Evaluating double integrals

At the end of last class, we discussed how we can approximate the value of a double integral using Riemann sums. Today we discuss a process called iterated integration that we can use to calculate many double integrals exactly.

However, before we get started with iterated integration, I want to mention some of the basic properties of the double integral.

Theorem. The integral satisfies some basic but important properties.

1. $\displaystyle\iint\limits_R f(x,y) + g(x,y)\, dA = \iint\limits_R f(x,y)\, dA + \iint\limits_R g(x,y)\, dA$
2. $\displaystyle\iint\limits_R c\, f(x,y)\, dA = c \iint\limits_R f(x,y)\, dA$ if c is a constant
3. If $f(x,y) \geq g(x,y)$ for all (x,y) in R, then
$$\iint\limits_R f(x,y)\, dA \geq \iint\limits_R g(x,y)\, dA.$$

Iterated integration

Example. Consider
$$\iint\limits_R \frac{y}{3}\, dA$$
where $R = \{(x,y)\,|\, 0 \leq x \leq 2,\ 0 \leq y \leq 3\}$.

We can generalize this slicing technique to obtain a method for calculating double integrals. Consider a positive function $f(x, y)$ and a rectangle

$$R = \{(x, y) \mid a \leq x \leq b,\ c \leq y \leq d\}$$

in the xy-plane. We can calculate the volume of the solid determined by R and $f(x, y)$ using x-slices or y-slices.

Pictures of x-slices and y-slices are available on the course web site and in your textbook as Figures 1 and 2 on pp. 850–851.

The basic idea is to compute the volume of the solid in question by integrating the areas of the slices. For example, suppose that we slice up the solid using y-slices. Let $A(y)$ denote the area of a y-slice. Then

$$\iint\limits_R f(x, y)\, dA = \int_c^d A(y)\, dy.$$

Moreover, for any given y, the area of the y-slice is

$$A(y) = \int_a^b f(x, y)\, dx.$$

We obtain an *iterated integral* that yields the volume of the solid. That is,

$$\iint\limits_R f(x, y)\, dA = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy.$$

We compute the inside integral treating y as a constant ("partial integration"), and then we compute the outside integral which only depends on y.

Of course, it also possible to use x-slices rather than y-slices.

Theorem. (Fubini's Theorem) If $f(x, y)$ is continuous on the rectangle

$$R = \{(x, y) \mid a \leq x \leq b,\ c \leq y \leq d\},$$

then

$$\begin{aligned} \iint\limits_R f(x, y)\, dA &= \int_a^b \int_c^d f(x, y)\, dy\, dx \\ &= \int_c^d \int_a^b f(x, y)\, dx\, dy. \end{aligned}$$

Let's return to the example discussed earlier.

Example. Consider

$$\iint\limits_R \frac{y}{3}\, dA$$

where

$$R = \{(x, y) \,|\, 0 \le x \le 2,\ 0 \le y \le 3\}.$$

The calculation that we did involved x-slices.

We can also calculate the integral using y-slices.

Example. Consider the double integral

$$\iint_R x\cos(xy)\, dA$$

where

$$R = \{(x, y) \,|\, 0 \leq x \leq \pi/4,\ 0 \leq y \leq 2\}.$$

Example. Calculate the average value of the function

$$f(x, y) = xe^{xy}$$

over the rectangle

$$R = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 2\}.$$