Directional derivatives and the gradient

Today we will discuss directional derivatives and related theoretical consequences.

Example. Last class we discussed three directional derivatives for the function

$$f(x, y) = 2x^2 + y^2$$

at the point $(2, 1)$. We have

$$D_{\mathbf{i}} f(2,1) = \frac{\partial f}{\partial x}(2,1) = 8$$

$$D_{\mathbf{j}} f(2,1) = \frac{\partial f}{\partial y}(2,1) = 2$$

$$D_{\mathbf{u}} f(2,1) = 5\sqrt{2} \approx 7.1 \text{ for } \mathbf{u} = \tfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}).$$

Definition of a Directional Derivative. We start with the two-variable case. Define the "directional derivative of $f(x, y)$ at the point (a, b) in the $\mathbf{u}$ direction" by parametrizing the line through (a, b) using the direction vector $\mathbf{u}$. In other words, if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$, then the line is written in vector form as

$$\mathbf{L}(h) = (a\mathbf{i} + b\mathbf{j}) + h\mathbf{u}$$

or in parametric form as $(x, y) = (a + u_1 h, b + u_2 h)$.

Then we compute

$$D_{\mathbf{u}} f(a, b) = \lim_{h \to 0} \frac{f(x, y) - f(a, b)}{h}.$$

Using vector notation with $\mathbf{P} = a\mathbf{i} + b\mathbf{j}$, the same limit is written as

$$D_{\mathbf{u}} f(\mathbf{P}) = \lim_{h \to 0} \frac{f(\mathbf{P} + h\mathbf{u}) - f(\mathbf{P})}{h}.$$

This vector notation generalizes nicely to functions of three variables or, in fact, to any number of variables.

There are two animations on the web site that illustrate the meaning of this limit.

Computing directional derivatives. A directional derivative for $f(x, y)$ at the point (a, b) can be computed by applying the Chain Rule to the composition $f(\mathbf{L}(h))$. Note that the vector $\mathbf{u} = u_1\,\mathbf{i} + u_2\,\mathbf{j}$ is used only to indicate a direction, and consequently, it is *always a unit vector*. In other words, $u_1^2 + u_2^2 = 1$.

Theorem. $D_{\mathbf{u}} f(a, b) = [\nabla f(a, b)] \cdot \mathbf{u}$.

Example. Calculate the directional derivative of $f(x, y) = e^x \sin y$ at the point $(\ln 2, \pi/6)$ in the direction of $2\mathbf{i} + \mathbf{j}$.

This theorem tells us know how a function changes in any given direction, and in particular, it indicates directions of most rapid increase or decrease for the function. Since $\mathbf{u}$ is a unit vector,

$$\begin{aligned} D_{\mathbf{u}} f(a,b) &= (\nabla f(a,b)) \cdot \mathbf{u} \\ &= |\nabla f(a,b)|\,|\mathbf{u}|\,\cos\theta \\ &= |\nabla f(a,b)|\,\cos\theta, \end{aligned}$$

where θ is the angle between $\mathbf{u}$ and the gradient vector $\nabla f(a,b)$.

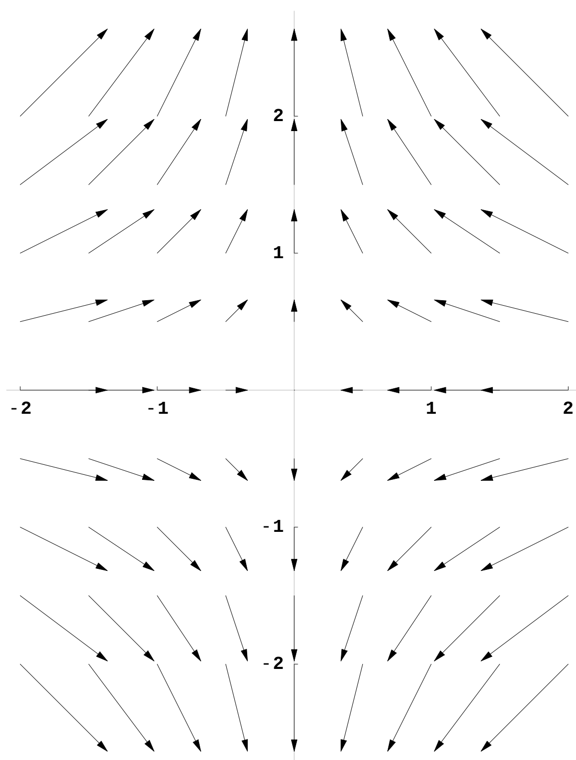
For what values of θ is this number largest? smallest? zero?

Theorem. The function $f(x,y)$ increases most rapidly in the direction of the gradient. The function is "constant" in directions perpendicular to the gradient.

Example. The gradient of $f(x,y) = \frac{1}{4}(y^2 - x^2)$ is

$$\nabla f(x,y) = \tfrac{1}{2}(-x\mathbf{i} + y\mathbf{j}).$$

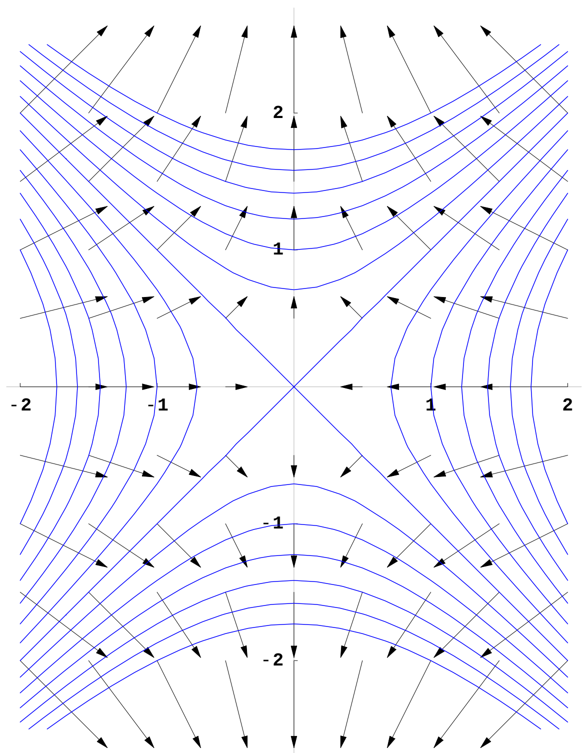
Here is the gradient *vector field* that it generates.



Another important theoretical application of the Chain Rule is the fact that the gradient vector is always perpendicular to its corresponding level set.

Theorem. The gradient vector $\nabla f(a, b)$ is perpendicular to the level set of level $f(a, b)$.

Example. Once again consider $f(x, y) = \frac{1}{4}(y^2 - x^2)$. Its level sets are hyperbolas that are perpendicular to the gradient vector field.



The same theorem holds for functions of any number of variables.

Example. Find an equation for the plane that is tangent to the surface

$$x^2 - y^2 + z^2 = 4$$

at the point $(2, -3, 3)$.