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The asymptotic behavior of quadratic forms in heavy-tailed strongly dependent random variables *†‡

Piotr S. Kokoszka
University of Utah

Murad S. Taqqu
Boston University

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Abstract

Suppose that $X_t = \sum_{j=0}^{\infty} c_j Z_{t-j}$ is a stationary linear sequence with regularly varying c_j's and with innovations $\{Z_j\}$ that have infinite variance. Such a sequence can exhibit both high variability and strong dependence. The quadratic form $Q_n = \sum_{t,s=1}^{n} \widehat{\eta}(t-s)X_t X_s$ plays an important role in the estimation of the intensity of strong dependence. In contrast with the finite variance case, $n^{-1/2}(Q_n - EQ_n)$ does not converge to a Gaussian distribution. We provide conditions on the c_j's and on $\widehat{\eta}$ for the quadratic form Q_n, adequately normalized and randomly centered, to converge to a stable law of index α, $1 < \alpha < 2$, as n tends to infinity.

1 Outline of the main ideas

The main result of this paper is a limit theorem for randomly centered quadratic forms

$$Q_n = \sum_{t,s=1}^{n} \widehat{\eta}(t-s)X_t X_s, \tag{1.1}$$

where $\{X_t\}$ is a strongly dependent linear process whose innovations have a heavy-tailed distribution. In (1.1), η is an integrable function on $[-\pi, \pi]$ which will be specified later, and the $\widehat{\eta}(t) = (2\pi)^{-1}\int_{-\pi}^{\pi} e^{i\lambda t}\eta(\lambda)d\lambda$ are its Fourier coefficients. Our goal is to characterize the asymptotic behavior of Q_n, in situations where $\{X_t\}$ has infinite variance.

The quadratic forms (1.1) were studied by Fox and Taqqu [6] when $\{X_t\}$ is a Gaussian sequence with long-range (or strong) dependence. They assumed that $\{X_t\}$ is a mean zero Gaussian sequence with covariances

$$E[X_t X_s] = \sigma^2 \int_{-\pi}^{\pi} e^{i(t-s)\lambda} f(\lambda)d\lambda \tag{1.2}$$

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whose normalized spectral density f satisfies

$$f(\lambda) = O(|\lambda|^{-a-\delta}) \tag{1.3}$$

for some $a < 1$ and arbitrarily small δ. The function η (a multiple of g in the notation of Fox and Taqqu [6]) was assumed to satisfy

$$\eta(\lambda) = O(|\lambda|^{-b-\delta}), \tag{1.4}$$

for some $b < 1$ such that $a + b < \frac{1}{2}$. Both f and η were assumed to be symmetric and to satisfy a very mild regularity condition, which was later shown by Giraitis and Surgailis [7] to be unnecessary. Under the above assumptions, Fox and Taqqu showed that the sequence

$$n^{-1/2}[Q_n - EQ_n] \tag{1.5}$$

converges in distribution to the normal distribution with mean zero and variance

$$4\pi\sigma^4 \int_{-\pi}^{\pi} [f(\lambda)\eta(\lambda)]^2 d\lambda. \tag{1.6}$$

The condition $a + b < 1/2$ ensures that the integral in (1.6) is finite. Observe that the spectral density f diverges at the origin when $a > 0$ (strong dependence).

Giraitis and Surgailis [7] extended the result of Fox and Taqqu to linear sequences

$$X_t = \sum_{j=-\infty}^{\infty} c_j Z_{t-j}, \tag{1.7}$$

where the innovations Z_t have finite fourth moment, and presented a simpler proof. Under their weaker assumptions on Z_t, the asymptotic variance of the sequence (1.5) is

$$4\pi\sigma^4 \int_{-\pi}^{\pi} [f(\lambda)\eta(\lambda)]^2 d\lambda + (\mathrm{Var}(Z_0^2) - 2\sigma^4) \left(\int_{-\pi}^{\pi} f(\lambda)\eta(\lambda) d\lambda \right)^2, \tag{1.8}$$

which clearly reduces to (1.6) if the Z_t are Gaussian.

The main motivation for the above results comes from their applicability to the study of the large sample properties of the Whittle's estimator of parameters of linear processes with long memory (see Fox and Taqqu [5] and Giraitis and Surgailis [7]). Recently Mikosch et al. [12] and Kokoszka and Taqqu [11] studied the Whittle's estimator for linear processes (1.7) with innovations Z_t having *infinite variance*. The ideas and tools used in the infinite variance case are very different from those used under the Gaussian or more general finite variance assumptions. In this paper we show that these methods can also be applied to the study of the asymptotic behavior of general quadratic forms (1.1) with the X_t of the form (1.7) and infinite variance Z_t. For simplicity, we set $c_j = 0$ for $j < 0$, that is, we focus only on causal moving averages (1.7).

We adopt here the normalization conventions of Harmonic Analysis, namely, $\widehat{b}_t = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{i\lambda t} b(\lambda) d\lambda$, $b(\lambda) = \sum_t e^{-i\lambda t} b_t$, and $\sum_t |b_t|^2 = (2\pi)^{-1} \int_{-\pi}^{\pi} |b(\lambda)|^2 d\lambda$ for Parseval's inequality, but use the probabilistic one, that is (1.2), to relate the covariances to the spectral density $f(\lambda)$, when these quantities are defined.

We now outline our approach, relegating the precise assumptions to Section 2. We will consider the following two cases at the same time: *Case I (infinite variance):* The tail probabilities $P\{|Z_1| > x\}$ behave like $x^{-\alpha}$, $1 < \alpha < 2$, as $x \to \infty$, and Z_1 is symmetric. *Case II (finite variance):* $EZ_1^2 = \sigma^2 < \infty$ and $EZ_1 = 0$. Set

$$a_n = \begin{cases} (n \log n)^{1/\alpha}, & \text{in Case I,} \\ n^{1/2}, & \text{in Case II,} \end{cases} \tag{1.9}$$

and consider the *periodogram*

$$I_{n,X}(\lambda) = n^{-2/\alpha} \left| \sum_{t=1}^{n} e^{-i\lambda t} X_t \right|^2, \tag{1.10}$$

with the understanding that $\alpha = 2$ in Case II. The periodogram of the noise sequence, $I_{n,Z}(\lambda)$, is defined correspondingly.

The main idea is to use the decomposition

$$I_{n,X}(\lambda) = 2\pi f(\lambda) I_{n,Z}(\lambda) + R_n(\lambda), \tag{1.11}$$

where $f(\lambda) = (2\pi)^{-1} |\sum_{j=0}^{\infty} c_j e^{-i\lambda j}|^2$ (the factor $(2\pi)^{-1}$ makes this definition consistent with (1.2) in the case $\alpha = 2$). The remainder term $R_n(\lambda)$ is defined in Section 2. Observe that for both cases (for Case II, set $\alpha = 2$ and ignore $\log n$ wherever it appears).

$$\begin{aligned} a_n^{-1} Q_n &= a_n^{-1} \sum_{t,s=1}^{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(t-s)\lambda} \eta(\lambda) d\lambda \right) X_t X_s \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (n \log n)^{-1/\alpha} \left| \sum_{t=1}^{n} e^{-i\lambda t} X_t \right|^2 \eta(\lambda) d\lambda \\ &= \left(\frac{n}{\log n} \right)^{1/\alpha} \frac{1}{2\pi} \int_{-\pi}^{\pi} I_{n,X}(\lambda) \eta(\lambda) d\lambda \\ &= \left(\frac{n}{\log n} \right)^{1/\alpha} \int_{-\pi}^{\pi} I_{n,Z}(\lambda) f(\lambda) \eta(\lambda) d\lambda + \left(\frac{n}{\log n} \right)^{1/\alpha} \frac{1}{2\pi} \int_{-\pi}^{\pi} R_n(\lambda) \eta(\lambda) d\lambda \end{aligned} \tag{1.12}$$

