Solution for problem 9.5.46

We’re given the plane with equation \( x + 2y - 2z = 1 \) and asked to find the two planes (one on each side of the given one) which are parallel to and at a distance of 2 units from it.

Recall that to find a plane, we need a point and a normal vector; leaving aside the point for now, let’s start with the normal vector: this is easy to determine though, as parallel planes share the same normal vectors. Moreover, a normal vector can be readily read off the plane equation by looking at the coefficients of the \( x, y, z \) terms; this normal vector will be \( <1, 2, -2> \).

Going back to the point now, we think as follows: since the distance between the 2 planes is 2, we can obtain a point on the unknown plane by starting form a point on the given plane and travelling along the direction normal to the plane for 2 units.

Now, the equation \( x + 2y - 2z = 1 \) gives all points on the plane, so we should be able to find at least one. The way to go about it is to fix 2 of the coordinates (say \( x = y = 0 \), though nothing is restrictive about this choice) and find the 3\(^{rd} \) one; for \( x = y = 0 \) we find via the equation of the plane that \( z = -\frac{1}{2} \). Thus, the point \( (0, 0, -\frac{1}{2}) \) is a point on the given plane (as can easily be checked by plugging these numbers into the plane equation).

The normal vector we found is not a unit vector; a unit vector along the same direction is \( <\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}} > \). Thus, distance 2 between the 2 planes means that if we add the position vector of the point \( (0, 0, -\frac{1}{2}) \) and twice the unit normal vector, we’ll get a point on the 2\(^{nd} \) plane. The other plane will be found in a similar manner, by subtracting twice the unit normal vector from the position vector \( <0, 0, -\frac{1}{2} > \) (so that we travel in the opposite direction).

Therefore, a point on one of the 2 planes is the one with position vector
< 0,0,−\frac{1}{2} > +2 < \frac{1}{3},\frac{2}{3},−\frac{2}{3} >= < \frac{2}{3},\frac{4}{3},−\frac{11}{6} >, while a point on the other plane has position vector < 0,0,−\frac{1}{2} > −2 < \frac{1}{3},\frac{2}{3},−\frac{2}{3} >= < −\frac{2}{3},−\frac{4}{3},\frac{5}{6} >.

From these facts, we see that the equation for the plane passing through the 1st point is < 1,2,−2 > \cdot < x,y,z >= < 1,2,−2 > \cdot < \frac{2}{3},\frac{4}{3},−\frac{11}{6} >, i.e.
x + 2y - 2z = 7, while the equation for the plane passing through the 2nd point is < 1,2,−2 > \cdot < x,y,z >= < 1,2,−2 > \cdot < −\frac{2}{3},−\frac{4}{3},\frac{5}{6} >, i.e. x + 2y - 2z = −5.