y, u + s + t - h_{n+m}(y))$. The same relation (3.7) implies that $0 \le u + s + t - h_{n+m}(y) < h(S^{n+m} y)$ and hence $\phi_{s+t}(y,u) = (S^{n+m} y, u + s + t - h_{n+m}(y))$. It follows that $\phi_s \circ \phi_t = \phi_{s+t}$.

Consider now any almost semi-additive functional $\{f_t\}_{t\in\mathbb{R}}$ for the special flow $\{\phi_t\}_{t\in\mathbb{R}}$ on Ω. We must prove that this $\{f_t\}_{t\in\mathbb{R}}$ has a version which is an (everywhere) semi-additive functional for $\{\phi_t\}_{t\in\mathbb{R}}$. A characteristic of the special flow (3.8) is that it mixes together the argument u and the time t and also the argument y through the intermediary of h. This will help us in our goal.

Let us denote for convenience $\{f_t(y,u)\}_{t\in\mathbb{R}}$ by $\{f(t,(y,u))\}_{t\in\mathbb{R}}$. Since this is an almost semi-additive functional for $\{\phi_t\}_{t\in\mathbb{R}}$, we have that, for all $s, t \in \mathbb{R}$,

$$f(s+t,(y,u)) = e^{-t} f(s,(y,u)) + f(t,(S^n y, u + s - h_n(y))) \tag{3.10}$$

a.e. dP for (y,u) such that $h_n(y) \le u + s < h_{n+1}(y)$. Fixing $u = u_0$ for which (3.10) holds would make u_0 depend on (s,t). To avoid this, we use Lemma 3.1, (i). Since f is jointly measurable in its all arguments, the relation (3.10) also holds a.e. $dsdtdP(y,u)$. Since h satisfies $h(y) \ge \theta > 0$ for $y \in Y$, we have $Y \times (0,\theta) \in \Omega$ and the relation (3.10) holds a.e. for $s, t \in \mathbb{R}, y \in Y, u \in (0,\theta)$ such that $h_n(y) \le u + s < h_{n+1}(y)$. Applying now Lemma 3.1, (ii), we conclude that there is $u_0 \in (0,\theta)$ such that

$$f(s+t,(y,u_0)) = e^{-t} f(s,(y,u_0)) + f(t,(S^n y, u_0 + s - h_n(y))) \tag{3.11}$$

a.e. for (s,t,y) such that $h_n(y) \le u_0 + s < h_{n+1}(y)$. Relation (3.11) will be used repeatedly in the sequel. For convenience to the reader, when applying (3.11), we shall indicate by horizontal braces the variables which play the roles of "s" and "t" (and sometimes "u_0").

Setting $n = 0$ and $s = u - u_0$ in (3.11), we can isolate the last term on the right-hand side of that relation and express it as

$$f(t,(y,u)) = f(t+u-u_0,(y,u_0)) - e^{-t} f(u-u_0,(y,u_0)). \tag{3.12}$$

Focus on the two terms on the right-hand side of (3.12). For the first term, observe that by (3.11),

$$\begin{aligned} f(t+u-u_0,(y,u_0)) &= f(\underbrace{h_n(y)-v}_{s} + \underbrace{t+u-u_0-h_n(y)+v}_{t},(y,u_0)) \\ &= e^{-(t+u-u_0-h_n(y)+v)} f(h_n(y)-v,(y,u_0)) \\ &\quad + f(t+u-u_0-h_n(y)+v,(S^m y, u_0+h_n(y)-v-h_m(y))) \end{aligned} \tag{3.13}$$

a.e. for (t,u,v,y) such that $h_m(y) \le u_0 + h_n(y) - v < h_{m+1}(y)$. Now, by Lemma 3.1, (ii), take $v_0 \in (0,u_0)$ for which (3.13) holds a.e. for (t,u,y). Since $u_0 \in (0,\theta)$, one has $v_0 \in (0,\theta)$ and hence $0 \le u_0 - v_0 < \theta < h(y)$ or $h_n(y) \le u_0 - v_0 + h_n(y) \le h_{n+1}(y)$, so that (3.13) holds with $m = n$. Therefore

$$\begin{aligned} f(t+u-u_0,(y,u_0)) &= e^{-(t+u-u_0-h_n(y)+v_0)} f(h_n(y)-v_0,(y,u_0)) \\ &\quad + f(t+u-u_0-h_n(y)+v_0,(S^n y, u_0-v_0)), \end{aligned} \tag{3.14}$$

which is an expression for the first term on the right-hand side of (3.12). To get an expression for the second term, we start with (3.11) again and observe that

$$f(\underbrace{u-u_0}_{t} + \underbrace{v_0}_{s},(y,\underbrace{u_0-v_0}_{u_0})) = e^{-(u-u_0)} f(v_0,(y,u_0-v_0)) + f(u-u_0,(y,u_0))$$

a.e. for (u,y), since here $n = 0$ because $u_0 - v_0 + v_0 = u_0 \in (0,\theta)$. Hence

$$f(u-u_0,(y,u_0)) = f(u-u_0+v_0,(y,u_0-v_0)) - e^{-(u-u_0)} f(v_0,(y,u_0-v_0)) \tag{3.15}$$

a.e. for (u,y), which gives an expression for the second term on the right-hand side of (3.12).

Set now

$$F(y,u) = f(u-u_0+v_0,(y,u_0-v_0)) \tag{3.16}$$

and

$$F_n(y) = e^{h_n(y)-v_0} f(h_n(y)-v_0,(y,u_0)) + f(v_0,(y,u_0-v_0)). \tag{3.17}$$

Then, by using (3.14) and (3.15), relation (3.12) can be rewritten as follows:

$$\begin{aligned} f(t,(y,u)) &= e^{-(t+u-u_0)} \left(e^{h_n(y)-v_0} f(h_n(y)-v_0,(y,u_0)) + f(v_0,(y,u_0-v_0)) \right) \\ &\quad + f(t+u-u_0-h_n(y)+v_0,(S^n y,u_0-v_0)) - e^{-t} f(u-u_0+v_0,(y,u_0-v_0)) \\ &= e^{-(t+u-u_0)} F_n(y) + F(\phi_t(y,u)) - e^{-t} F(y,u) \end{aligned} \tag{3.18}$$

a.e. (t,y,u) for $h_n(y) \le t+u < h_{n+1}(y)$, $n \in \mathbb{Z}$, where $F(\phi_t(y,u))$ uses the expressions (3.8) and (3.16). Hence f is the sum of two terms $e^{-(t+u-u_0)} F_n(y)$ and $F(\phi_t(y,u)) - e^{-t} F(y,u)$. The second term is a semi-additive functional by Example 3.3. Since f is assumed to be an *almost* semi-additive functional, the first term will also be so. We want to replace it by a semi-additive functional version $e^{-(t+u-u_0)}\widetilde{F}_n(y)$.

We shall now determine $\widetilde{F}_n$. Observe that we have simplified the problem considerably because the function F_n depends only on a single argument y. By using (3.7) and applying (3.11) twice, one gets

$$\begin{aligned}
F_{n+m}(y) &= e^{h_{n+m}(y)-v_0} f(h_{n+m}(y) - v_0, (y, u_0)) + f(v_0, (y, u_0 - v_0)) \\
&= e^{h_n(y)+h_m(S^n y)-v_0} f(\underbrace{h_n(y)}_{s} + \underbrace{h_m(S^n y) - v_0}_{t}, (y, u_0)) + f(v_0, (y, u_0 - v_0)) \\
&= e^{h_n(y)+h_m(S^n y)-v_0} \Big(e^{-h_m(S^n y)+v_0} f(h_n(y), (y, u_0)) \\
&\quad + f(h_m(S^n y) - v_0, (S^n(y), u_0 + h_n(y) - h_n(y))) \Big) + f(v_0, (y, u_0 - v_0)) \\
&= e^{h_n(y)} f(\underbrace{h_n(y) - v_0}_{s} + \underbrace{v_0}_{t}, (y, u_0)) \\
&\quad + e^{h_n(y)+h_m(S^n y)-v_0} f(h_m(S^n y) - v_0, (S^n(y), u_0)) + f(v_0, (y, u_0 - v_0)) \\
&= e^{h_n(y)} \Big(e^{-v_0} f(h_n(y) - v_0, (y, u_0)) + f(v_0, (S^n y, u_0 - v_0)) \Big) \\
&\quad + e^{h_n(y)+h_m(S^n y)-v_0} f(h_m(S^n y) - v_0, (S^n(y), u_0)) + f(v_0, (y, u_0 - v_0)) \\
&= F_n(y) + e^{h_n(y)} F_m(S^n y)
\end{aligned}$$

a.e. for y. Since $h_0(y) = 0$, this gives $F_0(y) = 0$ a.e., and by setting $m = 1$, one gets that F_n, $n \neq 0$, is determined by F_1. In fact,

$$F_n(y) = \widetilde{F}_n(y) \quad \text{a.e. for } y,$$

where $\widetilde{F}_n(y) = \sum_{k=0}^{n-1} e^{h_k(y)} F_1(S^k y)$, if $n \geq 1$, $\widetilde{F}_n(y) = 0$, if $n = 0$, and $\widetilde{F}_n(y) = \sum_{k=n}^{-1} e^{h_k(y)} F_1(S^k y)$, if $n \leq -1$. Observe that, by using (3.7), for all y (and, say, for $n + m \geq 1$),

$$\begin{aligned}
\widetilde{F}_{n+m}(y) &= \sum_{k=0}^{n+m-1} e^{h_k(y)} F_1(S^k y) \\
&= \sum_{k=0}^{n-1} e^{h_k(y)} F_1(S^k y) + \sum_{k=0}^{m-1} e^{h_{n+k}(y)} F_1(S^{n+k} y) \\
&= \sum_{k=0}^{n-1} e^{h_k(y)} F_1(S^k y) + e^{h_n(y)} \sum_{k=0}^{m-1} e^{h_k(S^n y)} F_1(S^k(S^n y)) \\
&= \widetilde{F}_n(y) + e^{h_n(y)} \widetilde{F}_m(S_n y).
\end{aligned} \tag{3.19}$$

Let us now verify that the function

$$\widetilde{F}_s(y, u) = e^{-(s+u-u_0)} \widetilde{F}_n(y), \quad \text{for} \quad h_n(y) \leq u + s < h_{n+1}(y), \tag{3.20}$$

is a semi-additive functional. Fix $s, t \in \mathbb{R}$ and (y, u). Let n be such that $0 \leq s + u - h_n(y) < h(y)$ and m be such that $h_m(S^n y) \leq t + s + u - h_n(u) < h_{m+1}(S^n y)$, so that, by (3.20),

$$\widetilde{F}_t(\phi_s(y, u)) = \widetilde{F}_t(S^n y, s + u - h_n(u)) = e^{-(t+s+u-h_n(u)-u_0)} \widetilde{F}_m(S^n y).$$

Using (3.20) again and also (3.19), we get

$$e^{-t}\widetilde{F}_s(y,u) + \widetilde{F}_t(\phi_s(y,u)) = e^{-(t+s+u-u_0)}(\widetilde{F}_n(y) + e^{h_n(u)}\widetilde{F}_m(S^n y)) = e^{-(t+s+u-u_0)}\widetilde{F}_{n+m}(y). \quad (3.21)$$

Since $h_{n+m}(y) \le t+s+u < h_{n+m+1}(y)$ (which follows from $h_m(S^n y) \le t+s+u-h_n(u) < h_{m+1}(S^n y)$ and (3.7)), we have from (3.20) that $e^{-(t+s+u-u_0)}\widetilde{F}_{n+m}(y) = \widetilde{F}_{s+t}(y,u)$ and hence (3.21) shows that $\widetilde{F}_s$ is a semi-additive functional.

In view of (3.18), define

$$\widetilde{f}(t,(y,u)) = \widetilde{f}_t(y,u)) = e^{-(t+u-u_0)}\widetilde{F}_n(y) + F(\phi_t(y,u)) - F(y,u),$$

for $h_n(y) \le u+t < h_{n+1}(y)$. Consequently, $\{\widetilde{f}_t\}_{t\in\mathbb{R}}$ is a semi-additive functional for the flow $\{\phi_t\}_{t\in\mathbb{R}}$. To show that it is a version, we must show that the relation

$$P(f(t,(y,u)) \ne \widetilde{f}(t,(y,u))) = 0,$$

which holds a.e. for $t \in \mathbb{R}$, holds in fact for all $t \in \mathbb{R}$. Proceeding as in Kubo (1970), set $\delta(t,(y,u)) = f(t,(y,u)) - \widetilde{f}(t,(y,u))$, $\Omega_t = \{(y,u) : \delta(t,(y,u)) = 0\}$ and, in view of (3.10), set also $\Omega_{s,t} = \{(y,u) : \delta(s+t,(y,u)) = e^{-t}\delta(s,(y,u)) + \delta(t,\phi_s(y,u))\}$. Then $P(\Omega_t) = 1$ a.e. for t and $P(\Omega_{s,t}) = 1$ for all $s,t \in \mathbb{R}$. If now $r \in \mathbb{R}$, then there are s and t such that $s+t = r$ and $P(\Omega_s) = P(\Omega_t) = 1$. Since ϕ_s is a one-to-one and onto map, one also has $P(\phi_{-s}\Omega_t) = 1$ Then, for $(y,u) \in \Omega_{s,t} \cap \Omega_s \cap \phi_{-s}\Omega_t$, we have $\delta(r,(y,u)) = e^{-t}\delta(s,(y,u)) + \delta(t,\phi_s(y,u)) = 0$. Therefore, $P(f(r,(y,u)) \ne \widetilde{f}(r,(y,u))) = 0$ for any $r \in \mathbb{R}$. That is, $\{\widetilde{f}_t\}_{t\in\mathbb{R}}$ is a version of $\{f_t\}_{t\in\mathbb{R}}$.