Now suppose that we can show

$$\left(\frac{n}{\log n} \right)^{1/\alpha} \int_{-\pi}^{\pi} R_n(\lambda) \eta(\lambda) d\lambda \xrightarrow{P} 0. \tag{1.13}$$

Then

$$\begin{aligned} &\left(\frac{n}{\log n} \right)^{1/\alpha} \int_{-\pi}^{\pi} I_{n,Z}(\lambda) f(\lambda) \eta(\lambda) d\lambda \\ &= 2\pi a_n^{-1} \left\{ \widehat{(f\eta)}(0) \sum_{t=1}^{n} Z_t^2 + 2 \sum_{k=1}^{n-1} \widehat{(f\eta)}(k) \sum_{t=1}^{n-k} Z_t Z_{t+k} \right\} \end{aligned} \tag{1.14}$$

is the leading term of $a_n^{-1} Q_n$. By Theorem 3.3 of Davis and Resnick [4] in Case I and by a standard central limit theorem for m-dependent random variables in Case II (details will be presented in Section 2), we have

$$4\pi a_n^{-1} \sum_{k=1}^{n-1} \widehat{(f\eta)}(k) \sum_{t=1}^{n-k} Z_t Z_{t+k} \xrightarrow{\mathcal{D}} 4\pi \left(\sum_{k=1}^{\infty} \left| \widehat{(f\eta)}(k) \right|^{\alpha} \right)^{1/\alpha} S \tag{1.15}$$

where, in Case I, S is a symmetric α-stable random variable to be defined in Section 2, and in Case II, S is $\mathcal{N}(0, \sigma^2)$. Combining (1.12), (1.13), (1.14) and (1.15), we obtain

$$a_n^{-1} \left[Q_n - 2\pi \widehat{(f\eta)}(0) \sum_{t=1}^{n} Z_t^2 \right] \xrightarrow{\mathcal{D}} 4\pi \left(\sum_{k=1}^{\infty} \left| \widehat{(f\eta)}(k) \right|^{\alpha} \right)^{1/\alpha} S, \tag{1.16}$$

which is the main result of this paper.

Remarks

1. The decomposition (1.11), which is standard in the spectral analysis of time series (see, for example, Chapter 10 of Brockwell and Davis [3]), has been shown to be extremely useful in the analysis of infinite variance time series in a series of papers by Claudia Klüppelberg and Thomas Mikosch (see Klüppelberg and Mikosch [9] and references therein).

2. While the idea behind the proof of (1.16) is simple, the difficulty is in verifying (1.13). This will be our main goal.

3. An advantage of the approach described above is that it works in both the infinite and finite variance cases. Moreover, in the finite variance case, only the existence of the second moment of the noise sequence is assumed. A disadvantage is that we must impose stronger conditions on the functions f and η than (1.3) and (1.4).

4. The random centering in (1.16), which is equivalent to removing the diagonal, is unavoidable in the Case I (infinite variance); since the mean of Q_n does not exist. In Case II, centering with the mean is possible if the fourth moment of the noise exists. We elaborate on this point in Section 2.

The remainder of the paper is organized as follows: Section 2 contains assumptions and the statement of the main results with their proofs. The proof of the crucial relation (1.13) is given in Section 3.

In the sequel, c stands for a constant whose value is not of interest and may vary from line to line. It may also depend on the small number $\delta > 0$ which frequently appears in the proofs.

2 Assumptions and main results

Consider the quadratic form (1.1) and assume that the X_t, $t \in \mathbf{Z}$ (integers) are the (one-sided) moving averages

$$X_t = \sum_{j=0}^{\infty} c_j Z_{t-j}. \tag{2.1}$$

where, either

(N1) Z_1 is symmetric and in the domain of normal attraction of an infinite variance symmetric α-stable law with $1 < \alpha < 2$,

or

(N2) $EZ_1^2 = \sigma^2 < \infty$ and $EZ_1 = 0$.

Recall that a symmetric random variable Z_1 is in the domain of normal attraction of a symmetric α-stable law Y with $0 < \alpha < 2$ if $n^{-1/\alpha} \sum_{j=1}^n Z_j \xrightarrow{\mathcal{D}} Y$, where $Z_2, Z_3, \ldots$, are i.i.d. copies of Z_1 and where $E\exp(i\theta Y) = \exp\{-\sigma^\alpha|\theta|^\alpha\}$. It is well known (see, for example, Gnedenko and Kolmogorov [8]) that this happens if and only if

$$\lim_{x\to\infty} x^\alpha P\{|Z_1| > x\} = \sigma^\alpha C_\alpha, \tag{2.2}$$

where σ is the scale parameter of the stable limit Y, and where

$$C_\alpha = \left(\alpha \int_0^\infty (1-\cos x)\frac{dx}{x^{\alpha+1}}\right)^{-1} = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\frac{\pi\alpha}{2})} & \text{if } \alpha \neq 1 \\ 2/\pi & \text{if } \alpha = 1. \end{cases} \tag{2.3}$$

CONDITIONS on the coefficients c_j in (2.1) and on the function η in (1.1):

(A1) $c_j = j^{d-1}L(j)$, $0 < d < 1 - 1/\alpha$ and L is a normalized slowly varying function. By "normalized," we mean that for any $\epsilon > 0$, $x^\epsilon L(x)$ is increasing and $x^{-\epsilon}L(x)$ is decreasing for sufficiently large x.

(A2) η is real and symmetric and there is a real number $e < 1$ such that, for any $\delta > 0$, $|\eta(\lambda)| \le c(\delta)|\lambda|^{-e-\delta}$.

(A3) Setting $C(\lambda) = \sum_{j=0}^{\infty} c_j e^{-i\lambda j}$, assume that $|C'(\lambda)| \le c(\delta)|\lambda|^{-d-1-\delta}$ and $|\eta'(\lambda)| \le c(\delta)|\lambda|^{-e-1-\delta}$.

Remarks

1. If the innovations Z_t satisfy (N2), set $\alpha = 2$ in (A1) and (A3).
2. By A(1), $C(\lambda) \sim$ const $\lambda^{-d}L(1/\lambda)$, as $\lambda \to 0+$, so, in particular

$$|C(\lambda)| \le c(\delta)|\lambda|^{-d-\delta}. \tag{2.4}$$

and

$$|f(\lambda)| \le c(\delta)|\lambda|^{-2d-2\delta}. \tag{2.5}$$

Thus, $2d$ corresponds to a in (1.3) and e to b in (1.4).

3. Condition (A3) is used to ensure a proper rate of decay of certain Fourier coefficients. It could be replaced by a number of weaker but more cumbersome conditions on the rate of decay of some Fourier coefficients, resulting in a loss of clarity in the assumptions.

4. The theorems below hold also if $\eta(\lambda)$ is real, but not symmetric, and if $\widehat{\eta}(t)$ is defined as the cosine transform, namely $\widehat{\eta}(t) = (2\pi)^{-1} \int_{\pi}^{\pi} (\cos \lambda t)\eta(\lambda) d\lambda$.

We now state and prove our main result.

