Two applications of the double integral

Double integrals are used to compute many different types of quantities—not just area and volume. Today we will discuss two different applications.

Centroids of lamina

Consider a flat plate (a lamina) with uniform density that occupies a region R in the plane. The center of mass of such an object is called its centroid. We can use double integrals to compute the centroid $(\overline{x}, \overline{y})$. The formulas are

$$\overline{x} = \frac{\displaystyle\iint_R x\, dA}{\displaystyle\iint_R dA} \qquad \text{and} \qquad \overline{y} = \frac{\displaystyle\iint_R y\, dA}{\displaystyle\iint_R dA}.$$

I am not going to go through the derivations of these formulas, but you can read about them in Section 12.5 of your text (pp. 868–871). The discussion in Section 12.5 includes a density function that varies over the region, but I am assuming that this density function is constant. In this case, we get the formulas given above. (You should check me on this.)

Your textbook also has information about moments and centers of mass on pp. 481–484. It is a good exercise for you to derive the formulas given in box 11 on p. 484 using double integrals.

Surface area

Another important application of double integrals is the calculation of surface area. For example, suppose we want to calculate the area of the saddle, that is, the area of the surface $z = y^2 - x^2$ over some region in the xy-plane such as the unit disk

$$R = \{(x, y) \mid x^2 + y^2 \leq 1\}.$$

Rather than derive a formula for surface area, we start with a definition.

Definition. Let S be the surface $z = f(x, y)$ where the points (x, y) come from a given region R in the xy-plane. Then

$$\text{Area}(S) = \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\; dA.$$

We make this formula a definition because there are many ways to derive wrong formulas, and all of these ways involve very subtle errors. An advanced course in analysis would explain why this formula is correct and why many others are wrong, but I would like to give a brief justification to motivate the formula.

Example. Calculate the area of that portion of the plane $z = 2 - x - 2y$ that lies in the first octant.

Example. Let's find the area of the portion of the saddle $z = y^2 - x^2$ that projects onto the unit circle $R = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

The integral formula for surface area can be expressed in differential notation. If we use the variable S to represent surface area, then the differential formulation of our integral formula is

$$dS = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\; dA$$

if the surface is the graph of $z = f(x, y)$. We should not forget that this is equivalent to the formula

$$dS = |\mathbf{N}|\; dA$$

where the vector

$$\mathbf{N} = \left(\frac{\partial z}{\partial x}\right)\mathbf{i} + \left(\frac{\partial z}{\partial y}\right)\mathbf{j} - \mathbf{k}$$

is the normal vector to the surface.

Your textbook also has a more general definition of surface area for parametric surfaces (Definition 4 on p. 879). You should convince yourself that the definition that we discussed today is consistent with that formula.