Consider now the case when μ is a σ-finite measure on the space X. Then X is a disjoint union of measurable sets X_n, $n \ge 1$, such that $\mu(X_n) < \infty$. Defining $\widetilde{\mu}(A) = \sum_{n=1}^{\infty}\mu(A\cap X_n)/(\mu(X_n)2^n)$ for $A \in \mathcal{X}$, we get a measure $\widetilde{\mu}$ on X equivalent to μ with $\widetilde{\mu}(X) = 1$. Then, since $\widetilde{\mu} \sim \mu$, $\{f_t\}_{t\in\mathbb{R}}$ is also an almost semi-additive functional with respect to $\widetilde{\mu}$. The result of the proposition then follows from the case considered earlier. □

Proposition 3.2 *Let $\{a_t\}_{t\in\mathbb{R}}$ be an almost cocycle for a flow $\{\phi_t\}_{t\in\mathbb{R}}$ and suppose that a_t takes values in $\{-1,1\}$. Then $\{a_t\}_{t\in\mathbb{R}}$ has a version which is a cocycle for $\{\phi_t\}_{t\in\mathbb{R}}$ and which also takes values in $\{-1,1\}$.*

Proof: As in the proof of Proposition 3.1, we may again prove the result in two cases: (1) the flow $\{\phi_t\}_{t\in\mathbb{R}}$ is an identity flow, and (2) the flow $\{\phi_t\}_{t\in\mathbb{R}}$ is a special flow built up by a function h such that $h(y) > \theta$ for all y and some $\theta > 0$. For the case (1), we have that, for all $s,t \in \mathbb{R}$, $a_{s+t}(x) = a_s(x)a_t(x)$ a.e. $\mu(dx)$. Then $a_t(x) = a_{t/2}(x)a_{t/2}(x)$ a.e. $\mu(dx)$ and, since $a_{t/2} \in \{-1,1\}$, $a_t(x) = 1$ a.e. $\mu(dx)$. (This elementary result also follows from the proofs of Proposition 5.1 in Rosiński (1995).) In other words, in the case where $\{\phi_t\}_{t\in\mathbb{R}}$ is an identity flow, $\{a_t\}_{t\in\mathbb{R}}$ is a version of a cocycle $\{\tilde{a}_t\}_{t\in\mathbb{R}}$ defined by $\tilde{a}_t(x) = 1$. The case (2) may be proved as in Proposition 3.1. The basic idea is to replace relations of the type (3.10) by a similar relation, where there is no factor e^{-t} and where the addition on the right-hand is replaced by a multiplication. In this way, one makes a semi-additive functional into a cocycle. □

Remarks

1. The results of Propositions 3.1 and 3.2 are clearly valid for multiplicative flows as well.

2. Proposition 3.2 is also proved in a more general setting by Zimmer (1984), Theorem B.9, p. 200. (This reference is indicated in Rosiński (2000).) There, almost cocycles are called cocycles and cocycles are called strict cocycles. This terminology is also used by Rosiński (1995, 2000).

The Hopf decomposition

It is important for the sequel that every nonsingular flow has the so-called Hopf decomposition (see, for example, Krengel (1985), p. 17, and Rosiński (1995), p. 1171). Consider a nonsingular map $V : X \mapsto X$. A set $B \in \mathcal{X}$ is called *wandering* if the sets $V^{-k}B = \{x \in X : V^k(x) \in B\}$, $k \geq 1$, are disjoint. The map V is called *conservative* if there is no wandering set of positive measure μ . Given any nonsingular map $V : X \mapsto X$ there exists a decomposition of X into two disjoint measurable sets C and D such that (i) C and D are V-invariant, that is, $V^{-1}C = C$ and $V^{-1}D = D$, (ii) the restriction of V to C is conservative, and (iii) $D = \cup_{k=-\infty}^{\infty} V^k B$ for some wandering set B. The decomposition of X into sets D and C is unique (modulo μ). It is called the *Hopf decomposition*. The sets D and C are called the *dissipative part* and the *conservative part*, respectively. If $\{\phi_t\}_{t\in\mathbb{R}}$ is a nonsingular flow, then for every $t \in \mathbb{R}$, each nonsingular map ϕ_t has the Hopf decomposition $X = D_t \cup C_t$. One can show (Krengel (1969), see Lemma 2.7) that all D_t, $t \neq 0$, are equal to each other modulo μ and that there is a set D, invariant under the flow, such that for every $t \neq 0$, $D = D_t$ modulo μ. The decomposition of the space X into the sets D and $C := X \setminus D$ is called the *Hopf decomposition for the flow* $\{\phi_t\}_{t\in\mathbb{R}}$. A flow is called *dissipative* if $X = D$ and *conservative* if $X = C$ (modulo μ). We also have the same notions for a nonsingular multiplicative flow $\{\psi_c\}_{c>0}$. A useful representation for the dissipative and the conservative parts of an additive flow $\{\phi_t\}_{t\in\mathbb{R}}$ is given by

$$D = \left\{x \in X : \int_{\mathbb{R}} g(\phi_t(x)) \frac{d(\mu \circ \phi_t)}{d\mu}(x) dt < \infty\right\} \quad \text{a.e. } d\mu, \tag{3.22}$$

$$C = \left\{x \in X : \int_{\mathbb{R}} g(\phi_t(x)) \frac{d(\mu \circ \phi_t)}{d\mu}(x) dt = \infty\right\} \quad \text{a.e. } d\mu, \tag{3.23}$$

where g is any $L^1(X, \mu)$ function such that $g \geq 0$ a.e. and $\text{supp}\{g\} = X$ a.e. This fact follows from the proof of Theorem 4.1 in Rosiński (1995) (see also Lemma 2.7 in Krengel (1969)).

4 Minimal kernels

Burnecki et al. (1998) have characterized the minimal kernel of self-similar processes. In this section, we specify the minimal kernel for the processes (1.7), that is, for self-similar processes that are mixed moving averages with stationary increments. These additional characteristics, which allow for a more detailed specification of a minimal kernel, render the proof more delicate. Proposition 3.1 in particular, will play a key role. We shall also use the following "rigidity" lemma mentioned in Section 2. It is due to Hardin (1982) and Rosiński (1994, 1995) (see Theorems 1.1 and 2.2, (b), in Rosiński (1995)).

Lemma 4.1 (Rigidity lemma) *Let $\alpha \in (0, 2)$ and $\{f_t^{(i)}\}_{t\in\mathbb{R}} \subset L^\alpha(E_i, \mathcal{E}_i, m_i)$, $i = 1, 2$, be two spectral representations of a $S\alpha S$ process $\{X(t)\}_{t\in\mathbb{R}}$, where $(E_i, \mathcal{E}_i, m_i)$ are standard Lebesgue spaces.*

(i) Suppose that $\text{supp}\{f_t^{(2)} : t \in \mathbb{R}\} = E_2$ *m_2-a.e. Then, for every σ-finite measure λ on $\mathbb{R}$, there are Borel functions $\Phi : E_2 \mapsto E_1$ and $h : E_2 \mapsto \mathbb{R} \setminus \{0\}$ such that*

$$f_t^{(2)}(x) = h(x) f_t^{(1)}(\Phi(x)) \tag{4.1}$$

a.e. $\lambda(dt)m_2(dx)$.

(ii) If both representations $\{f_t^{(i)}\}$, $i = 1, 2$, *are minimal for the process* X_α, *then there are unique modulo* m_2 *functions* $\Phi : E_2 \mapsto E_1$ *and* $h : E_2 \mapsto \mathbb{R} \setminus \{0\}$ *such that* Φ *is one-to-one, onto and bimeasurable, and for each* $t \in \mathbb{R}$, *the relation* (4.1) *holds a.e.* $m_2(dx)$ *and*

$$\frac{d(m_1 \circ \Phi)}{dm_2}(x) = |h(x)|^\alpha \quad \textit{a.e. } m_2(dx). \tag{4.2}$$

Minimality thus implies that the function Φ in (4.1) satisfies additional regularity conditions and that $|h(x)|^\alpha$ plays the role of the Jacobian (Radon-Nikodym derivative) of the transformation. The following theorem provides a necessary condition for a kernel to be minimal.

Theorem 4.1 *Let* $\alpha \in (0, 2)$, $H > 0$ *and* $\kappa = H - 1/\alpha$. *Suppose that the process* X_α, *given by* (1.7), *is self-similar with index* H *and that* $\{G_t\}_{t\in\mathbb{R}} \subset L^\alpha(X \times \mathbb{R}, \mu(dx)du)$ *given by* (1.8), *is its minimal spectral representation. Then there are a unique modulo* μ *measurable nonsingular multiplicative flow* $\{\psi_c\}_{c>0}$ *on the space* X, *a semi-additive functional* $\{g_c\}_{c>0}$ *for* $\{\psi_c\}_{c>0}$ *and a cocycle* $\{b_c\}_{c>0}$ *for* $\{\psi_c\}_{c>0}$ *taking values in* $\{-1, 1\}$ *such that, for any* $c > 0$ *and* $t \in \mathbb{R}$,

$$\begin{aligned} & c^{-\kappa}\left(G(x, c(t+u)) - G(x, cu)\right) \\ &= b_c(x)\left\{\frac{d(\mu \circ \psi_c)}{d\mu}(x)\right\}^{1/\alpha}\left(G(\psi_c(x), t+u+g_c(x)) - G(\psi_c(x), u+g_c(x))\right) \end{aligned} \tag{4.3}$$

almost everywhere $\mu(dx)du$.

Remark. The converse of Theorem 4.1 is not true. In other words, a kernel G which satisfies (4.3) need not to be minimal. Consider, for example, the process

$$\int_0^1 \int_{\mathbb{R}} \left((t+u)_+^\kappa - u_+^\kappa\right) M_\alpha(dx, du), \tag{4.4}$$

where $\kappa = H - 1/\alpha$, $\alpha \in (0, 2)$, $H \in (0, 1)$ and M_α has the Lebesgue control measure on $[0, 1) \times \mathbb{R}$. The kernel $G_t(x, u) = (t+u)_+^\kappa - u_+^\kappa$ in (4.4) satisfies (4.3) with $\psi_c(x) = x$ for all $x \in X = [0, 1)$, $b_c \equiv 1$, $g_c \equiv 0$ but it is not minimal because it does not satisfy condition (M2) in Definition 2.1.

PROOF: The self-similarity of the process X_α implies that

$$\int_X \int_{\mathbb{R}} c^{-\kappa}(G(x, c(t+u)) - G(x, cu))M_\alpha(dx, du) \stackrel{d}{=} \int_X \int_{\mathbb{R}} (G(x, t+u) - G(x, u))M_\alpha(dx, du), \ t \in \mathbb{R},$$

in the sense of the finite-dimensional distributions. Since the spectral representation $\{G_t\}_{t\in\mathbb{R}}$ is minimal, Lemma 4.1, (*ii*), implies that, for every $c > 0$, there are unique modulo $\mu(dx)du$ functions $\Phi_c(x, u) = (\Phi_c^1(x, u), \Phi_c^2(x, u)) : X \times \mathbb{R} \mapsto X \times \mathbb{R}$ and $\epsilon_c(x, u) : X \times \mathbb{R} \mapsto \mathbb{R} \setminus \{0\}$ such that Φ_c is one-to-one and onto, and, for each $t \in \mathbb{R}$,

$$\begin{aligned} & c^{-\kappa}\left(G(x, c(t+u)) - G(x, cu)\right) \\ &= \epsilon_c(x, u)\left(G(\Phi_c^1(x, u), t + \Phi_c^2(x, u)) - G(\Phi_c^1(x, u), \Phi_c^2(x, u))\right) \end{aligned} \tag{4.5}$$

and

$$|\epsilon_c(x,u)|^\alpha = \frac{d\ (\mu\otimes\mathbb{L})\circ\Phi_c}{d\ (\mu\otimes\mathbb{L})}(x,u) \tag{4.6}$$

a.e. $\mu(dx)du$, where $\mathbb{L}$ denotes the Lebesgue measure on $\mathbb{R}$. We divide the proof in a number of steps. In *Step 1* we show that one can take

$$\Phi_c(x,u) = (\Phi_c^1(x), u+\Phi_c^2(x)),$$
$$\epsilon_c(x,u) = \epsilon_c(x) \ = \ \Psi_c(x)\left\{\frac{d(\mu\circ\Phi_c^1)}{d\mu}(x)\right\}^{1/\alpha},$$

where the function Ψ_c takes values in $\{-1,1\}$. This yields expression (4.3). In *Step 2* we establish that, for any $c_1,c_2>0$,

$$\Phi^1_{c_1c_2}(x) \ = \ \Phi^1_{c_2}(\Phi^1_{c_1}(x)) \quad \text{a.e. } \mu(dx), \tag{4.7}$$
$$\Phi^2_{c_1c_2}(x) \ = \ c_2^{-1}\Phi^2_{c_1}(x)+\Phi^2_{c_2}(\Phi^1_{c_1}(x)) \quad \text{a.e. } \mu(dx), \tag{4.8}$$
$$\Psi_{c_1c_2}(x) \ = \ \Psi_{c_1}(x)\Psi_{c_2}(\Phi^1_{c_1}(x)) \quad \text{a.e. } \mu(dx). \tag{4.9}$$

Modulo the "a.e.", these relations state that Φ^1 is a multiplicative flow, Φ^2 is a semi-additive functional associated with Φ^1 and Ψ is a cocycle for the flow Φ^1. In *Step 3* we show that the functions Φ_c^1, Φ_c^2 and Ψ_c have versions ψ_c, g_c and b_c, respectively, that are measurable in (c,x) and satisfy the equations (4.7), (4.8) and (4.9), respectively, for all $c_1,c_2>0$ and all $x\in X$. This will conclude the proof.

