Using polar coordinates to calculate double integrals

Some important double integrals involve a lot of radial symmetry. Consequently, they are easier to evaluate if we use polar coordinates.

**Two basic facts.**

1. \[ \int_0^1 \sqrt{1 - u^2} \, du = \frac{\pi}{4} \]

2. The volume of a solid sphere of radius 1 is \( \frac{4}{3}\pi \).

**Nonexample.** Let’s try to calculate the volume of a hemisphere of radius 1 using polar coordinates.

What’s wrong with this calculation?
The problem with this calculation is that transforming to polar coordinates distorts area.

Recall the formula for area inside a polar curve (see the Sept 26 handout).

\[
\text{area} = \int_{\theta_1}^{\theta_2} \frac{(r(\theta))^2}{2} \, d\theta.
\]

Rewriting this formula as a double integral suggests the correct approach for converting double integrals to polar coordinates.
Differential form for area in terms of polar coordinates

\[ dA = r \, dr \, d\theta = r \, d\theta \, dr \]

**Example.** Let’s recalculate the volume of the hemisphere using this area adjustment factor.
Example. Calculate the volume of the region that is inside the sphere

\[ x^2 + y^2 + z^2 = 4 \]

and above the plane \( z = 1 \).