Two comments related to vector fields

At the end of class on Friday, we discussed a special type of vector field.

**Definition.** A gradient vector field is one that is the gradient of a function. That is,

\[ \mathbf{F} = \nabla f. \]

For a gradient vector field \( \mathbf{F}(x, y) \) in the \( xy \)-plane, we have

\[ \mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \left( \frac{\partial f}{\partial x} \right) \mathbf{i} + \left( \frac{\partial f}{\partial y} \right) \mathbf{j}. \]

For a gradient vector field \( \mathbf{F}(x, y, z) \) in \( xyz \)-space, we have

\[ \mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k} = \left( \frac{\partial f}{\partial x} \right) \mathbf{i} + \left( \frac{\partial f}{\partial y} \right) \mathbf{j} + \left( \frac{\partial f}{\partial z} \right) \mathbf{k}. \]

Keep the following figures in mind when dealing with gradient vector fields.

**Example.** (October 21 handout) Consider the function \( f(x, y) = \frac{1}{4}(y^2 - x^2) \) and its gradient vector field

\[ \mathbf{F}(x, y) = \nabla f(x, y) = \left( -\frac{x}{2} \right) \mathbf{i} + \left( \frac{y}{2} \right) \mathbf{j}. \]

The figure on the left is the gradient vector field alone while the figure on the right has the field superimposed on the level sets of \( f(x, y) \).

Last Friday’s handout also included a matching exercise that we did not get a chance to discuss. You should try it on your own.
Path integrals

Your textbook calls these integrals “line integrals along a curve” and sometimes you will also see the term “line integrals with respect to arc length.”

Recall that all of our integrals so far involved summing up a function.

**Definition.** A path integral of a function $f(x, y)$ along a curve $C$ in the $xy$-plane is an integral of the form

$$\int_C f(x, y) \, ds,$$

where $ds$ represents the differential of arc length (see the September 26 handout).

How do we compute such an integral?

If we parameterize the curve using a vector-valued function $\mathbf{r}(t)$ with $a \leq t \leq b$, then

$$\int_C f(x, y) \, ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| \, dt.$$

There is a similar definition for functions of three variables and curves in space.
Example. Consider a curved fence that sits on the ground along a parabolic path of the form

$$\mathbf{r}(t) = ti + t^2j.$$ 

Suppose that its height is $h(x,y) = x + \sqrt{y}$. What is the surface area for the part of the fence that sits along the parabolic arc from $(0,0)$ to $(2,4)$?
Path integrals are independent of parameterization. In other words, if both $r_1(t)$ and $r_2(t)$ trace out the same curve $C$, then

$$\int_C f(x, y) \, ds$$

can be calculated using either $r_1(t)$ or $r_2(t)$.

**Example.** Consider the path integral

$$\int_C x \, ds$$

where $C$ is the line segment from $(0,0)$ to $(1,1)$. There are many ways to parameterize $C$. For example, consider the three parameterizations

$$r_1(t) = ti + tj$$
$$r_2(t) = t^2i + t^2j$$
$$r_3(t) = (1 - t^2)i + (1 - t^2)j,$$

where $0 \leq t \leq 1$ in all three cases.
Example. Consider a semicircular piece of wire of radius $R$. Find its center of mass.