Step 1. Let $c>0$ be fixed. For any $h\in\mathbb{R}$, by replacing u by $u+h$ in (4.5), we have that

$$c^{-\kappa}\left(G(x,c(t+h+u))-G(x,c(h+u))\right)$$
$$=\epsilon_c(x,u+h)\Big(G(\Phi_c^1(x,u+h),t+\Phi_c^2(x,u+h))-G(\Phi_c^1(x,u+h),\Phi_c^2(x,u+h))\Big)$$

a.e. $\mu(dx)du$. On the other hand, by subtracting the relation (4.5) with $t=h$ from the same relation with $t+h$, we also have

$$c^{-\kappa}\left(G(x,c(t+h+u))-G(x,c(h+u))\right)$$
$$=\epsilon_c(x,u)\Big(G(\Phi_c^1(x,u),t+h+\Phi_c^2(x,u))-G(\Phi_c^1(x,u),h+\Phi_c^2(x,u))\Big)$$

The uniqueness of (Φ_c^1,Φ_c^2) and ϵ_c now implies that, for any $h>0$,

$$\Phi_c^1(x,u+h) \ = \ \Phi_c^1(x,u) \quad \text{a.e. } \mu(dx)du,$$
$$\Phi_c^2(x,u+h) \ = \ \Phi_c^2(x,u)+h \quad \text{a.e. } \mu(dx)du,$$
$$\epsilon_c(x,u+h) \ = \ \epsilon_c(x,u) \quad \text{a.e. } \mu(dx)du.$$

By Lemma 3.1, (i), we also have that these equations hold a.e. $dh\mu(dx)du$. Then, making the change of variables $z=u+h$, we have that $\Phi_c^1(x,z)=\Phi_c^1(x,u)$, $\Phi_c^2(x,z)=\Phi_c^2(x,u)+z-u$ and $\epsilon_c(x,z)=\epsilon_c(x,u)$ a.e. $dz\mu(dx)du$. Now, by Lemma 3.1, (ii), fix $u=u_0$ for which the equations hold a.e. $dz\mu(dx)$. We then get that, for some functions $\Phi_c^1(x)$, $\Phi_c^2(x)$ and $\epsilon_c(x)$, $\Phi_c^1(x,z)=\Phi_c^1(x)$, $\Phi_c^2(x,z)=\Phi_c^2(x)+z$ and $\epsilon_c(x,z)=\epsilon_c(x)$ a.e. $dz\mu(dx)$. We also have that

$$|\epsilon_c(x)|^\alpha = |\epsilon_c(x,z)|^\alpha = \frac{d\ (\mu\otimes\mathbb{L})\circ\Phi_c}{d\ (\mu\otimes\mathbb{L})}(x,z) = \frac{d(\mu\circ\Phi_c^1)}{d\mu}(x)$$

a.e. $dz\mu(dx)$, so that we can take

$$\epsilon_c(x) = \Psi_c(x) \left\{ \frac{d(\mu \circ \Phi_c^1)}{d\mu}(x) \right\}^{1/\alpha}, \tag{4.10}$$

where the function Ψ_c takes values in $\{-1, 1\}$. Finally, we note that (4.5) now becomes

$$\begin{aligned} &c^{-\kappa}(G(x, c(t+u)) - G(x, cu)) \\ &= \epsilon_c(x) \ \left(G(\Phi_c^1(x), t + u + \Phi_c^2(x)) - G(\Phi_c^1(x), u + \Phi_c^2(x)) \right), \end{aligned} \tag{4.11}$$

for each $t \in \mathbb{R}$ a.e. $\mu(dx)du$, where ϵ_c is given by (4.10).

Step 2. The following argument is standard but we repeat it for convenience to the reader. Let $c_1, c_2 > 0$. Then, by (4.11),

$$\begin{aligned} &(c_1c_2)^{-\kappa}(G(x, c_1c_2(t+u)) - G(x, c_1c_2u)) \\ &= \epsilon_{c_1c_2}(x) \ \left(G(\Phi_{c_1c_2}^1(x), t + u + \Phi_{c_1c_2}^2(x)) - G(\Phi_{c_1c_2}^1(x), u + \Phi_{c_1c_2}^2(x)) \right). \end{aligned}$$

On the other hand, by iterating the relation (4.11) twice we get that

$$\begin{aligned} &(c_1c_2)^{-\kappa}(G(x, c_1c_2(t+u)) - G(x, c_1c_2u)) \\ &= c_2^{-\kappa} \left(c_1^{-\kappa}(G(x, c_1(c_2t + c_2u)) - G(x, c_1(c_2u))) \right) \\ &= \epsilon_{c_1}(x)c_2^{-\kappa} \left(G(\Phi_{c_1}^1(x), c_2(t + u + c_2^{-1}\Phi_{c_1}^2(x))) - G(\Phi_{c_1}^1(x), c_2(u + c_2^{-1}\Phi_{c_1}^2(x))) \right) \\ &= \epsilon_{c_1}(x)\epsilon_{c_2}(\Phi_{c_1}^1(x)) \left(G(\Phi_{c_2}^1(\Phi_{c_1}^1(x)), t + u + c_2^{-1}\Phi_{c_1}^2(x) + \Phi_{c_2}^2(\Phi_{c_1}^1(x))) \right. \\ &\qquad \left. -G(\Phi_{c_2}^1(\Phi_{c_1}^1(x)), u + c_2^{-1}\Phi_{c_1}^2(x) + \Phi_{c_2}^2(\Phi_{c_1}^1(x))) \right) \end{aligned}$$

The uniqueness implies that the conditions (4.7) and (4.8) are satisfied. The condition (4.9) follows again from uniqueness and the expression (4.10), since

$$\frac{d(\mu \circ \Phi_{c_1c_2}^1)}{d\mu} = \frac{d(\mu \circ \Phi_{c_2}^1 \circ \Phi_{c_1}^1)}{d\mu} = \left(\frac{d(\mu \circ \Phi_{c_2}^1)}{d\mu} \circ \Phi_{c_1}^1 \right) \frac{d(\mu \circ \Phi_{c_1}^1)}{d\mu}.$$

Step 3. We have to deal with joint measurability and a.e. issues. Let

$$F_c(x, u) = \Phi_c \left(x, \frac{u}{c} \right) = \left(\Phi_c^1(x), \frac{u}{c} + \Phi_c^2(x) \right). \tag{4.12}$$

It is easy to see that, for all $c_1, c_2 > 0$,

$$F_{c_1c_2}(x, u) = F_{c_1}(F_{c_2}(x, u))$$

a.e. $\mu(dx)du$. As in the proof of Theorem 3.1 in Rosiński (1995), we conclude that there is a measurable nonsingular multiplicative flow $\{\varphi_c\}_{c>0}$ which is a version of $\{F_c\}_{c>0}$ and that we may also assume without loss of generality that the function $\epsilon_c(x)$ above is measurable in (c,x). The basic difficulty is to show that we can in fact take this flow $\{\varphi_c\}_{c>0}$ to be of the form $\varphi_c(x,u) = (\psi_c(x), \frac{u}{c} + b_c(x))$, that is, it can have the same form (4.12) as F_c.

Let us deal with the function Φ_c^1 first. Suppose that $\varphi_c(x,u) = (\phi_c^1(x,u), \phi_c^2(x,u))$. Then, for any $c > 0$, $\Phi_c^1(x) = \phi_c^1(x,u)$ a.e. $\mu(dx)du$ and $\Phi_c^1(x) = \int_0^1 \Phi_c^1(x)du = \int_0^1 \phi_c^1(x,u)du =: \tilde{\psi}_c(x)$. Since $\phi_c^1(x,u)$ is measurable in (c,x,u), the function $\tilde{\psi}_c(x)$ is measurable in (c,x). Moreover, $\{\tilde{\psi}_c\}_{c>0}$ is a version of $\{\Phi_c^1\}_{c>0}$ and it satisfies $\tilde{\psi}_{c_1c_2}(x) = \tilde{\psi}_{c_2}(\tilde{\psi}_{c_1}(x))$ a.e. $\mu(dx)$ for all $c_1, c_2 > 0$. By applying the same arguments as in the proof of Theorem 3.1 in Rosiński (1995), we conclude that there is a measurable nonsingular multiplicative flow $\{\psi_c\}_{c>0}$ which is a version of $\{\tilde{\psi}_c\}_{c>0}$. Observe that $\{\psi_c\}_{c>0}$ is also a version of $\{\Phi_c^1\}_{c>0}$.

Let us now deal with the functions Φ_c^2. Arguing as for the map Φ_c^1 above, we may assume without loss of generality that the map $\Phi_c^2(x) : (0,\infty) \times X \mapsto \mathbb{R}$ is measurable in (c,x). It satisfies the relation

$$\Phi_{c_1c_2}^2(x) = c_2^{-1}\Phi_{c_1}^2(x) + \Phi_{c_2}^2(\psi_{c_1}(x))$$

a.e. $\mu(dx)$ for all $c_1, c_2 > 0$. In other words, the collection of maps $\{\Phi_c^2\}_{c>0}$ is an almost semi-additive functional for the flow $\{\psi_c\}_{c>0}$ on (X,μ). It follows by Proposition 3.1 that $\{\Phi_c^2\}_{c>0}$ has a version $\{g_c\}_{c>0}$ which is a semi-additive functional for the flow $\{\psi_c\}_{c>0}$.

As for the functions $\Psi_c(x)$, we also have that

$$\Psi_{c_1c_2}(x) = \Psi_{c_1}(x)\Psi_{c_2}(\psi_{c_1}(x))$$

a.e. $\mu(dx)$. In other words, the collection $\{\Psi_c\}_{c>0}$ is an almost cocycle for the flow $\{\psi_c\}_{c>0}$ on (X,μ). Then, by Proposition 3.2, $\{\Psi_c\}_{c>0}$ has a version $\{b_c\}_{c>0}$ which is a cocycle for $\{\psi_c\}_{c>0}$ taking values in $\{-1,1\}$. $\Box$

The following lemma will be used a number of times. It concerns the support of G_t, $t \in \mathbb{R}$.

Lemma 4.2 *Let $(X,\mathcal{X},\mu)$ be a σ-finite measure space, $G : X \times \mathbb{R} \mapsto \mathbb{R}$ a measurable function on $(X \times \mathbb{R}, \mathcal{X} \otimes \mathcal{B}, \mu(dx)du)$ and $G_t(x,u) = G(x,t+u) - G(x,u)$, for $x \in X$, $u,t \in \mathbb{R}$. Then there is a set $X_0 \in \mathcal{X}$ such that*

$$\text{supp}\{G_t, t \in \mathbb{R}\} = X_0 \times \mathbb{R} \quad \text{a.e. } \mu(dx)du. \tag{4.13}$$

Moreover, if

$$X_\alpha(t) = \int_X \int_{\mathbb{R}} G_t(x,u) M_\alpha(dx,du),\ t \in \mathbb{R}, \tag{4.14}$$

as in (1.7)*, where M_α has the control measure $\mu(dx)du$, then we may suppose without loss of generality that*

$$\text{supp}\{G_t, t \in \mathbb{R}\} = X \times \mathbb{R} \quad \text{a.e. } \mu(dx)du. \tag{4.15}$$

PROOF: Observe that $\text{supp}\{G_t, t \in \mathbb{R}\} = \text{supp}\{G_{s,t}, s,t \in \mathbb{R}\}$, where $G_{s,t} = G_t - G_s$. Indeed, on one hand $G_{t,0} = G_t$, so that $\text{supp}\{G_t, t \in \mathbb{R}\} \subset \text{supp}\{G_{s,t}, s,t \in \mathbb{R}\}$; on the other hand, $\{G_{s,t} \neq 0\} \subset \{G_s \neq 0\} \cup \{G_t \neq 0\}$, so that $\text{supp}\{G_{s,t}, s,t \in \mathbb{R}\} \subset \text{supp}\{G_t, t \in \mathbb{R}\}$ as well. It is also clear that, for all $h \in \mathbb{R}$,

$$\text{supp}\{G_{s,t}, s,t \in \mathbb{R}\} = \text{supp}\{G_{s+h,t+h}, s,t \in \mathbb{R}\} = \text{supp}\{G_{s,t}, s,t \in \mathbb{R}\} + h, \tag{4.16}$$

where by $E+h$, for $E \in \mathcal{X} \otimes \mathcal{B}$ and $h \in \mathbb{R}$, we mean the set $\{(x, u+h) : (x,u) \in E\}$. Setting $E_0 = \mathrm{supp}\{G_{s,t}, s, t \in \mathbb{R}\}$ and using Lemma 3.1, (i), relation (4.16) implies that $1_{E_0}(x, u+h) = 1_{E_0}(x,u)$ a.e. $\mu(dx)dudh$. By making the change of variables $v = u+h$ and then fixing $v = v_0$ for which the relation holds a.e. $\mu(dx)du$, we get $1_{E_0}(x, v_0) = 1_{E_0}(x,u)$ a.e. $\mu(dx)du$. Since $1_{E_0}(x, v_0) = 1_{X_0\times\mathbb{R}}(x,u)$, where $X_0 = \{x : (x, v_0) \in E_0\}$, we obtain (4.13).

If (4.14) holds, then we may suppose (4.15) without loss of generality, since replacing X by X_0 does not change the distribution of the process. □

Example 4.1 Let $\alpha \in (0,2)$ and $\{X_\alpha(t)\}_{t\in\mathbb{R}}$ be a non-degenerate $S\alpha S$ process with stationary increments, having the representation

$$\{X_\alpha(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{ \int_{\mathbb{R}} (G(t+u) - G(u)) M_\alpha(du) \right\}_{t\in\mathbb{R}},$$

where M_α has the Lebesgue control measure. Denote $G_t(u) = G(t+u) - G(u)$, $u, t \in \mathbb{R}$. Observe that in contrast with (1.7) there is no "x" here. We will show that the spectral representation $\{G_t\}_{t\in\mathbb{R}}$ of X_α is minimal. Consequently, if the process X_α is also self-similar, then Theorem 4.1 implies that the kernel function G satisfies condition (4.3). This fact is used in Pipiras and Taqqu (2001) to characterize linear fractional stable motions, log-fractional stable motion and stable Lévy motion.

