More on using cylindrical and spherical coordinates

Last class we briefly discussed the Marilyn Vos Savant problem.

Marilyn Vos Savant problem: Pick a sphere ... any sphere. Bore a perfect hole through the center in such a way that the remaining solid is 6 inches high. What is the volume that remains?

If R is the radius of the sphere and $2h$ is the height that remains ($h = 3$ in the statement above), then the volume is

$$\int_0^{2\pi} \int_{\sqrt{R^2-h^2}}^{R} \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r \, dz \, dr \, d\theta.$$

Converting integrals to spherical coordinates

Whenever we use a new coordinate system to do a triple integral, we need a volume adjustment factor. In other words, we need to determine how the volume of a "cube" in the new coordinates converts to a volume in rectangular coordinates.

Consider a "spherical cube" with sides $\Delta\rho$, $\Delta\phi$, and $\Delta\theta$.

If we look at various cubes all with the same $\Delta\rho$, $\Delta\phi$, and $\Delta\theta$, we see that their volumes vary greatly. Let's see if we can approximate this variation.

In summary, our volume conversion factor for spherical coordinates is

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Example. Let R be the region inside the first octant as well as inside the sphere of radius 2 centered at the origin. Let's evaluate

$$\iiint\limits_R e^{(x^2+y^2+z^2)^{3/2}} \, dV.$$

Vector analysis and vector fields

Now we are ready to pursue the final topic of the course—the calculus of vector fields. Before I start introducing the new integrals that we will study, I want to be sure that you understand the notation involved and the typical applications. Therefore, let's start with a few examples that I will use throughout the remainder of the course.

Example. Gravitational Vector Field of Earth: Any mass in space is attracted to the Earth by a gravitational force and Newton's law specifies the strength of the attraction.

Newton's Law:

> Gravitational attraction of a particle of mass m which is a distance d from the center of the Earth is
> $$\frac{GMm}{d^2}$$
> where G is the gravitational constant and M is the mass of the Earth.

Example. Steady fluid flow: For example, consider water slowly flowing through a pipe. The motion of a particle can be represented by a streamline. In other words, it is realistic to assume that the velocity of a particle at a certain point in the pipe is independent of time. Then the fluid flow is entirely determined by the velocity field $\mathbf{V}$.