Theorem 2.1 *Suppose that the assumptions (A1)-(A3) hold, and that* $2d + e < 1 - 1/\alpha$.
(a) If the Z_t *satisfy (N1), then*

$$(2.6)\quad (n \log n)^{-1/\alpha} \left[Q_n - \left(\int_{-\pi}^{\pi} f(\lambda)\eta(\lambda) d\lambda \right) \sum_{t=1}^{n} Z_t^2 \right] \xrightarrow{\mathcal{D}} 4\pi\sigma^2 \left(\sum_{k=1}^{\infty} |\widehat{(f\eta)}(k)|^\alpha \right)^{1/\alpha} S_1$$

where $E \exp i\theta S_1 = \exp\{-C_\alpha |\theta|^\alpha\}$; σ *is defined in* (2.2) *and* C_α *in* (2.3)
(b) If the Z_t *satisfy (N2), then*

$$(2.7)\qquad n^{-1/2} \left[Q_n - \left(\int_{-\pi}^{\pi} f(\lambda)\eta(\lambda) d\lambda \right) \sum_{t=1}^{n} Z_t^2 \right] \xrightarrow{\mathcal{D}} 4\pi\sigma^2 \left(\sum_{k=1}^{\infty} |\widehat{(f\eta)}(k)|^2 \right)^{1/2} Y_1,$$

where now Y_1 *is* $\mathcal{N}(0,1)$ *and* $\sigma^2 = Var\ Z_1$.

Remarks

1. Since $\sum_{k=-\infty}^{\infty} [\widehat{(f\eta)}(k)]^2 = (2\pi)^{-1} \int_{-\pi}^{\pi} [(f\eta)(\lambda)]^2 d\lambda$ by Parseval's equality, and $\widehat{(f\eta)}(0) = (2\pi)^{-1} \int_{-\pi}^{\pi} (f\eta)(\lambda) d\lambda$, the variance of the RHS of (2.7) equals

$$(2.8)\qquad 4\pi\sigma^4 \int_{-\pi}^{\pi} [f(\lambda)\eta(\lambda)]^2 d\lambda - 2\sigma^4 \left(\int_{-\pi}^{\pi} f(\lambda)\eta(\lambda) d\lambda \right)^2 .$$

   It is smaller than the asymptotic variance (1.6) obtained when the Q_n are centered by their means.

2. If the assumption that η is symmetric is dropped, then (2.6) and (2.7) continue to hold with $\widehat{(f\eta)}(k)$ replaced by $(2\pi)^{-1} \int_{-\pi}^{\pi} \cos(\lambda k) f(\lambda)\eta(\lambda) d\lambda$. This follows immediately by appropriately modifying the RHS of (1.14).

3. We show in Theorem 2.2 below that the random centering in Theorem 2.1 can be replaced by a function of the observable sequence $X_1, \ldots, X_n$.

The proof of Theorem 2.1 uses the following two propositions:

Proposition 2.1 *The remainder* R_n *in* (1.11) *equals*

$$(2.9)\qquad R_n(\lambda) = |Y_n(\lambda)|^2$$

$$+ Y_n(\lambda) \left(\sum_{j=0}^{\infty} c_j e^{i\lambda j} \right) \left(n^{-1/\alpha} \sum_{l=1}^{n} Z_l e^{i\lambda l} \right) + Y_n(-\lambda) \left(\sum_{j=0}^{\infty} c_j e^{-i\lambda j} \right) \left(n^{-1/\alpha} \sum_{l=1}^{n} Z_l e^{-i\lambda l} \right),$$

where

$$(2.10)\qquad Y_n(\lambda) = n^{-1/\alpha} \sum_{j=0}^{\infty} c_j e^{-i\lambda j} \left(\sum_{k=1-j}^{n-j} Z_k e^{-i\lambda k} - \sum_{k=1}^{n} Z_k e^{-i\lambda k} \right),$$

and satisfies for all $1 < \alpha \leq 2$,

$$(2.11)\qquad n^{1/\alpha} \int_{-\pi}^{\pi} R_n(\lambda)\eta(\lambda) d\lambda \xrightarrow{P} 0.$$

Proposition 2.1 is proved in Section 3.

Proposition 2.2 *(a) If (N1) holds, then*

$$\left(n^{-2/\alpha}\sum_{t=1}^{n} Z_t^2,\ (n\log n)^{-1/\alpha}\sum_{t=1}^{n-1} Z_tZ_{t+1},\dots,(n\log n)^{-1/\alpha}\sum_{t=1}^{n-m} Z_tZ_{t+m}\right) \tag{2.12}$$

$$\xrightarrow{\mathcal{D}} C_\alpha^{2/\alpha}\sigma^2\cdot(Y_0,Y_1,\dots,Y_m),$$

where the $Y_k,\ k=0,1,\dots,m$ are independent, Y_0 is a positive $(\alpha/2)$-stable with scale parameter $C_{\alpha/2}^{-2/\alpha}$ and the $Y_k,\ k=1,\dots,m$ are i.i.d. symmetric α-stable with scale parameter $C_\alpha^{-1/\alpha}$.

(b) If (N2) holds, then

$$n^{-1/2}\left(\sum_{t=1}^{n-1} Z_tZ_{t+1},\dots,\sum_{t=1}^{n-m} Z_tZ_{t+m}\right)\xrightarrow{\mathcal{D}}\sigma^2\cdot(Y_1,\dots,Y_m), \tag{2.13}$$

where the $Y_k,\ k=1,\dots,m$ are i.i.d. $\mathcal{N}(0,1)$.

(c) If (N2) holds and $EZ_1^4<\infty$, then

$$\left(n^{-1/2}\sum_{t=1}^{n}(Z_t^2-\sigma^2),\ n^{-1/2}\sum_{t=1}^{n-1} Z_tZ_{t+1},\dots,n^{-1/2}\sum_{t=1}^{n-m} Z_tZ_{t+m}\right) \tag{2.14}$$

$$\xrightarrow{\mathcal{D}}(Y_0,\sigma^2Y_1,\dots,\sigma^2Y_m),$$

where the $Y_k,\ k\ge 1$ are i.i.d. $\mathcal{N}(0,1)$ and Y_0 is $\mathcal{N}(0,\mathrm{Var}(Z_0^2))$, independent of the remaining Y_k.

Proposition 2.2 has been used extensively by Klüppelberg and Mikosch. Part (a) is a direct consequence of Theorem 3.3 of Davis and Resnick [4], whereas parts (b) and (c) follow from a standard limit theorem for m-dependent vectors (see, for example, Billingsley [1] Section IV.20).

Proof of Theorem 2.1: The proof was given in Section 1. However, Relations (1.13) and (1.15) must still be verified and the scale parameter of S_1 (and variance of Y_1) must be determined. Relation (1.13) follows from Proposition 2.1.

To prove Relation (1.15), we use Proposition 2.2. Set $r_k=\widehat{(f\eta)}(k)$, and let Y_k be as in Proposition 2.2. By Theorem 4.2 of Billingsley [1], we must verify (under (N1)):

$$\sum_{k=1}^{m} r_kY_k\xrightarrow{\mathcal{D}}\sum_{k=1}^{\infty} r_kY_k,\quad\text{as}\quad m\to\infty; \tag{2.15}$$

$$\sum_{k=1}^{m} r_k a_n^{-1}\sum_{k=1}^{n-k} Z_tZ_{t+k}\xrightarrow{\mathcal{D}}\sum_{k=1}^{m} r_kC_\alpha^{2/\alpha}\sigma^2Y_k,\quad\text{as}\quad n\to\infty, \tag{2.16}$$

for each fixed m, and

$$\lim_{m\to\infty}\ \limsup_{n\to\infty} P\{a_n^{-1}\Big|\sum_{k=m+1}^{n-1} r_k\sum_{t=1}^{n-k} Z_tZ_{t+k}\Big|\ge\epsilon\}=0, \tag{2.17}$$

for each $\epsilon > 0$. Under (N2), we must verify the same relations, except that there is no $C_\alpha^{2/\alpha}$ in the RHS of (2.16). We will now show that relations (2.15), (2.16) and (2.17) hold if

$$\sum_{k=1}^{\infty} |r_k|^\mu < \infty, \quad \text{for some} \quad 1 < \mu < \alpha \tag{2.18}$$

(case $\alpha = 2$ included). Using (A1)-(A3), $2d + e < 1 - 1/\alpha$, and standard arguments used to evaluate the order of magnitude of Fourier coefficients, it can be shown that (see Appendix)

$$|\widehat{(f\eta)}(k)| \leq c(\delta) \begin{cases} k^{2d+e+\delta-1}, & \text{if } 2d + e \geq 0 \\ k^{-1}, & \text{if } 2d + e < 0, \end{cases} \tag{2.19}$$

which implies (2.18) since $r_k = \widehat{(f\eta)}(k)$.

