Flux integrals

Given a vector field in space

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

and a surface S, the flux of $\mathbf{F}$ across S is the surface integral

$$\iint_S (\mathbf{F} \cdot \mathbf{n}) \, dS,$$

where $\mathbf{n}$ is the unit normal vector at each point of S.

Special case: Assume that the surface S is the graph of a function $g(x,y)$ over a region R in the xy-plane. In this case, the differential of surface area is

$$dS = \sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} \, dA.$$

This differential can also be expressed as

$$dS = |\mathbf{N}| \, dA,$$

where the vector

$$\mathbf{N} = \left(\frac{\partial g}{\partial x}\right)\mathbf{i} + \left(\frac{\partial g}{\partial y}\right)\mathbf{j} - \mathbf{k}$$

is the normal vector to the surface (see the November 7 and November 9 handouts).

In this case, we compute the flux of $\mathbf{F}$ across S by

$$\iint_S (\mathbf{F}\cdot\mathbf{n})\, dS = \pm \iint_R \left(\mathbf{F}\cdot\frac{\mathbf{N}}{|\mathbf{N}|}\right) |\mathbf{N}|\, dA$$

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Example. Consider the surface S that is the boundary of the solid that is bounded by the paraboloid

$$z = 4 - x^2 - y^2$$

and the xy-plane. Also, consider the vector field

$$\mathbf{F}(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 3\mathbf{k}.$$

Let's calculate the flux across S in the direction of the outer normal.

The surface S splits nicely into two surfaces. Let's denote the part of S given by the paraboloid as S_1 and the part of S that lies in the xy-plane as S_2.

Flux across S_1:

Flux across S_2: