Conservative vector fields and potential functions

Last class we saw that the Fundamental Theorem of Calculus for line integrals simplified the calculation of line integrals.

Theorem. (Fundamental Theorem of Calculus for line integrals) If $\mathbf{F} = \nabla f$, then

$$\int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

So how can we tell if a given vector field is the gradient of a function? In other words, how can we tell if a vector field is conservative, and how can we construct a potential function if it is?

Necessary condition:

Suppose that $\mathbf{F}(x, y) = \nabla f(x, y)$, that is, suppose

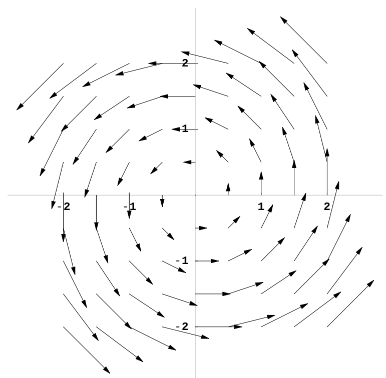
$$\mathbf{F}(x, y) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}.$$

Necessary condition: If the vector field $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$, then $\mathbf{F}(x,y)$ has a potential function only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Example. Consider the merry-go-round vector field

$$\mathbf{F}(x,y) = -y\mathbf{i} + x\mathbf{j}.$$



Sufficient condition: Is the condition

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

enough to guarantee that $\mathbf{F}(x,y)$ has a potential function? Sometimes ...

Theorem. Suppose that $P(x,y)$ and $Q(x,y)$ are defined and are continuous for all (x,y). If

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

for all (x,y), then $\mathbf{F}(x,y)$ has a potential function.

How do we find $f(x, y)$?

Example. Consider the vector field

$$\mathbf{F}(x, y) = (1 + e^y)\ \mathbf{i} + \left(xe^y + y^2\right)\ \mathbf{j}.$$

Example. Consider the vector field

$$\mathbf{F}(x, y) = \left(x^2 + y\right)\, \mathbf{i} + \left(\sin x + y^2\right)\, \mathbf{j}.$$

Green's Theorem

Green's Theorem relates line integrals of vector fields in the xy-plane to double integrals.

What is a positively-oriented, simple, closed curve in the plane?

Theorem. (Green's Theorem) Let C be a positively-oriented, simple, closed curve in the plane and let D denote the region it encloses. Then

$$\oint_C P\,dx + Q\,dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

Example. Let C be the perimeter of the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$. Calculate

$$\oint_C x\,dx + xy\,dy.$$

Note: If $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ has a potential function, then

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0.$$

Example. Compute the line integral

$$\oint -y^3\,dx + x^3\,dy$$

over the unit circle in the positively-oriented direction.