Relation (2.15) holds because $\sum_{k=1}^{\infty} |r_k|^\alpha < \infty$. Relation (2.16) follows immediately from Proposition 2.2. To prove (2.17) under (N1) we use Theorem 3.1 of Rosiński and Woyczyński [13] (actually we use only its corollary, Proposition 3.2 of Kokoszka and Taqqu [11]). We have for $1 < \mu < \alpha$:

$$P\left\{ a_n^{-1} \left| \sum_{k=m+1}^{n-1} r_k \sum_{t=1}^{n-k} Z_t Z_{t+k} \right| \geq \epsilon \right\} = P\left\{ \left| \sum_{t=1}^{n-m-1} Z_t \sum_{s=t+m+1}^{n} r_{s-t} Z_s \right| \geq \epsilon a_n \right\} \tag{2.20}$$

$$\begin{aligned}
&\leq c(\epsilon a_n)^{-\alpha}(1 + \log_+(\epsilon a_n)) \sum_{t=1}^{n-m-1} \sum_{s=t+m+1}^{n} |r_{s-t}|^\alpha (1 + \log_+ |r_{s-t}|^{-1}) \\
&\leq c\epsilon^{-\alpha} n^{-1} (\log n)^{-1} (1 + \alpha^{-1} \log n + \alpha^{-1} \log\log n) \sum_{t=1}^{n-m-1} \sum_{k=m+1}^{n-t} |r_k|^\mu \\
&\leq c\epsilon^{-\alpha} n^{-1} \sum_{t=1}^{n} \sum_{k=m+1}^{\infty} |r_k|^\mu \leq c\epsilon^{-\alpha} \sum_{k=m+1}^{\infty} |r_k|^\mu \to 0,
\end{aligned}$$

as $m \to \infty$. The verification of (2.17) under (N2) is even simpler; set $\mu = 2$ and use Chebyshev's inequality. This completes the proof of Theorem 2.1 ∎

The random centering in Theorem 2.1 involves $\sum_{t=1}^{\infty} Z_t^2$ which is not observable. Under some mild additional assumptions on the function f, it is possible to replace it by a function of the observable sequence $X_1, \ldots, X_n$. Setting

$$K(X_1, \ldots, X_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [2\pi f(\lambda)]^{-1} \left| \sum_{t=1}^{n} X_t e^{-i\lambda t} \right|^2 d\lambda, \tag{2.21}$$

we have

Theorem 2.2 *If in addition to the assumptions of Theorem 2.1 we assume that $f(\lambda) > 0$ for every λ and*

$$\left| \frac{d}{d\lambda} f^{-1}(\lambda) \right| \leq c(\delta) |\lambda|^{2d-1-\delta}, \tag{2.22}$$

then (2.6) and (2.7) hold with $\sum_{t=1}^{n} Z_t^2$ replaced by $K(X_2, \ldots, X_n)$ defined in (2.21).

PROOF: It suffices to show that

$$a_n^{-1}\left\{K(X_1,\ldots,X_n)-\sum_{t=1}^{n}Z_t^2\right\}\xrightarrow{P}0. \tag{2.23}$$

Observe that

$$K(X_1,\ldots,X_n)=\sum_{t=1}^{n}Z_t^2+(2\pi)^{-2}n^{2/\alpha}\int_{-\pi}^{\pi}R_n(\lambda)[f(\lambda)]^{-1}d\lambda, \tag{2.24}$$

so (2.23) reduces to showing that

$$n^{1/\alpha}\int_{-\pi}^{\pi}R_n(\lambda)[f(\lambda)]^{-1}d\lambda\xrightarrow{P}0. \tag{2.25}$$

It is verified in Kokoszka and Mikosch [10] (Lemma 5.1), that if f is positive then

$$|f^{-1}(\lambda)|\le c(\delta)|\lambda|^{2d-\delta}. \tag{2.26}$$

In view of (2.26) and (2.22), relation (2.25) immediately follows from Proposition 2.1. □

Remark. The assumptions of Theorem 2.2 are satisfied by the fractional ARIMA sequences (cf. Fox and Taqqu [5]).

Notice that, in the proof of Theorem 2.1, we used only parts (a) and (b) of Proposition 2.2. Part (c) yields the following corresponding result when centering with the mean. We must impose, however, additional restrictions on the range of d and e.

Theorem 2.3 *Suppose (A1)-(A3), $2d+e<1/2$ and (N2) hold. Assume, in addition, that $EZ_0^4<\infty$, $d<1/4$ and $e<1/2$. Then,*

$$n^{-1/2}(Q_n-EQ_n)\xrightarrow{\mathcal{D}}\mathcal{N}(0,\sigma_{asy}^2), \tag{2.27}$$

where

$$\sigma_{asy}^2=4\pi\sigma^2\int_{-\pi}^{\pi}(f(\lambda)\eta(\lambda))^2d\lambda+(\mathrm{Var}(Z_0^2)-2\sigma^4)\left(\int_{-\pi}^{\pi}f(\lambda)\eta(\lambda)d\lambda\right)^2. \tag{2.28}$$

Remarks

1. If the Z_t have the first four moments equal to those of the Gaussian distribution, then $\mathrm{Var}(Z_0^2)=2\sigma^4$ and σ_{asy}^2 coincides with the asymptotic variance (1.6) obtained by Fox and Taqqu [6].

2. The finite variance result, directly analogous to (2.6), is

$$n^{-1/2}\left(Q_n-n\sigma^2\int_{-\pi}^{\pi}f(\lambda)\eta(\lambda)d\lambda\right)\xrightarrow{\mathcal{D}}\mathcal{N}(0,\sigma_{asy}^2),$$

and, in this case, one needs not assume $d<1/4$ and $e<1/2$. In fact, using different methods, Giraitis and Surgailis (Theorem 2, [7]) show that these additional assumptions are not even required for (2.27) to hold. We have included Theorem 2.3 in order to show that it immediately follows from Proposition 2.1 and Proposition 2.2 (c).