To show that $\{G_t\}_{t\in\mathbb{R}}$ is minimal for X_α, one needs to verify conditions (M1) and (M2) stated in Definition 2.1. By Lemma 4.2, the support of the functions G_t, $t \in \mathbb{R}$, is either $\mathbb{R}$ or $\emptyset$ a.e. du (the support is $\emptyset$ a.e. du, if the set X_0 in Lemma 4.2 is $\emptyset$). If the support is $\mathbb{R}$, then condition (M1) holds. If it is $\emptyset$, then $G_t(u) = 0$ a.e. du and hence $X_\alpha \equiv 0$ which is a contradiction (X_α is degenerate). To verify condition (M2), by Theorem 3.8 (iii) in Rosiński (1998), it is enough to prove that, if for some measurable nonsingular map $\phi : \mathbb{R} \mapsto \mathbb{R}$, a measurable function $k : \mathbb{R} \mapsto \mathbb{R} \setminus \{0\}$ and all $t \in \mathbb{R}$,

$$G_t(\phi(u)) = G(t+\phi(u)) - G(\phi(u)) = k(u)(G(t+u) - G(u)) = k(u)G_t(u) \quad \text{a.e. } du, \tag{4.17}$$

then $\phi(u) = u$ a.e. du. By replacing t by $t+v$ in (4.17), we get, for all $t, v \in \mathbb{R}$,

$$G(t+v+\phi(u)) - G(\phi(u)) = k(u)(G(t+v+u) - G(u)) \quad \text{a.e. } du \tag{4.18}$$

and, by subtracting (4.17) from (4.18),

$$G(t+v+\phi(u)) - G(t+\phi(u)) = k(u)(G(t+v+u) - G(t+u)) \quad \text{a.e. } du. \tag{4.19}$$

By Lemma 3.1, (i), relation (4.19) holds also a.e. $dtdvdu$. By making the change of variables $t+u = z$, we then get

$$G(v+z+\phi(u)-u) - G(z+\phi(u)-u) = k(u)(G(v+z) - G(z)) \quad \text{a.e. } dzdvdu. \tag{4.20}$$

Let us show that, unless $\phi(u) = u$ a.e. du, relation (4.20) implies $G(u) = \mathrm{const}$ a.e. du and hence $X_\alpha \equiv 0$ which is a contradiction. Thus suppose there is a set of positive Lebesgue measure on which $\phi(u) \neq u$. Then, (4.20) implies there is a set V, whose complement has Lebesgue measure 0, such that for every $v \in V$, $G(v+z+\phi(u)-u) - G(z+\phi(u)-u) = k(u)(G(v+z) - G(z))$ a.e. $dzdu$. Now, for every $v \in V$, fix $u = u_0 = u_0(v)$ for which $\phi(u_0) \neq u_0$ and

$$G(v+z+\phi(u_0)-u_0) - G(z+\phi(u_0)-u_0) = k(u_0)(G(v+z) - G(z)) \quad \text{a.e. } dz. \tag{4.21}$$

By integrating both sides of (4.21), we get for all $v \in V$, $\int_{\mathbb{R}} |G(v + z + \phi(u_0) - u_0) - G(z + \phi(u_0) - u_0)|^\alpha dz = |k(u_0)|^\alpha \int_{\mathbb{R}} |G(v + z) - G(z)|^\alpha dz < \infty$. By the translation of the Lebesgue measure, the two integrals are equal. Then, either $\int_{\mathbb{R}} |G(v + z) - G(z)|^\alpha dz = 0$ which implies $G(v + z) = G(z)$ a.e. dz or $|k(u_0)| = 1$. In the latter case, however, by (4.21), the function $h(z) = |G(v + z) - G(z)|$ is periodic with a period $|\phi(u_0) - u_0| \neq 0$. Since $h \in L^\alpha(\mathbb{R})$, we necessarily have that $h(z) = 0$. In either case, the conclusion is that for all $v \in V$, $G(v + z) = G(z)$ a.e. dz. By Lemma 3.1, (i), $G(v + z) = G(z)$ a.e. $dzdv$ and, by making a change of variables, $G(v) = G(z)$ a.e. $dzdv$. By fixing v for which this equation holds a.e. dz, we obtain $G(z) = \text{const}$ a.e. dz which is what we wanted to show. This proves that $\{G_t\}_{t\in\mathbb{R}}$ is minimal for X_α.

The preceding example extends a result of Rosiński (1998) valid for stationary processes of the form $\int_{\mathbb{R}} G(t + u)M_\alpha(du)$ to processes with stationary increments. While not all representations are minimal, the following theorem states that one can always make a transformation and obtain a minimal representation. Note that we restrict the range of parameter α to $(1, 2)$. We comment on the case $\alpha \in (0, 1]$ in a remark after the proof.

Theorem 4.2 *Let $\alpha \in (1, 2)$, $H > 0$ and X_α be a $S\alpha S$ H-ss process with stationary increments given by* (1.7) *and* (1.8). *Then, there exists another standard Lebesgue space $(\widetilde{X}, \widetilde{\mathcal{X}}, \widetilde{\mu})$ and measurable function $\widetilde{G} : \widetilde{X} \times \mathbb{R} \mapsto \mathbb{R}$ such that*

$$\{X_\alpha(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{\int_{\widetilde{X}} \int_{\mathbb{R}} (\widetilde{G}(\widetilde{x}, t + u) - \widetilde{G}(\widetilde{x}, u))\widetilde{M}_\alpha(d\widetilde{x}, du)\right\}_{t\in\mathbb{R}} = \left\{\int_{\widetilde{X}} \int_{\mathbb{R}} \widetilde{G}_t(\widetilde{x}, u)\widetilde{M}_\alpha(d\widetilde{x}, du)\right\}_{t\in\mathbb{R}},$$

where $\widetilde{M}_\alpha$ has the control measure $\widetilde{\mu}(d\widetilde{x})du$, and that the spectral representation $\{\widetilde{G}_t\}_{t\in\mathbb{R}}$ is minimal for X_α.

PROOF: Consider the collection of functions $\{G_t(x, u) = G(x, t + u) - G(x, u), t \in \mathbb{R}\}$ where $G : X \times \mathbb{R} \mapsto \mathbb{R}$. By Lemma 4.2, we may suppose without loss of generality that $\text{supp}\{G_t, t \in \mathbb{R}\} = X \times \mathbb{R}$. Consider now a $S\alpha S$ stationary process Y_α given by

$$\begin{aligned} Y_\alpha(t) &= \int_{-\infty}^{t} e^{-(t-s)}(X(t) - X(s))ds \\ &= X_\alpha(t) - \int_{-\infty}^{t} e^{-(t-s)} X_\alpha(s)ds \qquad (4.22) \\ &= \int_X \int_{\mathbb{R}} \left(G(x, t + u) - \int_{-\infty}^{t} e^{-(t-s)} G(x, s + u)ds\right) M_\alpha(dx, du), \qquad (4.23) \end{aligned}$$

which can be rewritten as

$$Y_\alpha(t) = \int_X \int_{\mathbb{R}} g(x, t + u)M_\alpha(dx, du), \qquad (4.24)$$

where

$$g(x, u) = G(x, u) - e^{-u} \int_{-\infty}^{u} e^s G(x, s)ds.$$

The process Y_α is well-defined for $\alpha \in (1, 2)$ and the equality of (4.22) and (4.23) hold, since $\int_{-\infty}^{t} e^{-(t-s)} E|X_\alpha(s)|ds = E|X_\alpha(1)| \int_{-\infty}^{t} e^{-(t-s)} |s|^H ds < \infty$ (see Chapter 11 in Samorodnitsky and

Taqqu (1994)). Moreover, the relation (4.22) is invertible in the sense that

$$\begin{aligned} X_\alpha(t) &= Y_\alpha(t) - Y_\alpha(0) + \int_0^t Y_\alpha(s)ds & (4.25) \\ &= \int_X \int_{\mathbb{R}} \left(g(x,t+u) - g(x,u) + \int_0^t g(x,s+u)ds \right) M_\alpha(dx,du) & (4.26) \end{aligned}$$

and that

$$G(x,u) = g(x,u) + \int_0^u g(x,s)ds.$$

(These transformations between stationary processes and processes with stationary increments are used in Surgailis et al. (1998).) Let also $g_t(x,u) = g(x,t+u)$ and observe that, by Lemma 4.3 below, $\mathrm{supp}\{g_t, t \in \mathbb{R}\} = \mathrm{supp}\{G_t, t \in \mathbb{R}\} = X \times \mathbb{R}$ a.e. Now, since $Y_\alpha(t) = \int_X \int_{\mathbb{R}} g(x,t+u) M_\alpha(dx,du)$ is stationary, $\mathrm{supp}\{g(x,t+u), t \in \mathbb{R}\} = X \times \mathbb{R}$ a.e. $\mu(dx)du$ and $\int_{\mathbb{R}} |g(x,t+u)|^\alpha dt = \int_{\mathbb{R}} |g(x,t)|^\alpha dt < \infty$ a.e. $\mu(dx)$, Corollary 4.2 in Rosiński (1995) implies that the process Y_α is generated by a dissipative flow. Then, by Theorem 4.4 in Rosiński (1995) there is another standard Lebesgue space $(\widetilde{X}, \widetilde{\mu})$ and measurable function $\widetilde{g} : \widetilde{X} \times \mathbb{R} \mapsto \mathbb{R}$ such that

$$\{Y_\alpha(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{ \int_{\widetilde{X}} \int_{\mathbb{R}} \widetilde{g}(\widetilde{x}, t+u) \widetilde{M}_\alpha(d\widetilde{x}, du) \right\}_{t\in\mathbb{R}} = \left\{ \int_{\widetilde{X}} \int_{\mathbb{R}} \widetilde{g}_t(\widetilde{x}, u) \widetilde{M}_\alpha(d\widetilde{x}, du) \right\}_{t\in\mathbb{R}}$$

and the spectral representation $\{\widetilde{g}_t\}_{t\in\mathbb{R}}$ is minimal for Y_α. By (4.26), we have

$$\begin{aligned} \{X_\alpha(t)\}_{t\in\mathbb{R}} &\stackrel{d}{=} \left\{ \int_{\widetilde{X}} \int_{\mathbb{R}} (\widetilde{g}(\widetilde{x}, t+u) - \widetilde{g}(\widetilde{x}, u) + \int_0^t \widetilde{g}(\widetilde{x}, s+u)ds) \widetilde{M}_\alpha(d\widetilde{x}, du) \right\}_{t\in\mathbb{R}}, \\ &= \left\{ \int_{\widetilde{X}} \int_{\mathbb{R}} (\widetilde{G}(\widetilde{x}, t+u) - \widetilde{G}(\widetilde{x}, u)) \widetilde{M}_\alpha(d\widetilde{x}, du) \right\}_{t\in\mathbb{R}}, \end{aligned}$$

where $\widetilde{G}(\widetilde{x}, u) = \widetilde{g}(\widetilde{x}, u) + \int_0^u \widetilde{g}(\widetilde{x}, s)ds$. Let $\widetilde{G}_t(\widetilde{x}, u) = \widetilde{G}(\widetilde{x}, t+u) - \widetilde{G}(\widetilde{x}, u)$. To prove the theorem, it is enough to show that the spectral representation $\{\widetilde{G}_t\}_{t\in\mathbb{R}}$ is minimal for X_α. By Lemma 4.3 below, $\mathrm{supp}\{\widetilde{G}_t, t \in \mathbb{R}\} = \mathrm{supp}\{\widetilde{g}_t, t \in \mathbb{R}\} = \widetilde{X} \times \mathbb{R}$ a.e. $\widetilde{\mu}(d\widetilde{x})du$. Let $\mathcal{E}$ be the σ–algebra associated with $\widetilde{X} \times \mathbb{R}$. To establish minimality, it is sufficient to show that $\rho(\widetilde{G}_t, t \in \mathbb{R}) = \mathcal{E}$, where ρ is the ratio σ-algebra defined in Section 2. Since $\{\widetilde{g}_t\}_{t\in\mathbb{R}}$ is minimal, we have $\rho(\widetilde{g}_t, t \in \mathbb{R}) = \mathcal{E}$ and hence it is sufficient to show that $\rho(\widetilde{g}_t, t \in \mathbb{R}) \subset \rho(\widetilde{G}_t, t \in \mathbb{R})$. This follows from Lemma 4.3 below, since for each $t^{(1)}, t^{(2)} \in \mathbb{R}$, $\widetilde{g}_{t^{(1)}}/\widetilde{g}_{t^{(2)}}$ is an a.e. limit of $\sum c_n \widetilde{G}_{t_n^{(1)}} / \sum d_n \widetilde{G}_{t_n^{(2)}}$ type sums. $\Box$

Remark. When $\alpha \in (0,1]$, there exist processes X_α (having the representation (5.1)) for which the integral $\int_{-\infty}^t e^{-(t-s)} X(s)ds$ and hence the transformation (4.22) is not defined (see, for example, Samorodnitsky and Taqqu (1994), p. 510). As a consequence, when $\alpha \in (0,1]$ the proof of Theorem 4.2 works only for the subclass of processes X_α for which this integral is well-defined. Although we feel that the result of the theorem holds for $\alpha \in (0,1]$ without this restriction, we do not have a proof. Therefore, we will provide a decomposition of processes X_α for $\alpha \in (1,2)$ only. It is clear from the sequel that the results of this work hold for any X_α, $\alpha \in (0,1]$, for which the conclusion of Theorem 4.2 is valid.

The following lemma was used in the proof of Theorem 4.2 above. It states that the kernels G_t and g_t used in that proof can be approximated by each other.

Lemma 4.3 *Let $\alpha \in (1,2)$ and X_α be a $S\alpha S$, H-ss process given by (1.7) and (1.8). Let also Y_α be a $S\alpha S$, stationary process defined by (4.22) and having the representation (4.24). Then, for each $s,t \in \mathbb{R}$, there exist real numbers s_l^n, a_l^n, $n \geq 1$, $l = 1, \cdots, l_n$, and t_k^n, b_k^n, $n \geq 1$, $k = 1, \cdots, k_n$, such that*

$$\sum_{l=1}^{l_n} a_l^n G_{s_l^n} \to g_s \quad a.e., \qquad \sum_{k=1}^{k_n} b_k^n g_{t_k^n} \to G_t \quad a.e., \tag{4.27}$$

as $n \to \infty$, where $G_t(x,u) = G(x, t+u) - G(x,u)$ and $g_t(x,u) = g(x,t+u) - g(x,u)$ for $t, u \in \mathbb{R}$ and $x \in X$.

Proof: Let us show the first convergence in (4.27). For $n \geq 1$, let $k_n = \frac{k}{n}$ and set

$$X_\alpha^n(s) = \sum_{k=-\infty}^{\infty} X_\alpha(k_n) 1_{[k_n,(k+1)_n)}(s) = \int_X \int_{\mathbb{R}} G_s^n(x,u) M_\alpha(dx,du),$$

where

$$G_s^n(x,u) = \sum_{k=-\infty}^{\infty} (G(x, k_n + u) - G(x,u)) 1_{[k_n,(k+1)_n)}(s), \tag{4.28}$$

and

$$Y_\alpha^n(s) = X_\alpha^n(s) - \int_{-\infty}^{s} e^{-(s-t)} X_\alpha^n(t) dt = \int_X \int_{\mathbb{R}} g_s^n(x,u) M_\alpha(dx,du),$$

where

$$g_s^n(x,u) = G_s^n(x,u) - \int_{-\infty}^{s} e^{-(s-t)} G_t^n(x,u) dt. \tag{4.29}$$

Since $E|X_\alpha(t_2) - X_\alpha(t_1)| = |t_2 - t_1|^H E|X_\alpha(1)|$, we have

$$\int_{-\infty}^{s} e^{-(s-t)} E|X_\alpha(t) - X_\alpha^n(t)| dt = E|X_\alpha(1)| \int_{-\infty}^{s} e^{-(s-t)} \sum_{k=-\infty}^{\infty} |t - k_n|^H 1_{[k_n,(k+1)_n)}(t) dt = O\left(\frac{1}{n^H}\right),$$

as $n \to \infty$, since the sum is bounded by $1/n^H$ and $\int_{-\infty}^{s} e^{-(s-t)} dt = 1$. Then, for each $s \in \mathbb{R}$, $Y_\alpha^n(s) \to Y_\alpha(s)$ in $L^1(\Omega)$ and, hence, in probability and distribution as well. This implies that $g_s^n \to g_s$ in $L^\alpha(X \times \mathbb{R})$ and, hence, there exists a subsequence $\{n_k\}_{k \geq 1}$ such that the convergence holds a.e. Observe that by (4.29) and (4.28), the g_s^n's are of the type $\sum a_l^n G_{s_l^n}$, where the sums have an infinite number of terms. After truncating these sums to a finite number of terms, we obtain the result. The second convergence in (4.27) can be proved similarly by setting $g_s^n(x,u) = \sum_{k=-\infty}^{\infty} g(x, k_n + u) 1_{[k_n,(k+1)_n)}(s)$ with $k_n = \frac{k}{n}$ and using (4.26). $\Box$

Definition 4.1 A flow ψ, its related cocycle b and semi-additive functional g constitute the *triplet* (ψ, b, g).

