Geometric interpretation of partial derivatives

When we calculate a partial derivative of a function of many variables, we fix all but one variable, and we differentiate the function that is obtained by varying the remaining variable.

Example. Consider the function $f(x, y) = 9 - x^2 - y^2$ at the point $(1, 2)$. In what direction, the x-direction or the y-direction, does $f(x, y)$ decrease most rapidly?

Now let's discuss the geometric significance of the two numbers that we obtain from the partials of $f(x, y)$ at $(1, 2)$. For example, we can use these numbers to calculate the equation of the tangent plane to the graph of $z = f(x, y)$ at the point $(1, 2, 4)$.

Definition. Suppose that the partial derivatives

$$\frac{\partial f}{\partial x}(a,b) \qquad \text{and} \qquad \frac{\partial f}{\partial y}(a,b)$$

exist at the point (a,b). Then let

$$\mathbf{T}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x}(a,b)\right)\mathbf{k} \qquad \text{and} \qquad \mathbf{T}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y}(a,b)\right)\mathbf{k}.$$

The normal vector for $f(x,y)$ at the point (a,b) is

$$\mathbf{N} = \mathbf{T}_y \times \mathbf{T}_x.$$

The equation for the tangent plane can be written as

$$z - c = \left(\frac{\partial f}{\partial x}(a,b)\right)(x-a) + \left(\frac{\partial f}{\partial y}(a,b)\right)(y-b),$$

where $c = f(a,b)$.

Linear approximation

The equation for the tangent plane can also be thought of as a linear approximation to $f(x, y)$ for (x, y) near (a, b).

Let $\Delta x = x - a$, $\Delta y = y - b$, and $\Delta z = f(x, y) - f(a, b)$. Then the equation for the tangent plane yields the linear approximation

$$\begin{aligned}\Delta z &= f(x, y) - f(a, b) \\ &= z - c \\ &\approx \left(\frac{\partial f}{\partial x}(a, b)\right) \Delta x + \left(\frac{\partial f}{\partial y}(a, b)\right) \Delta y.\end{aligned}$$

Example. The linear approximation of the function

$$f(x, y) = 9 - x^2 - y^2$$

centered at the point $(1, 2)$ is

$$f(x, y) - 4 \approx -2(x - 1) - 4(y - 2).$$

Another way to write this approximation is as

$$f(x, y) \approx 4 - 2\,\Delta x - 4\,\Delta y.$$

Second partials

Just as there is a second derivative for a function of one variable, there are four second partial derivatives for a function of two variables.

Example. Consider $g(x, y) = y \ln(xy) + y$ as discussed last class. We have already calculated that

$$\frac{\partial g}{\partial x} = \frac{y}{x} \qquad \text{and} \qquad \frac{\partial g}{\partial y} = 2 + \ln(xy).$$

Consequently,

$$\frac{\partial^2 g}{\partial x^2} = -\frac{y}{x^2} \qquad \text{and} \qquad \frac{\partial^2 g}{\partial y^2} = \frac{1}{y}.$$

What about the other two partials

$$\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) \qquad \text{and} \qquad \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right)?$$

Clairaut's Theorem. If $f(x, y)$ and its partial derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}$$

are continuous, then the order of partial differentiation is irrelevant. In other words,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

There is a link on the course web page to a discussion of an example for which the conclusion of Clairaut's Theorem does not hold. We will do our best to avoid such functions in this course.