Proof of Theorem 2.3: We have, as in Section 1,

$$
\begin{aligned}
(2.29)\qquad n^{-1/2}(Q_n - EQ_n) &= 4\pi \sum_{k=1}^{n-1} \widehat{(f\eta)}(k)\; n^{-1/2} \sum_{t=1}^{n-k} Z_t Z_{t+k} \\
&+ n^{1/2}(2\pi)^{-1} \int_{-\pi}^{\pi} R_n(\lambda)\eta(\lambda)d\lambda \\
&+ 2\pi\widehat{(f\eta)}(0) n^{-1/2} \left(\sum_{t=1}^{n} (Z_t^2 - \sigma^2) \right) \\
&+ n^{-1/2}\left(2\pi\widehat{(f\eta)}(0) n\sigma^2 - EQ_n \right).
\end{aligned}
$$

If it can be shown that

$$(2.30)\qquad n^{-1/2}\left(2\pi\widehat{(f\eta)}(0) n\sigma^2 - EQ_n \right) \to 0,$$

then Proposition 2.1 and Proposition 2.2(c) yield, as in the proof of Theorem 2.1,

$$(2.31)\qquad n^{-1/2}(Q_n - EQ_n) \xrightarrow{\mathcal{D}} 4\pi\sigma^2 \sum_{k=1}^{\infty} \widehat{(f\eta)}(k) Y_k + 2\pi\widehat{(f\eta)}(0) Y_0,$$

where the $Y_k,\ k = 0, 1, \dots$ are as in Proposition 2.2(c). Using Parseval's equality it is easy to verify that the RHS of (2.31) has variance (2.28). It remains, therefore, to verify (2.30). Setting

$$(2.32)\qquad r(k) = E[X_0 X_k] = \sigma^2 \int_{-\pi}^{\pi} e^{i\lambda k} f(\lambda) d\lambda = 2\pi\sigma^2 \widehat{f}(k),$$

we have

$$(2.33)\qquad EQ_n = \sum_{t,s=1}^{n} \widehat{\eta}(t-s) r(t-s) = 2\pi\sigma^2 n \sum_{|k|<n} \left(1 - \frac{|k|}{n} \right) \widehat{\eta}(k)\widehat{f}(k).$$

Since $f, \eta \in L^2[-\pi, \pi]$ (recall that we assume $2d < 1/2$, $e < 1/2$),

$$(2.34)\qquad \widehat{(f\eta)}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\lambda)\eta(\lambda) d\lambda = \sum_{k=-\infty}^{\infty} \widehat{\eta}(k)\widehat{f}(k).$$

In view of (2.33) and (2.34), (2.30) boils down to the verification of

$$(2.35)\qquad n^{-1/2} \sum_{k=1}^{n} k |\widehat{\eta}(k)| |\widehat{f}(k)| \to 0$$

and

$$(2.36)\qquad n^{1/2} \sum_{k>n} |\widehat{\eta}(k)| |\widehat{f}(k)| \to 0.$$

The conditions imposed on the functions η and f imply (see Appendix)

$$(2.37)\qquad |\widehat{\eta}(k)| \le c \begin{cases} k^{e-1+\delta}, & \text{if } e \ge 0 \\ k^{-1}, & \text{if } e < 0 \end{cases}$$

and

$$|\widehat{f}(k)| \le c \begin{cases} k^{2d-1+\delta}, & \text{if } d \ge 0 \\ \\ k^{-1}, & \text{if } d < 0. \end{cases} \tag{2.38}$$

Using (2.37) and (2.38), it is not difficult to verify that (2.35) and (2.36) hold provided $2d < 1/2$, $e < 1/2$ and $2d + e < 1/2$. ■

3 Proof of Proposition 2.1

The outline of the proof of Proposition 2.1 is similar to that of Relation (3.45) in Kokoszka and Taqqu [11]. The situation here, however, is much more complicated because the function η may not be bounded.

We shall often use below the following result which is a direct consequence of Theorem 3.1 of Rosiński and Woyczyński [13] (see Corollary 3.1 of Kokoszka and Taqqu [11]).

Proposition 3.1 *Consider the sequence of quadratic forms*

$$Q_n = \sum_{\substack{j,k=1, \\ j\neq k}}^{n} q_n(j,k) Z_j Z_k,$$

with the Z_k satisfying (N1). If for some $1 < \mu < \alpha$,

$$\lim_{n\to\infty} \sum_{\substack{j,k=1 \\ j\neq k}}^{n} |q_n(j,k)|^{\mu} = 0,$$

then

$$\sum_{\substack{j,k=1 \\ j\neq k}}^{n} q_n(j,k) Z_j Z_k \xrightarrow{P} 0, \quad \text{as} \quad n \to \infty.$$

If the Z_k satisfy (N2), then the above result clearly remains true, with $\mu = \alpha = 2$ (use Chebyshev's inequality). In the sequel, we will frequently use Proposition 3.1 or its finite variance counterpart without specifically mentioning it. Also, in the sequel, we set $1 < \mu < \alpha$ *under (N1) and* $\mu = \alpha = 2$ *under (N2).*

The expression (2.9) of the remainder R_n can be verified as in the case of finite variance ARMA processes (see, for example, the proof of Theorem 10.3.1 of Brockwell and Davis [3]. Hence the proof of (2.11) reduces to the verification of

$$n^{-1/\alpha} \int_{-\pi}^{\pi} |\Delta_n(\lambda)|^2 |\eta(\lambda)| d\lambda \xrightarrow{P} 0 \tag{3.1}$$

and

$$n^{-1/\alpha} \int_{-\pi}^{\pi} \Delta_n(\lambda) C(\lambda) \left(\sum_{l=1}^{n} Z_l e^{i\lambda l} \right) \eta(\lambda) d\lambda \xrightarrow{P} 0, \tag{3.2}$$

$1 < \alpha \le 2$, where

$$\Delta_n(\lambda) = n^{1/\alpha} Y_n(\lambda) = \sum_{j=0}^{\infty} c_j e^{-i\lambda j} \left(\sum_{k=1-j}^{n-j} Z_k e^{-i\lambda k} - \sum_{k=1}^{n} Z_k e^{-i\lambda k} \right).$$

Verification of (3.1):

We split $\Delta_n(\lambda)$ as follows: $\Delta_n(\lambda) = \sum_{u=1}^{4} \Gamma_{un}(\lambda)$ where

$$\Gamma_{1n}(\lambda) = \sum_{k=0}^{n-2} \left(\sum_{j=k+1}^{n+k} c_j e^{i(k-j)\lambda} \right) Z_{-k}, \tag{3.3}$$

$$\Gamma_{2n}(\lambda) = \sum_{k=n-1}^{\infty} \left(\sum_{j=k+1}^{n+k} c_j e^{i(k-j)\lambda} \right) Z_{-k}, \tag{3.4}$$

$$\Gamma_{3n}(\lambda) = -e^{-in\lambda} \sum_{k=0}^{n-2} \left(\sum_{j=k+1}^{n-1} c_j e^{i(k-j)\lambda} \right) Z_{n-k}, \tag{3.5}$$

$$\Gamma_{4n}(\lambda) = -\left(\sum_{j=n}^{\infty} c_j e^{-i\lambda j} \right) \sum_{t=1}^{n} Z_t e^{-i\lambda t}, \tag{3.6}$$

and verify that

$$n^{-1/\alpha} \int_{-\pi}^{\pi} |\Gamma_{un}(\lambda)|^2 |\eta(\lambda)| d\lambda \xrightarrow{P} 0, \tag{3.7}$$

for $u = 1, 2, 3, 4$. The verification of (3.7) relies heavily on the following result:

Lemma 3.1 *Suppose the c_j satisfy $c_j = j^{d-1}L(j)$, where $0 < d < 1$ and L is normalized slowly varying. Then, for $\lambda \in [-\pi, \pi]\backslash\{0\}$, any integers $0 \le n_1 < n_2$ and arbitrarily small $\delta > 0$*

$$\left| \sum_{j=n_1}^{n_2} c_j e^{i\lambda j} \right| \le c(\delta) n_1^{d-1+\delta} |\lambda|^{-1}, \tag{3.8}$$

$$\left| \sum_{j=n_1}^{n_2} c_j e^{i\lambda j} \right| \le c(\delta) |\lambda|^{-d-\delta}. \tag{3.9}$$

Proof: Since L is normalized slowly varying, it is quasi-monotone so the arguments used in the proof of Theorem 4.3.2 on p. 208 of Bingham et al. [2] lead to (3.8). Relation (3.9) can be proved by modifying the proof of Theorem 2.6 on p. 187 of Zygmund [14]. ∎

In the sequel we will frequently use Lemma 3.1 without specifically mentioning it.