The next result states that triplets corresponding to different minimal spectral representations of the same process are equivalent in the sense of the following definition (see also Definition 3.2 in Rosiński (1995)).

Definition 4.2 Triplets $(\psi^{(1)}, b^{(1)}, g^{(1)})$ and $(\psi^{(2)}, b^{(2)}, g^{(2)})$, where $b^{(i)} = \{b_c^{(i)}\}_{c>0}$ is a cocycle and $g^{(i)} = \{g_c^{(i)}\}_{c>0}$ is a semi-additive functional for a measurable nonsingular multiplicative flow $\psi^{(i)} = \{\psi_c^{(i)}\}_{c>0}$ on (X_i, μ_i), $i = 1, 2$, are said to be *equivalent*, denoted by $(\psi^{(1)}, b^{(1)}, g^{(1)}) \sim (\psi^{(2)}, b^{(2)}, g^{(2)})$, if there exists a measurable map $\Phi : X_2 \mapsto X_1$ such that

(i) there is $N_i \subset X_i$ with $\mu_i(N_i) = 0$, $i = 1, 2$, such that Φ is one-to-one, onto and bimeasurable between $X_2 \setminus N_2$ and $X_1 \setminus N_1$,

(ii) μ_1 and $\mu_2 \circ \Phi^{-1}$ are mutually absolutely continuous,

(iii) *relation between flows*: for all $c > 0$, $\psi_c^{(1)} \circ \Phi = \Phi \circ \psi_c^{(2)}$ a.e. $d\mu_2$,

(iv) *relation between cocycles*: the cocycle $\{b_c^{(1)} \circ \Phi\}_{c>0}$ is cohomologous to $\{b_c^{(2)}\}_{c>0}$, that is, there is a measurable function $b : X_2 \mapsto \{-1, 1\}$ such that, for all $c > 0$, $b_c^{(1)} \circ \Phi = b_c^{(2)} \cdot (b \circ \psi_c^{(2)})/b$ a.e. $d\mu_2$,

(v) *relation between semi-additive functionals*: there is a measurable function $g : X_2 \mapsto \mathbb{R}$ such that $g_c^{(1)} \circ \Phi = g_c^{(2)} + g \circ \psi_c^{(2)} - c^{-1} g$ a.e. $d\mu_2$.

Theorem 4.3 *Let $\alpha \in (0, 2)$ and $\{G^{(i)}(x_i, t + u) - G^{(i)}(x_i, u), x_i \in X_i, u \in \mathbb{R}\}_{t\in\mathbb{R}} \subset L^\alpha(X_i \times \mathbb{R}, \mu_i(dx_i)du)$ be two minimal representations for a $S\alpha S$, self-similar mixed moving average X_α. Let also $(\psi^{(i)}, b^{(i)}, g^{(i)})$, $i = 1, 2$, be the triplets corresponding to these minimal spectral representation by Theorem 4.1. Then, $(\psi^{(1)}, b^{(1)}, g^{(1)}) \sim (\psi^{(2)}, b^{(2)}, g^{(2)})$.*

Proof: By using Lemma 4.1, (ii), we obtain that there are unique modulo $\mu_2(dx_2)du$ functions $\mathbf{\Phi} = (\Phi^1(x_2, u), \Phi^2(x_2, u)) : X_2 \times \mathbb{R} \mapsto X_1 \times \mathbb{R}$ and $h : X_2 \times \mathbb{R} \mapsto \mathbb{R} \setminus \{0\}$ such that $\mathbf{\Phi}$ is one-to-one, onto and bimeasurable, and for all $t \in \mathbb{R}$,

$$G^{(2)}(x_2, t + u) - G^{(2)}(x_2, u)$$

$$= h(x_2, u)\Big(G^{(1)}(\Phi^1(x_2, u), t + \Phi^2(x_2, u)) - G^{(1)}(\Phi^1(x_2, u), \Phi^2(x_2, u))\Big), \tag{4.30}$$

$$h(x_2, u) = b(x_2, u) \left\{ \frac{d(\mu_1 \otimes \mathbb{L}) \circ \mathbf{\Phi}}{d(\mu_2 \otimes \mathbb{L})}(x_2, u) \right\}^{1/\alpha} \tag{4.31}$$

a.e. $\mu_2(dx_2)du$ with some $b : X_2 \times \mathbb{R} \mapsto \{-1, 1\}$. Arguing as in Step 1 in the proof of Theorem 4.1, we conclude that

$$\mathbf{\Phi} = (\Phi^1(x_2, u), \Phi^2(x_2, u)) = (\Phi(x_2), u + g(x_2)), \quad b(x_2, u) = b(x_2) \tag{4.32}$$

a.e. $\mu_2(dx_2)du$, where $\Phi : X_2 \mapsto X_1$, $g : X_2 \mapsto \mathbb{R}$ and $b : X_2 \mapsto \{-1, 1\}$ are some measurable functions. Changing variables, we get $\mathbf{\Phi}^* = (\Phi^1(x_2, u - g(x)), \Phi^2(x_2, u - g(x))) = (\Phi(x_2), u)$ a.e. $\mu_2(dx_2)du$. Since $\mathbf{\Phi}^*$ is also one-to-one, we deduce that there is a set $N_2 \subset X_2$ with $\mu_2(N_2) = 0$ such that Φ is one-to-one on $X_2 \setminus N_2$. Then, the function $(\Phi(x_2), u + g(x_2))$ is one-to-one on $(X_2 \setminus N_2) \times \mathbb{R}$ and, since it is equal a.e. to a bimeasurable and nonsingular function $\mathbf{\Phi}$, we conclude that the image of Φ on $X_2 \setminus N_2$ is $X_1 \setminus N_1$ with $\mu_1(N_1) = 0$ and that Φ^{-1} on $X_1 \setminus N_1$ is measurable as well. By using the notation X_1 and X_2 for the spaces $X_1 \setminus N_1$ and $X_2 \setminus N_2$, respectively, we have from (4.30)–(4.32) that there are a one-to-one, onto and bimeasurable map $\Phi : X_2 \mapsto X_1$ and $h : X_2 \mapsto \mathbb{R} \setminus \{0\}$ such that, for all $t \in \mathbb{R}$,

$$G^{(2)}(x_2, t + u) - G^{(2)}(x_2, u) = h(x_2)\Big(G^{(1)}(\Phi(x_2), t + u + g(x_2)) - G^{(1)}(\Phi(x_2), u + g(x_2))\Big), \tag{4.33}$$

$$h(x_2) = b(x_2)\left\{\frac{d(\mu_1 \circ \Phi)}{d\mu_2}(x_2)\right\}^{1/\alpha} \tag{4.34}$$

a.e. $\mu_2(dx_2)du$ with some $b : X_2 \mapsto \{-1, 1\}$. On the other hand, we know from Theorem 4.1 that, for $i = 1, 2$, and all $c > 0$,

$$c^{-\kappa}\left(G^{(i)}(x_i, c(t+u)) - G(x_i, cu)\right)$$

$$= b_c^{(i)}(x_i)\left\{\frac{d(\mu_i \circ \psi_c^{(i)})}{d\mu_i}(x_i)\right\}^{1/\alpha}\left(G^{(i)}(\psi_c^{(i)}(x_i), t+u+g_c^{(i)}(x_i)) - G(\psi_c^{(i)}(x_i), u+g_c^{(i)}(x_i))\right) \tag{4.35}$$

a.e. $\mu_i(dx_i)du$. Then (see explanations below), for all $c > 0$,

$$c^{-\kappa}\left(G^{(2)}(x_2, c(t+u)) - G(x_2, cu)\right)$$

$$= c^{-\kappa}h(x_2)\Big(G^{(1)}(\Phi(x_2), c(t+u)+g(x_2)) - G^{(1)}(\Phi(x_2), cu+g(x_2))\Big) \tag{4.36}$$

$$= h(x_2)b_c^{(1)}(\Phi(x_2))\left\{\frac{d(\mu_1 \circ \psi_c^{(1)})}{d\mu_1}(\Phi(x_2))\right\}^{1/\alpha}\cdot$$

$$\cdot\Big(G^{(1)}(\psi_c^{(1)}(\Phi(x_2)), t+u+c^{-1}g(x_2)+g_c^{(1)}(\Phi(x_2))) - G(\psi_c^{(1)}(\Phi(x_2)), u+c^{-1}g(x_2)+g_c^{(1)}(\Phi(x_2)))\Big) \tag{4.37}$$

$$= h(x_2)(b_c^{(1)} \circ \Phi)(x_2)\left\{\Big(\frac{d\mu_1 \circ \psi_c^{(1)}}{d\mu_1} \circ \Phi\Big)(x_2)\right\}^{1/\alpha}\Big((h \circ \Phi^{-1} \circ \psi_c^{(1)} \circ \Phi)(x_2)\Big)^{-1}\cdot$$

$$\cdot\Big(G^{(2)}((\Phi^{-1} \circ \psi_c^{(1)} \circ \Phi)(x_2), t+u+c^{-1}g(x_2)+(g_c^{(1)} \circ \Phi)(x_2) - (g \circ \Phi^{-1} \circ \psi_c^{(1)} \circ \Phi)(x_2))$$

$$-G^{(2)}((\Phi^{-1} \circ \psi_c^{(1)} \circ \Phi)(x_2), u+c^{-1}g(x_2)+(g_c^{(1)} \circ \Phi)(x_2) - (g \circ \Phi^{-1} \circ \psi_c^{(1)} \circ \Phi)(x_2))\Big) \tag{4.38}$$

a.e. $\mu_2(dx_2)du$. For (4.36), we used (4.33); for (4.37), we used (4.35) with $i = 1$, where x_1 is replaced by $\Phi(x_2)$ and u by $u+c^{-1}g(x_2)$; for (4.38), we used (4.33) again, where x_2 is replaced by $(\Phi^{-1} \circ \psi_c^{(1)} \circ \Phi)(x_2)$ and then u by $u + c^{-1}g(x_2) + (g_c^{(1)} \circ \Phi)(x_2) - (g \circ \Phi^{-1} \circ \psi_c^{(1)} \circ \Phi)(x_2)$. By comparing (4.38) to (4.35) with $i = 2$ and by using the uniqueness in Theorem 4.1, we conclude that for all $c > 0$,

$$\psi_c^{(2)} = \Phi^{-1} \circ \psi_c^{(1)} \circ \Phi \quad \text{a.e. } d\mu_2, \tag{4.39}$$

$$b_c^{(1)} \circ \Phi = \frac{b \circ \psi_c^{(2)}}{b} b_c^{(2)} \quad \text{a.e. } d\mu_2, \tag{4.40}$$

$$g_c^{(1)} \circ \Phi = g_c^{(2)} + g \circ \psi_c^{(2)} - \frac{g}{c} \quad \text{a.e. } d\mu_2 \tag{4.41}$$

(to obtain the equality (4.40), we also used (4.34) and

$$\frac{d\mu_1 \circ \Phi}{d\mu_2}\left(\frac{d\mu_1 \circ \psi_c^{(1)}}{d\mu_1} \circ \Phi\right)\left(\frac{d\mu_1 \circ \Phi}{d\mu_2} \circ \Phi^{-1} \circ \psi_c^{(1)} \circ \Phi\right)^{-1} = \frac{d\mu_2 \circ \psi_c^{(2)}}{d\mu_2} \quad \text{a.e. } d\mu_2,$$

which follows by using (4.39)). Then, in view of relations (4.39)–(4.41) and Definition 4.2, we have $(\psi^{(1)}, b^{(1)}, g^{(1)}) \sim (\psi^{(2)}, b^{(2)}, g^{(2)})$. $\Box$

5 Decomposition of the process in two components

Theorem 4.3 and part (*iii*) of Definition 4.2 show that the flows corresponding to two different minimal representations of a self-similar mixed moving average are "a.e. isomorphic". If two flows are a.e. isomorphic and one of them is dissipative (conservative, resp.), then so is the other one. (This fact can be proved by using the representations (3.22) and (3.23) for the dissipative and the conservative parts of a flow.) Based on this observation, one may then classify self-similar mixed moving averages into those whose minimal representations are associated with a dissipative flow and into those whose minimal representations are associated with a conservative flow. In practice, however, it is not easy to say when a representation is minimal and therefore to determine whether a process is generated by a dissipative or a conservative flow. Since minimal representation kernels are of the form (4.3), it is best to use (not necessarily minimal) kernels of that form (4.3) as a starting point and derive the properties of the corresponding processes. In fact, many commonly used kernels (e.g. the kernel of linear fractional stable motion in Example 6.2 below) are already of the form (4.3).

Let then $\alpha \in (0,2)$, $H > 0$ and $\kappa = H - 1/\alpha$. Let also $(X, \mathcal{X}, \mu)$ denote as before a standard Lebesgue space and M_α be a $S\alpha S$ random measure on $X \times \mathbb{R}$ with the control measure $\mu(dx)du$.

