Divergence of vector fields

**Definition.** Given a vector field in space

\[ \mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k}, \]

the divergence of \( \mathbf{F} \) is the scalar field (scalar function) defined by

\[ \text{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}. \]

Shorthand notation: \( \text{div} \mathbf{F} = \nabla \cdot \mathbf{F} \).

**Example.** Calculate \( \text{div} \mathbf{F} \) for \( \mathbf{F}(x, y, z) = x^2y \mathbf{i} + yz^2 \mathbf{j} + x^2z \mathbf{k} \).

To understand what divergence measures in the case of our velocity field of a planar fluid, we consider a different path integral. Given a simple, closed curve \( C \) in the plane, consider the path integral

\[ \oint_C \mathbf{F} \cdot \mathbf{n} \, ds, \]

where \( \mathbf{n} \) is a unit normal vector to \( C \) that points outside the region enclosed by \( C \).

**Theorem.** (Another vector version of Green’s Theorem) Let \( C \) be a positively-oriented, simple, closed curve in the \( xy \)-plane and let \( D \) be the region that is enclosed by \( C \). Then

\[ \oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \, dA. \]

This identity justifies the name “divergence.”
Surface integrals

A path integral is how we “add up” a function over a curve in the plane or in space. Similarly, a surface integral is how we add up a function \( f(x, y, z) \) over a surface in space.

We want an integral such that

\[
\iint_S f(x, y, z) \, dS = \text{the “sum” of } f(x, y, z) \text{ over the surface } S
\]

\[= \text{(the average of } f(x, y, z) \text{ over } S) (\text{surface area}(S)).\]

Special case: Assume that the surface \( S \) is the graph of a function \( g(x, y) \) over a region \( R \) in the \( xy \)-plane. In this case, the differential of surface area is

\[
dS = \sqrt{1 + \left( \frac{\partial g}{\partial x} \right)^2 + \left( \frac{\partial g}{\partial y} \right)^2} \, dA.
\]

This differential can also be expressed as

\[
dS = |N| \, dA,
\]

where the vector

\[
N = \left( \frac{\partial g}{\partial x} \right) i + \left( \frac{\partial g}{\partial y} \right) j - k
\]

is the normal vector to the surface (see the November 7 and November 9 handouts).

Consequently, we have

\[
\iint_S f(x, y, z) \, dS = \iint_R f(x, y, g(x, y)) \sqrt{1 + \left( \frac{\partial g}{\partial x} \right)^2 + \left( \frac{\partial g}{\partial y} \right)^2} \, dA.
\]
Example. Let $S$ be the portion of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 2$. Let’s find its center of mass.

If the center of mass is denoted $(\bar{x}, \bar{y}, \bar{z})$, then we know that $\bar{x} = \bar{y} = 0$. Also,

$$
\bar{z} = \frac{\iint z \, dS}{\text{area}(S)}.
$$