1) *Verifications of* (3.7) *with* $u = 1$: Observe that

$$\int_{-\pi}^{\pi} |\Gamma_{1n}(\lambda)|^2 |\eta(\lambda)| d\lambda = \sum_{k=0}^{n-2} \nu_n(k,k) Z_{-k}^2 + \sum_{\substack{k,t=0 \\ k \neq t}} \nu_n(k,t) Z_{-k} Z_{-t}, \tag{3.10}$$

where

$$\nu_n(k,t) = \int_{-\pi}^{\pi} \left(\sum_{j=k+1}^{n+k} c_j e^{i(k-j)\lambda} \right) \left(\sum_{j=t+1}^{n+t} c_j e^{i(j-t)\lambda} \right) |\eta(\lambda)| d\lambda. \tag{3.11}$$

In order to verify that

$$n^{-1/\alpha} \sum_{k=0}^{n-2} \nu_n(k,k) Z_{-k}^2 \xrightarrow{P} 0 \tag{3.12}$$

and

$$n^{-1/\alpha} \sum_{0 \le k \ne t \le n-2} \nu_n(k,t) Z_{-k} Z_{-t} \xrightarrow{P} 0, \tag{3.13}$$

we need appropriate upper bounds on the $\nu_n(k,t)$.

(a) If $d+e<0$, $\nu_n(k,t)$ is bounded above by (use Lemma 3.1):

$$c \int_0^\pi k^{d-1+\delta} \lambda^{-1} \lambda^{-d-\delta} \lambda^{-e-\delta} d\lambda = c k^{d-1+\delta}, \quad \text{for sufficiently small } \delta. \tag{3.14}$$

(b) If $d+e \ge 0$, we split the integral in (3.11) into $\int_{\{|\lambda|<\pi/n\}} + \int_{\{|\lambda|>\pi/n\}}$. We have for sufficiently small δ,

$$\begin{aligned} \int_{|\lambda|<\pi/n} \dots d\lambda &\le c \int_0^{\pi/n} n k^{d-1+\delta} \lambda^{-d-\delta} \lambda^{-e-\delta} d\lambda \\ &\le c n k^{d-1+\delta} n^{(d+e-1+2\delta)} \\ &= c k^{d-1+\delta} n^{d+e+\delta}. \end{aligned} \tag{3.15}$$

Also, for $\delta>0$, setting $x = n\lambda$, we have

$$\begin{aligned} \int_{\pi/n<|\lambda|\le\pi} \dots d\lambda &\le c \int_{\pi/n}^{\pi} k^{d-1+\delta} \lambda^{-1} \lambda^{-d-\delta} \lambda^{-e-\delta} d\lambda \\ &\le c k^{d-1+\delta} \int_\pi^{n\pi} \left(\frac{x}{n}\right)^{-(1+d+e+2\delta)} \frac{dx}{n} \\ &\le c k^{d-1+\delta} n^{d+e+2\delta} \int_\pi^\infty x^{-(1+d+e+2\delta)} dx \\ &= c k^{d-1+\delta} n^{d+e+\delta}. \end{aligned} \tag{3.16}$$

We have thus verified that

$$|\nu_n(k,t)| \le c \begin{cases} (k \vee t)^{d-1+\delta} n^{d+e+\delta}, & \text{if } d+e \ge 0 \\ (k \vee t)^{d-1+\delta}, & \text{if } d+e<0. \end{cases} \tag{3.17}$$

To prove (3.12) and (3.13) we verify that

$$E\left| n^{-1/\alpha} \sum_{k=0}^{n-2} \nu_n(k,k) Z_{-k}^2 \right|^{\mu/2} \to 0 \tag{3.18}$$

and

$$\sum_{0 \le t < k \le n-2} |n^{-1/\alpha} \nu_n(k,t)|^\mu \to 0. \tag{3.19}$$

We shall focus on the more difficult case $d+e \geq 0$. By (3.17), the expectation in (3.18) is bounded above by

$$\begin{aligned}
n^{-\mu/2\alpha} E\Big|\sum_{k=0}^{n-2} \nu_n(k,k) Z_{-k}^2\Big|^{\mu/2} &\leq cn^{-\mu/2\alpha} \sum_{k=0}^{n-2} |\nu_n(k,k)|^{\mu/2} \\
&\leq cn^{-\mu/2\alpha} n^{(d+e+\delta)\mu/2} \sum_{k=0}^{n-2} k^{(d-1+\delta)\mu/2} \\
&\leq cn^{\frac{\mu}{2}(d+e+\delta-1/\alpha+d-1+\delta)+1},
\end{aligned}$$

which tends to 0 as $n \to \infty$. Indeed, if $\mu = \alpha$ and $\delta = 0$, the exponent of n is negative because of the assumption $2d + e < 1 - 1/\alpha$.

The double sum in (3.19) is bounded above by

$$\begin{aligned}
cn^{-\mu/\alpha} \int_1^n dt \int_t^n dk (k^{d-1+\delta} n^{d+e+\delta})^\mu &\leq cn^{\mu(d+e+\delta-1/\alpha)} \int_1^n dt\; t^{(d-1+\delta)\mu+1} \\
&\leq cn^{\mu(2d+e-1/\alpha-1+\delta)+2}.
\end{aligned}$$

Again the exponent is negative for $\mu = \alpha$ and $\delta = 0$, because of the assumption $2d + e < 1 - 1/\alpha$.

This completes the verification of (3.7) for $u = 1$.

2) *Verification of* (3.7) *with* $u = 2$:

Here we must show that

$$n^{-1/\alpha} \sum_{k=n-1}^{\infty} \nu_n(k,k) Z_{-k}^2 \xrightarrow{P} 0 \tag{3.20}$$

and

$$n^{-1/\alpha} \sum_{n-1 \leq t \neq k < \infty} \nu_n(k,t) Z_{-k} Z_{-t} \xrightarrow{P} 0 \tag{3.21}$$

with $\nu_n(k,t)$ still defined by (3.11). Because of the different range of t and k than in the case $u = 1$, we need here other upper bounds on the $\nu_n(k,t)$. We derive different bounds depending on whether $e \geq -1$ or $e < -1$.

If $e \geq -1$, we split the integral in (3.11) into the integral over $|\lambda| < \pi/n$ and $|\lambda| > \pi/n$, and get $|\nu_n(k,t)| \leq ck^{d-1+\delta} t^{d-1+\delta} n^{e+1+\delta}$. If $e < -1$, the integral in (3.11) is bounded above by $c\int_0^\pi k^{d-1+\delta} t^{d-1+\delta} \lambda^{-2} \lambda^{-e-\delta} d\lambda \leq ck^{d-1+\delta} t^{d-1+\delta}$, provided δ is so small that $e + \delta < -1$. Summarizing, we have verified that

$$|\nu_n(k,t)| \leq c \begin{cases} k^{d-1+\delta} t^{d-1+\delta} n^{e+1+\delta}, & \text{if } e \geq -1 \\ k^{d-1+\delta} t^{d-1+\delta}, & \text{if } e < -1. \end{cases} \tag{3.22}$$

Using (3.22) and the condition $2d + e < 1 - 1/\alpha$ in case $e \geq -1$, and the condition $\alpha(d-1) < -1$ in case $e < -1$, it is not difficult to verify that

$$E|n^{-1/\alpha} \sum_{k=n-1}^{\infty} \nu_n(k,k) Z_{-k}^2|^{\mu/2} \to 0 \tag{3.23}$$

and

$$n^{-\mu/\alpha} \sum_{n-1 \le t < k < \infty} |\nu_n(k,t)|^\mu \to 0, \tag{3.24}$$

relation which imply (3.20) and (3.21), respectively.