Definition 5.1 Let $\alpha \in (0,2)$. A $S\alpha S$ H-ss process X_α with stationary increments, given by the spectral representation

$$\{X_\alpha(t)\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{\int_X \int_{\mathbb{R}} G_t(x,u) M_\alpha(dx,du)\right\}_{t\in\mathbb{R}} \tag{5.1}$$

$$= \left\{\int_X \int_{\mathbb{R}} (G(x,t+u) - G(x,u)) M_\alpha(dx,du)\right\}_{t\in\mathbb{R}}, \tag{5.2}$$

is *generated by a nonsingular measurable multiplicative flow* $\{\psi_c\}_{c>0}$ *on* (X,μ) *associated with* G (or simply generated by a flow $\{\psi_c\}_{c>0}$) if, for all $t \in \mathbb{R}$ and $c > 0$,

$$c^{-\kappa}\left(G(x, c(t+u)) - G(x, cu)\right)$$

$$= b_c(x) \left\{\frac{d(\mu\circ\psi_c)}{d\mu}(x)\right\}^{1/\alpha} \left(G(\psi_c(x), t+u+g_c(x)) - G(\psi_c(x), u+g_c(x))\right) \tag{5.3}$$

a.e. $\mu(dx)du$, where $\{b_c\}_{c>0}$ is a cocycle for the flow $\{\psi_c\}_{c>0}$ taking values in $\{-1,1\}$, $\{g_c\}_{c>0}$ is a semi-additive functional for the flow $\{\psi_c\}_{c>0}$, and

$$\operatorname{supp}\{G_t, t\in\mathbb{R}\} = X \times \mathbb{R}, \tag{5.4}$$

a.e. $\mu(dx)du$.

Remarks

1. The flow is related to the kernel G and hence to the representation. Sometimes, for clarity, it may be necessary to say "generated by a flow $\{\psi_c\}_{c>0}$ associated with G".

2. When a self-similar mixed moving average X_α is generated (through its representation) by a flow $\psi = \{\psi_c\}_{c>0}$ and a related cocycle $b = \{b_c\}_{c>0}$ and a semi-additive functional $g = \{g_c\}_{c>0}$, we shall also say that X_α is generated by (or associated with) the triplet (ψ, b, g).

3. If the process X_α is given by (5.1)–(5.2), then $\mathrm{supp}\{G_t, t \in \mathbb{R}\} = X_0 \times \mathbb{R}$ a.e. for some $X_0 \in \mathcal{X}$ and, by replacing X by X_0 in (5.1) and (5.2), we may suppose without loss of generality that (5.4) holds (see Lemma 4.2). In this case, to verify that the process X_α is generated by a flow, one needs to check condition (5.3) only.

Proposition 5.1 *Condition* (5.3) *is equivalent to the condition that, for all* $c > 0$,

$$c^{-\kappa} G(x, cu) = b_c(x) \left\{ \frac{d(\mu \circ \psi_c)}{d\mu}(x) \right\}^{1/\alpha} G(\psi_c(x), u + g_c(x)) + J(x, c) \tag{5.5}$$

a.e. $\mu(dx)du$, *for some measurable function* J.

Proof: It is obvious that (5.5) implies (5.3). For the converse implication, note that, by Lemma 3.1, (i), (5.3) holds also a.e. $\mu(dx)dtdu$. But, by making the change of variables $v = t + u$, we get that

$$c^{-\kappa} G(x, cu) = b_c(x) \left\{ \frac{d(\mu \circ \psi_c)}{d\mu}(x) \right\}^{1/\alpha} G(\psi_c(x), u + g_c(x)) + J(x, c, v)$$

a.e. $\mu(dx)dvdu$. By fixing $v = v_0$, for which this equation holds a.e. $\mu(dx)du$ we obtain (5.5). $\Box$

If the spectral representation $\{G_t\}_{t \in \mathbb{R}}$ of X_α is minimal (as in Example 4.1), then X_α is always generated by a flow in the sense of Definition 5.1. But since we require in (5.1) only equality in the sense of the finite-dimensional distributions, it is not necessary that the spectral representation $\{G_t\}_{t \in \mathbb{R}}$ be minimal.

Theorem 5.1 *For* $\alpha \in (1, 2)$, *any* $S\alpha S$ H*-ss process* X_α *with stationary increments, having the representation* (5.1)–(5.2), *is generated by a multiplicative flow in the sense of Definition 5.1 (with possibly a new kernel* $\widetilde{G}$*). Conversely, if the process* X_α *has the representation* (5.1)–(5.2) *with the kernel function* G *satisfying* (5.3), *then it is self-similar with exponent* H *and has stationary increments.*

Proof: The first part of the theorem follows from Theorems 4.2 and 4.1. To show the second part of the theorem, use the computations in (2.4). $\Box$

The next result shows that there is a map relating the various spectral representations.

Theorem 5.2 *Let* $\alpha \in (0, 2)$ *and consider a* $S\alpha S$ H*-ss process* $\{X_\alpha(t)\}_{t \in \mathbb{R}}$, *given by* (5.1)–(5.2), *with* $\mathrm{supp}\{G_t, t \in \mathbb{R}\} = X \times \mathbb{R}$ *a.e.* $\mu(dx)du$. *Suppose that* $\{X_\alpha(t)\}_{t \in \mathbb{R}}$ *has another spectral representation*

$$\{X_\alpha(t)\}_{t \in \mathbb{R}} \stackrel{d}{=} \left\{ \int_{\widetilde{X}} \int_{\mathbb{R}} \widetilde{G}_t(\widetilde{x}, u) \widetilde{M}_\alpha(d\widetilde{x}, du) \right\}_{t \in \mathbb{R}} = \left\{ \int_{\widetilde{X}} \int_{\mathbb{R}} (\widetilde{G}(\widetilde{x}, t + u) - \widetilde{G}(\widetilde{x}, u)) \widetilde{M}_\alpha(d\widetilde{x}, du) \right\}_{t \in \mathbb{R}},$$

where $(\widetilde{X}, \widetilde{\mathcal{X}}, \widetilde{\mu})$ *is also a standard Lebesgue space,* $\{\widetilde{G}_t\}_{t \in \mathbb{R}} \subset L^\alpha(\widetilde{X} \times \mathbb{R}, \widetilde{\mu}(d\widetilde{x})du)$, *a* $S\alpha S$ *random measure* $\widetilde{M}_\alpha$ *has the control measure* $\mu(d\widetilde{x})du$. *Then there exist measurable functions* $\Phi_1 : X \mapsto \widetilde{X}$, $h : X \mapsto \mathbb{R} \setminus \{0\}$ *and* $\Phi_2, \Phi_3 : X \mapsto \mathbb{R}$ *such that*

$$G(x, u) = h(x) \widetilde{G}(\Phi_1(x), u + \Phi_2(x)) + \Phi_3(x) \tag{5.6}$$

a.e. $\mu(dx)du$.

Proof: By applying Lemma 4.1, (i), we obtain that there exist functions $\widetilde{\Phi}_1 : X \times \mathbb{R} \mapsto \widetilde{X}$, $\widetilde{h} : X \times \mathbb{R} \mapsto \mathbb{R} \setminus \{0\}$ and $\widetilde{\Phi}_2 : X \times \mathbb{R} \mapsto \mathbb{R}$ such that

$$G(x, t+u) - G(x,u) = \widetilde{h}(x,u)\left(\widetilde{G}(\widetilde{\Phi}_1(x,u), t + \widetilde{\Phi}_2(x,u)) - \widetilde{G}(\widetilde{\Phi}_1(x,u), \widetilde{\Phi}_2(x,u))\right) \tag{5.7}$$

a.e. $\mu(dx)dtdu$. Now, by making the change of variables $v = t+u$ in (5.7), we get that

$$G(x,v) = \widetilde{h}(x,u)\widetilde{G}(\widetilde{\Phi}_1(x,u), v - u + \widetilde{\Phi}_2(x,u)) + \widetilde{\Phi}_3(x,u) \tag{5.8}$$

a.e. $\mu(x)dvdu$, for some measurable function $\widetilde{\Phi}_3$. By Lemma 3.1, (ii), fix $u = u_0$, for which (5.8) holds a.e. $\mu(dx)dv$ (and for which the functions involved are measurable) to obtain the result of the theorem after the proper change of notation. □

Corollary 5.1 *Assume in Theorem 5.2 that the spectral representation* $\{\widetilde{G}_t\}_{t\in\mathbb{R}} \subset L^\alpha(\widetilde{X}\times\mathbb{R}, \widetilde{\mu}(d\widetilde{x})du)$ *is minimal and* $\mathrm{supp}\{G_t, t\in\mathbb{R}\} = X \times \mathbb{R}$. *Then there are measurable functions* $\Phi_1 : X \mapsto \widetilde{X}$, $h : X \mapsto \mathbb{R}\setminus\{0\}$ *and* $\Phi_2, \Phi_3 : X \mapsto \mathbb{R}$ *such that* (5.6) *holds a.e.* $\mu(dx)du$, *with* $\widetilde{\mu} = \mu_h \circ \Phi_1^{-1}$ *on* $\widetilde{\mathcal{X}}$, *where* $\mu_h(dx) = |h(x)|^\alpha \mu(dx)$

Proof: By Lemma 4.1, (i), above and Remark 2.5 in Rosiński (1995), there exist unique modulo $\mu(dx)du$ functions $\widetilde{\Phi}_1 : X \times \mathbb{R} \mapsto \widetilde{X}$, $\widetilde{h} : X \times \mathbb{R} \mapsto \mathbb{R}\setminus\{0\}$ and $\widetilde{\Phi}_2 : X\times\mathbb{R} \mapsto \mathbb{R}$ such that the relation (5.7) holds and $\widetilde{\mu}(d\widetilde{x})du = (\mu_{\widetilde{h}} \circ \widetilde{\Phi}^{-1})(d\widetilde{x}, du)$, where $\widetilde{\Phi} = (\widetilde{\Phi}_1, \widetilde{\Phi}_2)$ and $\mu_{\widetilde{h}}(dx, du) = |\widetilde{h}(x,u)|^\alpha \mu(dx)du$. Arguing as in Step 1 of the proof of Theorem 4.1, one gets that $\widetilde{\Phi}_1(x,u) = \Phi_1(x)$, $\widetilde{\Phi}_2(x,u) = u + \Phi_2(x)$ and $\widetilde{h}(x,u) = h(x)$ a.e. $\mu(dx)du$. Then $(\mu_{\widetilde{h}} \circ \widetilde{\Phi}^{-1})(d\widetilde{x}, du) = (\mu_h \circ \Phi_1^{-1})(d\widetilde{x})du$, where $\mu_h(dx) = |h(x)|^\alpha\mu(dx)$, and the relation (5.6) follows from (5.7) as in the proof of Theorem 5.2. □

The following result shows that the dissipative or conservative character of a flow is an invariant. It is analogous to Theorem 4.1 of Rosiński (1995) valid for stationary processes.

Theorem 5.3 *If the process* X_α, $\alpha \in (0,2)$, *given by* (5.1)–(5.4), *is generated by a dissipative(conservative, resp.) flow, then in any other representation* (5.1)–(5.4) *of* X_α, *the multiplicative flow must be dissipative (conservative, resp.).*

Proof: Suppose that the process X_α with the spectral representation $\{G_t\}_{t\in\mathbb{R}}$ is generated by a multiplicative flow $\{\psi_c\}_{c>0}$ as in Definition 5.1. Let $X = D \cup C$ be the Hopf decomposition of the flow $\{\psi_c\}_{c>0}$ (C and D denote the conservative and dissipative parts of the flow, respectively). Set also $F_t(x) = \int_{\mathbb{R}} |G_t(x,u)|^\alpha du$ and note that $F_t \in L^1(X,\mu)$. We will show that

$$\begin{aligned} D &= \left\{x \in X : \int_0^\infty \int_{\mathbb{R}} |G(\psi_c(x), 1+u) - G(\psi_c(x), u)|^\alpha du\ \lambda_c(x)\ c^{-1}dc < \infty\right\} && (5.9)\\ &= \left\{x \in X : \int_0^\infty F_1(\psi_c(x))\ \lambda_c(x)\ c^{-1}dc < \infty\right\}, && (5.10)\end{aligned}$$

and

$$\begin{aligned} C &= \left\{x \in X : \int_0^\infty \int_{\mathbb{R}} |G(\psi_c(x), 1+u) - G(\psi_c(x), u)|^\alpha du\ \lambda_c(x)\ c^{-1}dc = \infty\right\} && (5.11)\\ &= \left\{x \in X : \int_0^\infty F_1(\psi_c(x))\ \lambda_c(x)\ c^{-1}dc = \infty\right\} && (5.12)\end{aligned}$$

and a.e. $\mu(dx)$, where $\lambda_c = d(\mu \circ \psi_c)/d\mu$. Let D_0 and C_0 denote the right-hand side of (5.9) and (5.11), respectively. We want to show that $D = D_0$ and $C = C_0$ a.e. $\mu(dx)$. As in the proof of Theorem 4.1 in Rosiński (1995), by making the change of variables $c = e^v$, one can show that $C \cap \mathrm{supp}\{F_1\} \subset C_0$ and $D \subset D_0$ a.e. $\mu(dx)$. Since, for $t > 0$, $G(\psi_c(x), t+u) = G(\psi_c(x), t(1+t^{-1}u))$, we can apply (5.3) and get

$$\int_0^\infty F_t(\psi_c(x))\ \lambda_c(x)\ c^{-1}dc = \int_0^\infty \int_{\mathbb{R}} |G(\psi_c(x), t+u) - G(\psi_c(x), u)|^\alpha du\ \lambda_c(x)\ c^{-1}dc$$

$$= \int_0^\infty \int_{\mathbb{R}} |G(\psi_t(\psi_c(x)), 1+t^{-1}u+g_t(\psi_c(x))) - G(\psi_t(\psi_c(x)), t^{-1}u+g_t(\psi_c(x)))|^\alpha du\ \lambda_t(\psi_c(x))\lambda_c(x)\ c^{-1}dc$$

$$= \int_0^\infty \int_{\mathbb{R}} |G(\psi_{ct}(x), 1+u) - G(\psi_{ct}(x), u)|^\alpha du\ t\lambda_{ct}(x)\ c^{-1}dc$$

$$= t\int_0^\infty \int_{\mathbb{R}} |G(\psi_c(x), 1+u) - G(\psi_c(x), u)|^\alpha du\ \lambda_c(x)\ c^{-1}dc.$$

We can therefore replace F_1 by F_t, $t > 0$, in (5.12) and (5.10). This is also true for $t < 0$, since, by making the change of variables $t + u = v$ below,

$$\int_0^\infty \int_{\mathbb{R}} |G(\psi_c(x), t+u) - G(\psi_c(x), u)|^\alpha du\ \lambda_c(x)\ c^{-1}dc$$

$$= \int_0^\infty \int_{\mathbb{R}} |G(\psi_c(x), -t+v) - G(\psi_c(x), v)|^\alpha dv\ \lambda_c(x)\ c^{-1}dc.$$

Hence, we get that $C \cap \mathrm{supp}\{F_t\} \subset C_0$ for all $t \in \mathbb{R}$. Since condition (5.4) implies that $\mathrm{supp}\{F_t, t \in \mathbb{R}\} = X$ a.e. $\mu(dx)$, it follows that $C \subset C_0$ a.e. $\mu(dx)$. Since X is a disjoint union of D and C and since D_0 and C_0 are disjoint, we get $D = D_0$ and $C = C_0$ a.e. $\mu(dx)$.

