Parametric Surfaces and 3D Coordinate Systems

A parameterized surface in space is a set of points in space described by a (position) vector-valued function of the form

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k},$$

where the functions $x(u,v)$, $y(u,v)$, and $z(u,v)$ are defined on some region in the uv-plane.

Example. The elliptic cylinder

$$x^2 + \frac{y^2}{4} = 1$$

is parameterized by the function

$$\mathbf{r}(\theta, z) = (\cos\theta)\,\mathbf{i} + (2\sin\theta)\,\mathbf{j} + z\,\mathbf{k}.$$

In addition to rectangular coordinates in space, there are two other coordinate systems that are used frequently.

Cylindrical Coordinates

Cylindrical coordinates consist of polar coordinates in the xy-plane along with the usual rectangular coordinate z. Unlike polar coordinates, we often restrict our attention to the situation where $r \geq 0$.

Example. Use cylindrical coordinates to parameterize the hyperboloid of one sheet

$$x^2 + y^2 - z^2 = 1.$$

Spherical coordinates

Another 3D coordinate system that is often convenient to use is the spherical coordinate system.

Example. Parameterize the unit sphere using spherical coordinates.