3) *The verification of* (3.7) *with* $u = 3$ is the same as with $u = 1$.

4) *Verification of* (3.7) *with* $u = 4$*:*

We have

$$\int_{-\pi}^{\pi} |\Gamma_{4n}(\lambda)|^2 |\eta(\lambda)| d\lambda = \kappa_n(k,k) \sum_{k=1}^n Z_k^2 + \sum_{1 \le t \ne k \le n} \kappa_n(k,t) Z_k Z_t, \tag{3.25}$$

where

$$\kappa_n(k,t) = \int_{-\pi}^{\pi} \left| \sum_{j=n}^{\infty} c_j e^{-i\lambda j} \right|^2 |\eta(\lambda)| e^{i\lambda(k-t)} d\lambda. \tag{3.26}$$

To find an appropriate upper bound on the $\kappa_n(k,t)$ we consider here two cases: $d + e < 0$ and $d + e \ge 0$.

If $d+e<0$, the integral in (3.26) is bounded above by $c\int_0^\pi n^{d-1+\delta}\lambda^{-1}\lambda^{-d-\delta}\lambda^{-e-\delta}d\lambda \le cn^{d-1+\delta}$, provided δ is so small that $d+e+2\delta<0$. If $d+e \ge 0$, we again split the integral in (3.26) into integrals over $|\lambda| < \pi/n$ and $\frac{\pi}{n} < |\lambda| \le \pi$. We have $\int_{|\lambda|<\pi/n} \dots d\lambda \le \int_0^{\pi/n} \lambda^{-2(d+\delta)}\lambda^{-e-\delta} d\lambda \le cn^{2d+e-1+3\delta}$, because $1-2d-e-3\delta > 0$ for sufficiently small δ. The integral over $\pi/n < |\lambda| \le \pi$ is again estimated by setting $\lambda = x/n$: $\int_{\pi/n<|\lambda|\le\pi} \dots d\lambda \le cn^{d-1+\delta}\int_{\pi/n}^{\pi}\lambda^{-1}\lambda^{-d-\delta}\lambda^{-e-\delta}d\lambda = cn^{2d+e-1+2\delta}\int_\pi^\infty x^{-(1+d+e+2\delta)}dx \le cn^{2d+e-1+2\delta}$.

Summarizing, we have shown that

$$|\kappa_n(k,t)| \le c \begin{cases} n^{d-1+\delta}, & \text{if } d+e<0 \\ n^{2d+e-1+\delta}, & \text{if } d+e \ge 0 \end{cases} \tag{3.27}$$

Since $d-1 < -1/\alpha$ and $2d+e-1 < -1/\alpha$, we conclude from (3.27) that

$$|\kappa_n(k,t)| \le cn^p, \quad \text{for some } \ p < -1/\alpha. \tag{3.28}$$

With (3.28) in hand, it is easy to verify that

$$n^{-1/\alpha}\kappa_n(k,k)\sum_{k=1}^n Z_k^2 \xrightarrow{P} 0 \tag{3.29}$$

and

$$n^{-1/\alpha} \sum_{1 \le t \ne k \le n} \kappa_n(k,t) Z_k Z_t \xrightarrow{P} 0. \tag{3.30}$$

Indeed, to verify (3.29) observe that by (3.28)

$$\left| n^{-1/\alpha}\kappa_n(k,k)\sum_{k=1}^n Z_k^2 \right| \le cn^{(1/\alpha - p)} \left(n^{-2/\alpha} \sum_{k=1}^{\infty} Z_k^2 \right)$$

and recall that by Proposition 2.2 $n^{-2/\alpha}\sum_{k=1}^{n} Z_k^2$ converges in distribution. To verify (3.30) observe that $\sum_{1\le t<k\le n} |n^{-1/\alpha}\kappa_n(k,t)|^\mu$ is bounded above by $cn^{\mu(p-1/\alpha)+2}$.

This completes the verification of (3.7) for $u=4$, and hence that of (3.1).

VERIFICATION OF (3.2):

Now we split $\Delta_n(\lambda)$ as $\Delta_n(\lambda) = \Delta_{1n}(\lambda) + \Delta_{2n}(\lambda) + \Delta_{3n}(\lambda)$, where

$$\begin{aligned}
\Delta_{1n}(\lambda) &= \Gamma_{1n}(\lambda) + \Gamma_{2n}(\lambda) = \sum_{t=1}^{n-1} e^{-i\lambda t}\left(\sum_{j=t}^{\infty} c_j Z_{t-j}\right)\\
\Delta_{2n}(\lambda) &= \Gamma_{3n}(\lambda) = -e^{-i\lambda n}\sum_{t=1}^{n-1} e^{-i\lambda t}\left(\sum_{j=t}^{n-1} c_j Z_{n+t-j}\right),\\
\Delta_{3n}(\lambda) &= \Gamma_{4n}(\lambda) = -\left(\sum_{j=n}^{\infty} c_j e^{-i\lambda j}\right)\sum_{t=1}^{n} Z_t e^{-i\lambda t},
\end{aligned}$$

and verify that

$$n^{-1/\alpha}\int_{-\pi}^{\pi} \Delta_{un}\left(\sum_{l=1}^{n} Z_l e^{i\lambda l}\right) C(\lambda)\eta(\lambda)d\lambda \xrightarrow{P} 0 \tag{3.31}$$

for $u=1,2,3$. The verification of (3.31) for $u=3$ is essentially identical to that of (3.7) for $u=4$, so we omit it.

The idea behind the verification of (3.31) for $u=1$ and $u=2$ is to view $\Delta_{1n}(\lambda)$ and $\Delta_{2n}(\lambda)$ as trigonometric polynomials with random coefficients, a method used in Kokoszka and Taqqu [11]. Denote by h_s the Fourier coefficients of the function $C\eta$. It was shown in Kokoszka and Taqqu [11] that (3.31) holds with $u=1$ if

$$n^{-\mu/\alpha}\sum_{s=1}^{n-1}|h_{-s}|^\mu\sum_{t=1}^{n-s}\sum_{j=t}^{\infty}|c_j|^\mu \to 0 \tag{3.32}$$

and

$$n^{-\mu/\alpha}\sum_{s=0}^{n-2}|h_s|\sum_{t=1}^{n-1-s}\sum_{j=t+s}^{\infty}|c_j|^\mu \to 0, \tag{3.33}$$

and (3.31) holds with $u=2$ if

$$n^{-\mu/\alpha}\sum_{s=1}^{n-1}|h_{n-s}|^\mu\sum_{t=1}^{n-s}\sum_{j=t}^{n-1}|c_j|^\mu \to 0 \tag{3.34}$$

and

$$n^{-\mu/\alpha}\sum_{s=0}^{n-2}|h_{n+s}|^\mu\sum_{t=1}^{n-1-s}\sum_{j=t+s}^{n-1}|c_j|^\mu \to 0 \tag{3.35}$$

and

$$n^{-1/\alpha}\sum_{m=2}^{n}\nu_n(m)Z_m^2 \xrightarrow{P} 0, \tag{3.36}$$

where

$$\nu_n(m)=\frac{1}{2\pi}\int_{-\pi}^{\pi}C(\lambda)\eta(\lambda)\left(\sum_{j=n-m+1}^{n-1}c_je^{-i\lambda j}\right)d\lambda. \tag{3.37}$$

(The quadratic form defined by the integral in (3.31) has no diagonal elements for $u=1$ and has the diagonal (3.36) for $u=2$.)