We will now show that the dissipative and conservative character of a flow is an invariant. By Theorem 4.2, the process X_α has also a minimal spectral representation $\{\widetilde{G}_t\}_{t\in\mathbb{R}}$ on the space $(\widetilde{X} \times \mathbb{R}, \widetilde{\mu}(d\widetilde{x})du)$ and, by Theorem 4.1, it is generated by a multiplicative flow $\{\widetilde{\psi}_c\}_{c>0}$ on $(\widetilde{X}, \widetilde{\mathcal{X}})$ associated with the kernel $\widetilde{G}$. By Corollary 5.1, there are measurable functions $\Phi_1 : X \mapsto \widetilde{X}$, $h : X \mapsto \mathbb{R} \setminus \{0\}$ and $\Phi_2, \Phi_3 : X \mapsto \mathbb{R}$ such that

$$G(x, u) = h(x)\widetilde{G}(\Phi_1(x), u + \Phi_2(x)) + \Phi_3(x) \tag{5.13}$$

a.e. $\mu(dx)du$ and

$$\widetilde{\mu} = \mu_h \circ \Phi_1^{-1}, \tag{5.14}$$

where $\mu_h(dx) = |h(x)|^\alpha \mu(dx)$. By using (5.3) and the relation $-H\alpha = -\kappa\alpha - 1$, we have a.e. $\mu(dx)$,

$$\int_0^\infty \int_{\mathbb{R}} |G(\psi_c(x), 1+u) - G(\psi_c(x), u)|^\alpha du\ \frac{d(\mu \circ \psi_c)}{d\mu}(x)\ c^{-1}dc$$

$$= \int_0^\infty \int_{\mathbb{R}} |G(x, c(1+u)) - G(x, cu)|^\alpha du\ c^{-H\alpha}dc \tag{5.15}$$

$$= |h(x)|^\alpha \int_0^\infty \int_{\mathbb{R}} |\widetilde{G}(\Phi_1(x), c(1+u)) - \widetilde{G}(\Phi_1(x), cu)|^\alpha du\ c^{-H\alpha}dc$$

$$= |h(x)|^\alpha \int_0^\infty \int_{\mathbb{R}} |\widetilde{G}(\widetilde{\psi}_c(\Phi_1(x)), 1+u) - \widetilde{G}(\widetilde{\psi}_c(\Phi_1(x)), u)|^\alpha du \; \frac{d(\widetilde{\mu} \circ \widetilde{\psi}_c)}{d\widetilde{\mu}}(\Phi_1(x)) \;\; c^{-1} dc. \tag{5.16}$$

(Observe that the last equality in (5.16) holds a.e. $\mu(dx)$, since it holds a.e. $\widetilde{\mu}(d\widetilde{x})$ with $\widetilde{x} = \Phi_1(x)$ and since, by (5.14), $\widetilde{\mu}(\widetilde{N}) = 0$ for any $\widetilde{N} \in \widetilde{\mathcal{X}}$ implies that $\mu(\Phi_1^{-1}(\widetilde{N})) = 0$.) Relation (5.16) implies that

$$\Phi_1^{-1}(\widetilde{D}_0) = D_0 \quad \text{and} \quad \Phi_1^{-1}(\widetilde{C}_0) = C_0 \tag{5.17}$$

a.e. $\mu(dx)$, where D_0 and C_0 are the sets on the right-hand side of (5.9) and (5.11), respectively, and $\widetilde{D}_0$ and $\widetilde{C}_0$ are defined in the same way by replacing X, G, ψ_c, λ_c by $\widetilde{X}$, $\widetilde{G}$, $\widetilde{\psi}_c$, $\widetilde{\lambda}_c$. Hence, by (5.17), the flow $\{\psi_c\}_{c>0}$ is dissipative (conservative, resp.) if and only if the flow $\{\widetilde{\psi}_c\}_{c>0}$ is dissipative (conservative, resp.). Indeed, for example, if the flow $\{\psi_c\}_{c>0}$ is dissipative, then $\mu(C) = 0$ and hence $\mu(C_0) = 0$, which implies by (5.17) that $\mu(\Phi_1^{-1}(\widetilde{C}_0)) = 0$ and, by (5.14), that $\widetilde{\mu}(\widetilde{C}_0) = 0$ or $\widetilde{\mu}(\widetilde{C}) = 0$. Now, if the process X_α is generated by yet another flow $\{\widehat{\psi}_c\}_{c>0}$ associated with a kernel $\widehat{G}$, then we conclude as above that the flow $\{\widehat{\psi}_c\}_{c>0}$ is dissipative (conservative, resp.) if and only if the flow $\{\widetilde{\psi}_c\}_{c>0}$ is dissipative (conservative, resp.), and consequently, if and only if the flow $\{\psi_c\}_{c>0}$ is dissipative (conservative, resp.). This concludes the proof. □

Remarks

1. To get a feeling for (5.11), note that a conservative flow $\psi_c(x)$ comes back again and again to the same values as c grows. Hence, the integral of a positive function of $\psi_c(x)$ over $(0, \infty)$, which is the range of c, should diverge.

2. Theorem 5.3 implies that, if the process X_α is generated by a dissipative flow and the process Y_α is generated by a conservative flow, then the processes X_α and Y_α have different finite-dimensional distributions.

3. By changing the variable $c = e^t$, $t \in \mathbb{R}$, in the sets D_0 and C_0 on the right-hand sides of (5.9) and (5.11), respectively, and also by denoting $\phi_t = \psi_{e^t}$, $t \in \mathbb{R}$, the representations (5.9) and (5.11) look like (3.22) and (3.23), respectively, where $g = F_1$. (These changes of variable transform the multiplicative flow ψ_c into the additive flow ϕ_t.) The function F_1, however, need not have full support in X and this is why it does not, in general, correspond to the g in (3.22) and (3.23). We were nevertheless able to conclude that $D = D_0$ and $C = C_0$ a.e. by using special properties of G. Hence, in particular, (3.22) and (3.23) may hold with a g which does not have full support in X.

The next result provides a criterion for determining whether a flow is dissipative or conservative.

Theorem 5.4 *Let $\alpha \in (1, 2)$ and X_α be a $S\alpha S$ H-ss process with stationary increments given by (5.1)–(5.2). Suppose that* $\text{supp}\{G_t, t \in \mathbb{R}\} = X \times \mathbb{R}$. *Then the process X_α is generated by a dissipative (conservative, resp.) flow in the sense of Definition 5.1 (with possibly a new kernel $\widetilde{G}$) if and only if the integral*

$$I(x) = \int_0^\infty c^{-H\alpha} \int_{\mathbb{R}} |G(x, c(1+u)) - G(x, cu)|^\alpha \, du \; dc \tag{5.18}$$

is finite (infinite, resp.) a.e. $\mu(dx)$.

Proof: By Theorem 4.2, the process X_α has a minimal spectral representation with the kernel function $\widetilde{G}$ and, by Theorem 4.1, there is a flow $\{\widetilde{\psi}_c\}_{c>0}$ associated with $\widetilde{G}$ which generates X_α in the sense of Definition 5.1. By Theorem 5.3, X_α is generated by a dissipative (conservative, resp.) flow if and only if the flow $\{\widetilde{\psi}_c\}_{c>0}$ is dissipative (conservative, resp.) and, by Corollary 5.1, the a.e finiteness of the integral (5.18) is equivalent to the a.e. finiteness of a similar integral where G is replaced by $\widetilde{G}$. By applying (5.3) with $\widetilde{G}$, we see as in (5.15) that the integral (5.18) with $\widetilde{G}$ equals to the integral

$$\int_0^\infty \int_{\mathbb{R}} |\widetilde{G}(\widetilde{\psi}_c(\widetilde{x}), 1+u) - \widetilde{G}(\widetilde{\psi}_c(\widetilde{x}), u)|^\alpha du \ \frac{d(\widetilde{\mu}\circ\widetilde{\psi}_c)}{d\widetilde{\mu}}(\widetilde{x}) \ c^{-1}dc.$$

The conclusion then follows from (5.9) ((5.11), resp.) with G, ψ_c and D (C, resp.) replaced by $\widetilde{G}$, $\widetilde{\psi}_c$ and $\widetilde{D}$ ($\widetilde{C}$, resp.). $\square$

Theorem 5.5 *Let $\alpha \in (1,2)$ and suppose that a process X_α is generated by a nonsingular multiplicative flow $\{\psi_c\}_{c>0}$ as in Definition 5.1. Let also $X = D \cup C$ be the Hopf decomposition of the flow $\{\psi_c\}_{c>0}$. Then, we have*

$$X_\alpha \stackrel{d}{=} X_\alpha^D + X_\alpha^C, \tag{5.19}$$

where

$$X_\alpha^D(t) = \int_D \int_{\mathbb{R}} G_t(x,u) M_\alpha(dx,du), \tag{5.20}$$

$$X_\alpha^C(t) = \int_C \int_{\mathbb{R}} G_t(x,u) M_\alpha(dx,du). \tag{5.21}$$

The processes X_α^D and X_α^C are independent, and are both H-ss and have stationary increments. The process X_α^D is generated by a dissipative flow and the process X_α^C is generated by a conservative flow in the sense of Definition 5.1. The decomposition (5.19)*, moreover, is unique in distribution, that is, it does not depend on the representation $\{G_t\}_{t\in\mathbb{R}}$ in Definition 5.1.*

Proof: The processes X_α^D and X_α^C are independent because their kernels have disjoint support (Theorem 3.5.3 in Samorodnitsky and Taqqu (1994)). The process X_α^D is generated by a dissipative flow and the process X_α^C is generated by a conservative flow in the sense of Definition 5.1 because D and C are invariant under the flow.

To prove the uniqueness in distribution, let $\{\widetilde{G}_t\}_{t\in\mathbb{R}} \subset L^\alpha(\widetilde{X}\times\mathbb{R}, \widetilde{\mu}(dx)du)$ be the minimal spectral representation of the process X_α obtained in Theorem 4.2. Suppose that this representation is generated by a multiplicative flow $\{\widetilde{\psi}_c\}_{c>0}$ on $(\widetilde{X}, \widetilde{\mu})$ as in Theorem 4.1. Let $\widetilde{D}$ and $\widetilde{C}$ be the dissipative part and the conservative parts of the flow $\{\widetilde{\psi}_c\}_{c>0}$, respectively. The kernels G and $\widetilde{G}$ can be related as in (5.13) and (5.14). Moreover, by (5.17), $\Phi_1^{-1}(\widetilde{D}_0) = D_0$ and $\Phi_1^{-1}(\widetilde{C}_0) = C_0$, where the sets D_0, C_0, $\widetilde{D}_0$ and $\widetilde{C}_0$ are defined in the proof of Theorem 5.3. Then, since $C = C_0$ μ-a.e. and $\widetilde{C} = \widetilde{C}_0$ $\widetilde{\mu}$-a.e., we have for every $a_1, \cdots, a_n \in \mathbb{R}$ and $t_1, \cdots, t_n \in \mathbb{R}$, $n \geq 1$,

$$\begin{aligned}
\int_C \int_{\mathbb{R}} \left| \sum_{k=1}^n a_k G_{t_k}(x,u) \right|^\alpha \mu(dx)du &= \int_{C_0} \int_{\mathbb{R}} \left| \sum_{k=1}^n a_k G_{t_k}(x,u) \right|^\alpha \mu(dx)du \\
&= \int_{C_0} \int_{\mathbb{R}} \left| \sum_{k=1}^n a_k \widetilde{G}_{t_k}(\Phi_1(x), u+\Phi_2(x)) \right|^\alpha |h(x)|^\alpha \mu(dx)du
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\Phi_1^{-1}(\widetilde{C}_0)} \int_{\mathbb{R}} \left| \sum_{k=1}^{n} a_k \widetilde{G}_{t_k}(\Phi_1(x), u) \right|^{\alpha} |h(x)|^{\alpha} \mu(dx) du \\
&= \int_{\widetilde{C}_0} \int_{\mathbb{R}} \left| \sum_{k=1}^{n} a_k \widetilde{G}_{t_k}(\widetilde{x}, u) \right|^{\alpha} \widetilde{\mu}(d\widetilde{x}) du \\
&= \int_{\widetilde{C}} \int_{\mathbb{R}} \left| \sum_{k=1}^{n} a_k \widetilde{G}_{t_k}(\widetilde{x}, u) \right|^{\alpha} \widetilde{\mu}(d\widetilde{x}) du. \qquad (5.22)
\end{aligned}
$$

This implies that $X_\alpha^C \stackrel{d}{=} X_\alpha^{\widetilde{C}}$, where $X_\alpha^{\widetilde{C}}$ is defined analogously to X_α^C. Similarly, $X_\alpha^D \stackrel{d}{=} X_\alpha^{\widetilde{D}}$. It follows that the decomposition (5.19) does not depend on the representation $\{G_t\}_{t\in\mathbb{R}}$. $\square$

Corollary 5.2 *When $\alpha \in (1,2)$, every H-ss process X_α having the representation* (5.1)–(5.2) *can be uniquely decomposed into two independent processes: one generated by a dissipative flow and the other generated by a conservative flow.*

PROOF: By Theorem 5.1, when $\alpha \in (1,2)$, every H-ss process X_α having the representation (5.1)–(5.2) is generated by a multiplicative flow in the sense of Definition 5.1 (with possibly a new kernel $\widetilde{G}$). Then apply Theorem 5.5. $\square$

We shall call the two processes obtained in Corollary 5.2 the *dissipative and conservative components* of the process X_α. Observe that they are defined in distribution. An alternative way to obtain the decomposition of the process X_α into its dissipative and conservative components is as follows.

