Two comments related to vector fields

At the end of class on Friday, we discussed a special type of vector field.

Definition. A gradient vector field is one that is the gradient of a function. That is,

$$\mathbf{F} = \nabla f.$$

For a gradient vector field $\mathbf{F}(x, y)$ in the xy-plane, we have

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \left(\frac{\partial f}{\partial x}\right)\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)\mathbf{j}.$$

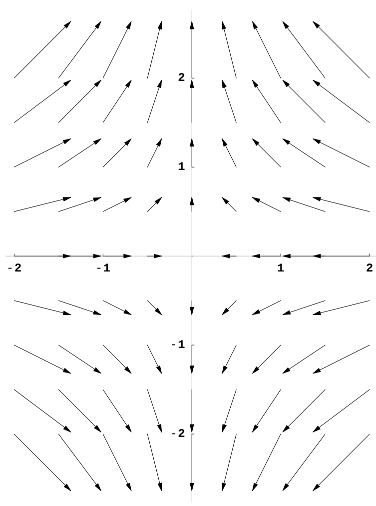
For a gradient vector field $\mathbf{F}(x, y, z)$ in xyz-space, we have

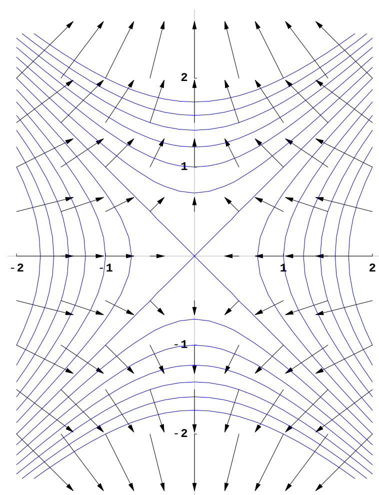
$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k} = \left(\frac{\partial f}{\partial x}\right)\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)\mathbf{j} + \left(\frac{\partial f}{\partial z}\right)\mathbf{k}.$$

Keep the following figures in mind when dealing with gradient vector fields.

Example. (October 21 handout) Consider the function $f(x, y) = \frac{1}{4}(y^2 - x^2)$ and its gradient vector field

$$\mathbf{F}(x, y) = \nabla f(x, y) = \left(-\frac{x}{2}\right)\mathbf{i} + \left(\frac{y}{2}\right)\mathbf{j}.$$





The figure on the left is the gradient vector field alone while the figure on the right has the field superimposed on the level sets of $f(x, y)$.

Last Friday's handout also included a matching exercise that we did not get a chance to discuss. You should try it on your own.

Path integrals

Your textbook calls these integrals "line integrals along a curve" and sometimes you will also see the term "line integrals with respect to arc length."

Recall that all of our integrals so far involved summing up a function.

Definition. A path integral of a function $f(x,y)$ along a curve C in the xy-plane is an integral of the form

$$\int_C f(x,y)\,ds,$$

where ds represents the differential of arc length (see the September 26 handout).

How do we compute such an integral?

If we parameterize the curve using a vector-valued function $\mathbf{r}(t)$ with $a \le t \le b$, then

$$\int_C f(x,y)\,ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt.$$

There is a similar definition for functions of three variables and curves in space.

Example. Consider a curved fence that sits on the ground along a parabolic path of the form

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}.$$

Suppose that its height is $h(x, y) = x + \sqrt{y}$. What is the surface area for the part of the fence that sits along the parabolic arc from $(0, 0)$ to $(2, 4)$?

Path integrals are independent of parameterization. In other words, if both $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ trace out the same curve C, then

$$\int_C f(x,y)\,ds$$

can be calculated using either $\mathbf{r}_1(t)$ or $\mathbf{r}_2(t)$.

Example. Consider the path integral

$$\int_C x\,ds$$

where C is the line segment from $(0,0)$ to $(1,1)$. There are many ways to parameterize C.

For example, consider the three parameterizations

$$\begin{aligned}\mathbf{r}_1(t) &= t\mathbf{i} + t\mathbf{j}\\ \mathbf{r}_2(t) &= t^2\mathbf{i} + t^2\mathbf{j}\\ \mathbf{r}_3(t) &= (1-t^2)\mathbf{i} + (1-t^2)\mathbf{j},\end{aligned}$$

where $0 \le t \le 1$ in all three cases.

Example. Consider a semicircular piece of wire of radius R. Find its center of mass.