The verifications of Relations (3.32) through (3.35) are essentially identical. They are based on the estimates

$$|h_s|\le c\begin{cases}|s|^{d+e-1+\delta}, & \text{if } d+e\ge 0\\ |s|^{-1}, & \text{if } d+e<0,\end{cases} \tag{3.38}$$

which are an immediate consequence of Proposition 4.1 below. We shall verify, for example, Relation (3.34) (the others can be verified in a similar manner).

Suppose $d+e\ge 0$. Then the LHS of (3.34) is bounded above by (set $u=n-s$)

$$cn^{-\mu/\alpha}\sum_{u=1}^{n-1}u^{(d+e-1+\delta)\mu}\sum_{t=1}^{u}\sum_{j=t}^{n-1}j^{(d-1+\delta)\mu}\le cn^{-\mu/\alpha}\sum_{u=1}^{n-1}u^{(d+e-1+\delta)\mu}\sum_{t=1}^{u}t^{(d-1+\delta)\mu+1} \tag{3.39}$$

$$\le cn^{-\mu/\alpha}\begin{cases}\sum_{u=1}^{n-1}u^{(2d+e-2+\delta)\mu+2}, & \text{if } (d-1)\alpha+2\ge 0\\ \sum_{u=1}^{n-1}u^{(d+e-1+\delta)\mu}, & \text{if } (d-1)\alpha+2<0.\end{cases}$$

If $(d-1)\alpha+2\ge 0$, (3.39) is further bounded above by

$$cn^{-\mu/\alpha}\begin{cases}n^{(2d+e-2+\delta)\mu+3}, & \text{if } (2d+e-2)\alpha+3\ge 0\\ 1, & \text{if } (2d+e-2)\alpha+3<0.\end{cases} \tag{3.40}$$

If $(2d+e-2)\alpha+3<0$, then clearly (3.40) tends to zero. If $(2d+e-2)\alpha+3\ge 0$, then (3.40) tends to zero provided $2d+e<2(1-1/\alpha)$. The last condition is satisfied because $2d+e<1-1/\alpha$ and $1-1/\alpha>0$.

If $(d-1)\alpha+2<0$, (3.39) is bounded above by

$$cn^{-\mu/\alpha}\begin{cases}n^{(d+e-1+\delta)\mu+1}, & \text{if } (d+e-1)\alpha+1\ge 0\\ 1, & \text{if } (d+e-1)\alpha+1<0.\end{cases} \tag{3.41}$$

Since $d+e<1$, (3.41) tends to zero.

Suppose now $d+e<0$. Using (3.38), the LHS of (3.34) is now bounded above by

$$cn^{-\mu/\alpha}\sum_{u=1}^{n-1}u^{-\mu}\sum_{t=1}^{u}\sum_{j=t}^{n-1}j^{(d-1+\delta)\mu}\le cn^{-\mu/\alpha}\sum_{u=1}^{n-1}u^{-\mu}\sum_{t=1}^{u}t^{(d-1+\delta)\mu+1} \tag{3.42}$$

$$\leq cn^{-\mu/\alpha}\begin{cases}\sum_{u=1}^{n-1} u^{-\mu+(d-1+\delta)\mu+2}, & \text{if } (d-1)\alpha+2\geq 0\\ \\ \sum_{u=1}^{n-1} u^{-\mu}, & \text{if } (d-1)\alpha+2<0.\end{cases}$$

In the case $(d-1)\alpha+2<0$, (3.42) clearly tends to zero. If $(d-1)\alpha+2\geq 0$, it suffices to consider only the case $(d-1)\alpha-\alpha+3\geq 0$, in which case (3.42) is bounded by $cn^{-\frac{\mu}{\alpha}+(d-1+\delta)\mu-\mu+3}$. The exponent is negative if $(d-1)\alpha<\alpha-2$. The last condition holds because $(d-1)\alpha<-1$ and $\alpha-2>-1$.

It remains to verify (3.36). This follows from the estimates

$$|\nu_n(m)|\leq c\begin{cases}(n-m+1)^{d-1+\delta}n^{d+e+\delta}, & \text{if } d+e\geq 0\\ \\ (n-m+1)^{d-1+\delta}, & \text{if } d+e<0\end{cases} \tag{3.43}$$

(see the proof of (3.17)) and from $n^{-\mu/\alpha}\sum_{m=2}^{n}|\nu_n(m)|^{\mu}\to 0$.

4 Appendix

Relations (2.19), (2.37), (2.38) and (3.38) follow from the conditions imposed on the functions C and η and the following proposition:

Proposition 4.1 *Suppose H is a differentiable function on $[-\pi,\pi]$ such that for some $p<1$ and any $\delta\geq 0$*

$$|H(\lambda)|\leq c(\delta)|\lambda|^{-(p+\delta)} \tag{4.1}$$

and

$$|H'(\lambda)|\leq c(\delta)|\lambda|^{-(p+1+\delta)}. \tag{4.2}$$

Then the Fourier coefficients $\widehat{H}(k)$ of H satisfy

$$|\widehat{H}(k)|\leq c(\delta)\begin{cases}k^{p-1+\delta}, & \text{if } p\geq 0\\ \\ k^{-1}, & \text{if } p<0.\end{cases} \tag{4.3}$$

Proof: If $p<0$, then H has an integrable derivative, so (4.3) holds. Thus, we can assume $0\leq p<1$, and use the same ideas as in the proof of Lemma 4 of Fox and Taqqu [5]. Formula (4.1) on p. 45 of Zygmund [14] states that

$$|\widehat{H}(k)|\leq \frac{1}{\alpha}\omega_1\left(\frac{\pi}{k};H\right), \tag{4.4}$$

where ω_1 is the integral modulus of continuity:

$$\omega_1(x;H)=\sup_{0<h\leq x}\frac{1}{2\pi}\int_{-\pi}^{\pi}|H(\lambda+h)-H(\lambda)|d\lambda. \tag{4.5}$$

To determine $\omega_1(x;H)$, we split the integral in (4.5) as follows:

$$\int_{-\pi}^{\pi}|H(\lambda+h)-H(\lambda)|d\lambda=I_1(h)+I_2(h)+I_3(h),$$

where

$$I_1(h) = \int_{-\pi}^{-2h} \dots d\lambda, \quad I_2(h) = \int_{-2h}^{h} \dots d\lambda, \quad I_3(h) = \int_{h}^{\pi} \dots d\lambda.$$

By (4.2) and the Mean Value Theorem,

$$I_1(h) + I_3(h) \leq c \int_h^\pi h\lambda^{-(p+1+\delta)} d\lambda \leq ch^{1-p-\delta}. \tag{4.6}$$

By (4.1), $I_2(h) \leq 2\int_{-2h}^{2h} |H(\lambda)| d\lambda \leq c\int_0^h \lambda^{-(p+\delta)} d\lambda \leq ch^{1-p-\delta}$ for $p + \delta < 1$, and hence $I_2(h) \leq ch^{1-p-\delta}$ for any $\delta > 0$, with a different c. Combining this relation with (4.6), we get

$$\omega_1(x, H) \leq c(\delta) x^{1-p-\delta} \qquad (0 \leq p < 1), \tag{4.7}$$

which together with (4.4) yields (4.3).

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Piotr Kokoszka, University of Utah, Department of Mathematics, 233 Widtsoe Building, Salt Lake City, UT 84112, USA. *email:* kokoszka@math.utah.edu

Murad S. Taqqu, Boston University, Department of Mathematics, 111 Cummington Street, Boston, MA 02215, USA. *email:* murad@math.bu.edu