Corollary 5.3 *Let $\alpha \in (1,2)$ and suppose that the H-ss process X_α has the representation* (5.1)–(5.2) *with* $\mathrm{supp}\{G_t, t\in\mathbb{R}\} = X \times \mathbb{R}$. *Define the sets*

$$
D = \{x \in X : I(x) < \infty\} \quad \textit{and} \quad C = \{x \in X : I(x) = \infty\}, \qquad (5.23)
$$

where I is the integral defined in (5.18), *and define the process X_α^D and X_α^C as in* (5.20) *and* (5.21) *but using the sets in* (5.23). *Then the processes X_α^D and X_α^C are (in distribution) the dissipative and conservative components of the process X_α.*

PROOF: It is enough to show that

$$
X_\alpha^D =_d X_\alpha^{\widetilde{D}} \quad \text{and} \quad X_\alpha^C =_d X_\alpha^{\widetilde{C}}, \qquad (5.24)
$$

where the processes $X_\alpha^{\widetilde{D}}$ and $X_\alpha^{\widetilde{C}}$ are the dissipative and conservative components of X_α defined in the proof of the uniqueness in Theorem 5.5. That proof also shows that (5.24) holds as long as

$$
D = \Phi_1^{-1}(\widetilde{D}_0) \quad \text{and} \quad C = \Phi_1^{-1}(\widetilde{C}_0) \qquad (5.25)
$$

μ-a.e. (compare with (5.17)). These last relations hold, since by applying (5.13) and (5.3) with $\widetilde{G}$, we get as in (5.15)–(5.16) that a.e. $\mu(dx)$,

$$
\begin{aligned}
I(x) &= \int_0^\infty c^{-H\alpha} \int_{\mathbb{R}} |G(x, c(1+u)) - G(x, cu)|^\alpha \, du \, dc \\
&= |h(x)|^\alpha \int_0^\infty c^{-H\alpha} \int_{\mathbb{R}} \left| \widetilde{G}(\Phi_1(x), c(1+u)) - \widetilde{G}(\Phi_1(x), cu) \right|^\alpha du \, dc \\
&= |h(x)|^\alpha \int_0^\infty \int_{\mathbb{R}} |\widetilde{G}(\widetilde{\psi}_c(\Phi_1(x)), 1+u) - \widetilde{G}(\widetilde{\psi}_c(\Phi_1(x)), u)|^\alpha du \; \frac{d(\widetilde{\mu} \circ \widetilde{\psi}_c)}{d\widetilde{\mu}}(\Phi_1(x)) \;\; c^{-1} dc.
\end{aligned}
$$

$\square$

Remark. The sets D and C in (5.23) are not related to a flow. But as indicated in the proof of the corollary, the processes X_α^D and X_α^C have the same distributions as $X_\alpha^{\widetilde{D}}$ and $X_\alpha^{\widetilde{C}}$, where $\widetilde{D}$ and $\widetilde{C}$ are the dissipative and conservative parts of a flow. This flow, which can be (but is not necessarily) associated with the minimal spectral representation of the process X_α, may live on a space which is different from X.

Depending on the practical situation, one can either apply Theorem 5.5 or Corollary 5.3. Corollary 5.3 is handy because it requires only that the process X_α is self-similar and has a mixed moving average representation (5.1)–(5.2). If the representation satisfies the conditions (5.1)–(5.4), then one can apply Theorem 5.5 and determine not only the character of the process but also the nature of the flow associated with the representation. Both approaches are illustrated in the following section.

6 Examples

In this section we provide some examples.

Example 6.1 For $\alpha \in (1,2)$ and $H \in (1/\alpha, 1)$, consider the process

$$X_\alpha(t) = \int_0^\infty \int_{\mathbb{R}} (((t+u)\wedge 0 + x)_+ - (u \wedge 0 + x)_+)\ x^{H-\frac{2}{\alpha}-1} M_\alpha(dx,du), \tag{6.1}$$

where M_α is a $S\alpha S$ random measure on $(0,\infty)\times\mathbb{R}$ with the Lebesgue control measure. The process X_α appears in the so-called renewal-reward problem as the limit of a properly normalized superposition of renewal reward processes (see Levy and Taqqu (2000) and Pipiras and Taqqu (2000)). It is an H-ss, $S\alpha S$ process with stationary increments which has the representation (1.7) with $X = (0,\infty)$, $\mu = \mathbb{L}$ (the Lebesgue measure) and

$$G(x,u) = (u \wedge 0 + x)_+\ x^{H-\frac{2}{\alpha}-1}$$

for $x > 0$, $u \in \mathbb{R}$. Let us show that the process X_α is generated by a dissipative flow in the sense of Definition 5.1. For $c > 0$ and $\kappa = H - 1/\alpha$, we have that

$$c^{-\kappa} G(x,cu) = c^{-H+\frac{1}{\alpha}} (cu \wedge 0 + x)_+ x^{H-\frac{2}{\alpha}-1} = c^{-H+\frac{1}{\alpha}} c c^{H-\frac{2}{\alpha}-1} (u \wedge 0 + c^{-1}x)_+ (c^{-1}x)^{H-\frac{2}{\alpha}-1}$$

$$= c^{-\frac{1}{\alpha}} G(c^{-1}x, u) = \left\{ \frac{d(\mathbb{L}\circ\psi_c)}{d\mathbb{L}}(x) \right\}^{1/\alpha} G(\psi_c(x), u),$$

where

$$\psi_c(x) = c^{-1}x, \quad c > 0, x > 0.$$

Hence the condition (5.3) is satisfied (see also (5.5)) with

$$b_c(x) \equiv 1 \quad \text{and} \quad g_c(x) \equiv 0.$$

It is clear that $\{\psi_c\}_{c>0}$ is a nonsingular measurable multiplicative flow on $(0,\infty)$ and that $\{b_c\}_{c>0}$ is a cocycle and $\{g_c\}_{c>0}$ is a semi-additive functional for the flow $\{\psi_c\}_{c>0}$. Set $G_t(x,u) = G(x,t+u) - G(x,u)$. Since $\mathrm{supp}\{G_t\} = \{(x,u) : x > 0, -t-x < u < -t\}$ for $t < 0$ and $\mathrm{supp}\{G_t\} = \{(x,u) : x > 0, -t-x < u < 0\}$ for $t > 0$, one has $\mathrm{supp}\{G_t, t \in \mathbb{R}\} = (0,\infty)\times\mathbb{R}$. Therefore, the process

X_α is generated by the flow $\{\psi_c\}_{c>0}$ in the sense of Definition 5.1. Moreover, this flow $\{\psi_c\}_{c>0}$ is dissipative, since, if $V(x) = \psi_c(x) = c^{-1}x$ for $x > 0$, $c \neq 1$, then $(0,\infty) = \cup_{k=-\infty}^{\infty} V^k B$ for a wandering set $B = (1, c]$ (see Section 3).

The fact that the process X_α is generated by a dissipative flow follows also from Corollary 5.3 since, by making the change of variables $c = x/v$ below (for fixed x),

$$\int_0^\infty c^{-\alpha H} \int_{\mathbb{R}} |G(x, c(1+u)) - G(x, cu)|^\alpha du \; dc$$

$$= x^{\alpha H - 2 - \alpha} \int_0^\infty c^{-\alpha H} \int_{\mathbb{R}} |(c(1+u) \wedge 0 + x)_+ - (cu \wedge 0 + x)_+|^\alpha du \; dc$$

$$= x^{-1} \int_0^\infty \int_{\mathbb{R}} |((1+u) \wedge 0 + v)_+ - (u \wedge 0 + v)_+|^\alpha v^{\alpha H - 2 - \alpha} du \; dv = x^{-1} \ln(-Ee^{iX_\alpha(1)}) < \infty,$$

for all $x > 0$.

Example 6.2 Let $a, b \in \mathbb{R}$, $H \in (0,1)$, $\alpha \in (0,2)$ and suppose that $\kappa = H - 1/\alpha \neq 0$. Define

$$X_\alpha(t) = \int_{\mathbb{R}} \left(a \left((t+u)_+^\kappa - u_+^\kappa \right) + b \left((t+u)_-^\kappa - u_-^\kappa \right) \right) M_\alpha(du), \; t \in \mathbb{R}, \tag{6.2}$$

where M_α is a $S\alpha S$ random measure with the control measure $m_\alpha(du) = du$. The processes X_α are called linear fractional stable motions (see Chapter 7 in Samorodnitsky and Taqqu (1994)). They are $H = \kappa + 1/\alpha$–ss, $S\alpha S$ processes with stationary increments and have the representation (1.7) with

$$X = \{1\}, \; \mu = \delta_{\{1\}} \quad \text{and} \quad G(1,u) = au_+^\kappa + bu_-^\kappa.$$

Since $c^{-\kappa} G(1, cu) = G(1,u)$ and $\mathrm{supp}\{G_t, t \in \mathbb{R}\} = \mathbb{R}$, linear fractional stable motions are processes generated by the following flow, related cocycle and semi-additive functional, respectively: for all $c > 0$,

$$\psi_c(1) = 1, \quad b_c(1) = 1, \quad g_c(1) = 0.$$

The flow $\{\psi_c\}_{c>0}$ is clearly conservative. These results also follow from Corollary 5.3, since

$$\int_0^\infty c^{-\alpha H} \int_{\mathbb{R}} |G(1, c(1+u)) - G(1, cu)|^\alpha du \; dc = \int_0^\infty c^{-1} dc \int_{\mathbb{R}} |G(1, 1+u) - G(1,u)|^\alpha du = \infty.$$

Example 6.3 For $\alpha \in (0,2)$, consider the $S\alpha S$ Lévy motion

$$X_\alpha(t) = \int_{\mathbb{R}} 1_{[0,t)}(u) M_\alpha(du) \overset{d}{=} \int_{\mathbb{R}} 1_{[0,t)}(-u) M_\alpha(du) = \int_{\mathbb{R}} \left(1_{(0,\infty)}(t+u) - 1_{(0,\infty)}(u) \right) M_\alpha(du), \; t \in \mathbb{R},$$

where M_α is again a $S\alpha S$ random measure with the control measure $m_\alpha(du) = du$ (see, for example, Samorodnitsky and Taqqu (1994)). The process X_α has stationary independent increments and is self-similar with exponent $H = 1/\alpha$. It has representation (1.7) with

$$X = \{1\}, \; \mu = \delta_{\{1\}} \quad \text{and} \quad G(1,u) = 1_{(0,\infty)}(u)$$

and, since $G(1, cu) = G(1,u)$ for $c > 0$, it is generated by the conservative flow $\psi_c(1) = 1$ and a related cocycle $b_c(1) = 1$ and a semi-additive functional $g_c(1) = 0$. This conclusion also follows from Corollary 5.3.

Example 6.4 Let $\alpha \in (1,2)$ and define

$$X_\alpha(t) = \int_{\mathbb{R}} (\ln|t+u| - \ln|u|)\, M_\alpha(du),\ t \in \mathbb{R},$$

where M_α is a $S\alpha S$ random measure with the control measure $m_\alpha(du) = du$. The process X_α is called log-fractional stable motion (see Chapter 7 in Samorodnitsky and Taqqu (1994)). Like the stable Lévy motion, it has stationary increments and is also self-similar with exponent $H = 1/\alpha$. However, since its increments are dependent, it has different finite-dimensional distributions than the stable Lévy motion. The log-fractional stable motion has representation (1.7) with

$$X = \{1\},\ \mu = \delta_{\{1\}} \quad \text{and} \quad G(1,u) = \ln|u|.$$

In this case $\kappa = H - 1/\alpha = 0$ and, for any $c > 0$, $G(1,cu) = G(1,u) + \ln c$. Then (5.5) is satisfied with

$$\psi_c(1) = 1,\ b_c(1) = 1,\ g_c(1) = 0 \quad \text{and} \quad J(1,c) = \ln c.$$

Therefore, the process X_α is generated by the conservative flow $\psi_c(1) = 1$. Alternatively, this conclusion also follows from Corollary 5.3 since $\int_0^\infty c^{-\alpha H} dc = \infty$. Observe that the log-fractional stable motion has the same triplet as the stable Lévy motion.

7 Conclusions

We focused, in this paper, on self-similar $S\alpha S$ (non-Gaussian) processes with stationary increments which have the mixed moving average representation (1.7), ss mma processes, in short. Examples of such processes include the well-known linear fractional stable, stable Lévy and log-fractional stable motions, as well as the limit of the so-called renewal reward processes. Since there are many *different* $S\alpha S$ ss mma processes, one would like to find characteristics that can be used to classify them and, in particular, to say when two such processes have different finite-dimensional distributions. We provided examples of such characteristics.

A $S\alpha S$ self-similar process can have many different mma representations. For example, the "mixing" space X in the representation (1.7) can always be mapped into some other space Y. The so-called minimal representations are of particular interest. By Theorem 4.1, a minimal representation can be associated with a triplet (ψ, b, g), where $\psi = \{\psi_c\}_{c>0}$ is a nonsingular multiplicative measurable flow, $b = \{b_c\}_{c>0}$ is a cocycle and $g = \{g_c\}_{c>0}$ is a semi-additive functional for the flow ψ. Moreover, by Theorem 4.3, triplets corresponding to different minimal representations of the same process are equivalent in the sense of Definition 4.2. In particular, the condition (iii) of that definition states that the flows are a.e. isomorphic. Since a dissipative (conservative, resp.) flow can be a.e. isomorphic only to a dissipative (conservative, resp.) flow, one can classify $S\alpha S$, self-similar mixed moving averages into two classes: those whose minimal representations are associated with dissipative flows and those whose minimal representation are associated with conservative flows.

Since it is not typically easy to say when a given representation is minimal, one cannot readily determine whether a process is associated with a dissipative or a conservative flow in the above sense. On the other hand, many well-known ss mma kernels (not necessarily minimal) may be associated with triplets (ψ, b, g) as it is the case for minimal kernels. The question then is whether these flows ψ share some properties with the class of a.e. isomorphic flows corresponding to minimal representations of the process. In particular, is the conservative or the dissipative character of the flow preserved? The answer is positive (Theorem 5.3). Thus, one may classify $S\alpha S$, self-similar mixed moving averages according to the nature of their flows, whether the representation of the process is minimal or not. When $\alpha \in (1, 2)$, any such process can be decomposed uniquely into two independent processes one of which is associated with a dissipative flow and the other one is associated with a conservative flow.

Let us emphasize that a triplet (ψ, b, g) associated with a ss mma process as in Definition 5.1 does not determine the process itself. In fact, the same triplet may correspond to different representations and even to different processes. For example, the stable Lévy motion in Example 6.3 and the log-fractional stable motion in Example 6.4 are both associated with an identity flow (and the corresponding cocycle and the semi-additive functional). Therefore, a triplet only captures a characteristic shared by a class of ss mma processes. A given process can also have many different representations each with its own triplet. Nevertheless, as noted above, the conservative or the dissipative character of the flow does not change from one representation to another. This is why one can classify ss mma processes into two different subclasses, namely, those associated with dissipative flows and those associated with conservative flows. A finer decomposition can be found in Pipiras and Taqqu (2001), which provides further insights into the structure of $S\alpha S$ self-similar mixed moving average processes.

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