A little more about lines

Last class we used vector addition and scalar multiplication to describe a line in space. Given two points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ on the line, we first calculate a direction vector $\mathbf{D}$ for the line. It is

$$\mathbf{D} = (b_1 - a_1)\mathbf{i} + (b_2 - a_2)\mathbf{j} + (b_3 - a_3)\mathbf{k}.$$

If $\mathbf{P}_1$ is a position vector representing any point on the line, then a (position) vector equation for the line is

$$\mathbf{L}(t) = \mathbf{P}_1 + t\mathbf{D}.$$

Example. Find a vector equation for the line containing $(1, 1, 1)$ and $(1 + \sqrt{2}, 2, 2)$.

A direction vector is $\mathbf{D} = \sqrt{2}\,\mathbf{i} + \mathbf{j} + \mathbf{k}$. So

$$\begin{aligned} \mathbf{L}(t) &= \mathbf{i} + \mathbf{j} + \mathbf{k} + t(\sqrt{2}\,\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= (1 + \sqrt{2}\,t)\mathbf{i} + (1 + t)\mathbf{j} + (1 + t)\mathbf{k}. \end{aligned}$$

There is also a parametric form:

$$\begin{cases} x(t) = 1 + \sqrt{2}\,t \\ y(t) = 1 + t \\ z(t) = 1 + t \end{cases}$$

We can also give an "equation" for the line that does not contain the parameter t. We solve for t and get

$$\frac{x - 1}{\sqrt{2}} = y - 1 = z - 1.$$

These equations are called a *symmetric form for the line.*

Here's another example that has a slightly different symmetric form:

Example. Find an equation of the line through $(2, 4, 6)$ and $(1, 6, 6)$.

Dot Product

Given two vectors $\mathbf{A}$ and $\mathbf{B}$, their **dot product** is defined to be

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\ |\mathbf{B}|\ \cos\theta,$$

where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$.

Important: Note that we start with two vectors and end up with a scalar. Therefore, we cannot take the dot product of three vectors in a row. Before we do a few examples, I would like to make a few observations.

1. Suppose $|\mathbf{A}|,\ |\mathbf{B}| \neq 0$. Then the two vectors $\mathbf{A}$ and $\mathbf{B}$ are perpendicular (orthogonal) $\iff\ \mathbf{A} \cdot \mathbf{B} = 0$.

2. There is a close relationship between projections and the dot product.

$$\mathrm{comp}_{\mathbf{A}}(\mathbf{B}) = |\mathbf{B}|\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}|} \qquad \mathrm{proj}_{\mathbf{A}}(\mathbf{B}) = (|\mathbf{B}|\cos\theta)\frac{\mathbf{A}}{|\mathbf{A}|} = \frac{(\mathbf{A} \cdot \mathbf{B})}{|\mathbf{A}|^2}\mathbf{A}$$

3. We can derive an algebraic formula for the dot product using the law of cosines. Let $\mathbf{C} = \mathbf{B} - \mathbf{A}$.

The law of cosines is

$$|\mathbf{C}|^2 = |\mathbf{A}|^2 + |\mathbf{B}|^2 - 2|\mathbf{A}||\mathbf{B}|\cos\theta.$$

Therefore,

$$\begin{aligned} 2(\mathbf{A} \cdot \mathbf{B}) &= |\mathbf{A}|^2 + |\mathbf{B}|^2 - |\mathbf{C}|^2 \\ &= a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 \\ &\quad -(b_1 - a_1)^2 - (b_2 - a_2)^2 - (b_3 - a_3)^2. \end{aligned}$$

So $2(\mathbf{A}\cdot\mathbf{B}) = 2(a_1b_1 + a_2b_2 + a_3b_3)$, and therefore,

$$\mathbf{A}\cdot\mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3.$$

This equation gives us an algebraic formula with geometric applications.

Examples. Let $\mathbf{A} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{B} = 3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$.

1. Compute the angle between $\mathbf{A}$ and $\mathbf{B}$.

2. Compute the length of the projection of $\mathbf{A}$ in the direction of $\mathbf{B}$.

Now let's use the dot product to derive the triangle inequality. Recall that

$$|\mathbf{A}|^2 = a_1^2 + a_2^2 + a_3^2,$$

so

$$|\mathbf{A}|^2 = \mathbf{A}\cdot\mathbf{A}.$$

We can therefore derive

$$\begin{aligned}
|\mathbf{A}+\mathbf{B}|^2 &= (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) \\
&= \mathbf{A}\cdot\mathbf{A} + 2(\mathbf{A}\cdot\mathbf{B}) + \mathbf{B}\cdot\mathbf{B} \\
&= |\mathbf{A}|^2 + 2|\mathbf{A}||\mathbf{B}|\cos\theta + |\mathbf{B}|^2 \\
&\le |\mathbf{A}|^2 + 2|\mathbf{A}||\mathbf{B}| + |\mathbf{B}|^2 \\
&\le (|\mathbf{A}| + |\mathbf{B}|)^2
\end{aligned}$$

and arrive at triangle inequality

$$|\mathbf{A}+\mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}|.$$

The length function $|\mathbf{A}|$ has four important properties:

1. $|\mathbf{A}| \geq 0$ for all $\mathbf{A}$
2. $|\mathbf{A}| = 0 \iff \mathbf{A} = \mathbf{0}$
3. $|r\mathbf{A}| = |r||\mathbf{A}|$
4. $|\mathbf{A}+\mathbf{B}| \leq |\mathbf{A}| + |\mathbf{B}|$

The dot product has the following properties:

1. $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$
2. $\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2$
3. $r(\mathbf{A} \cdot \mathbf{B}) = (r\mathbf{A}) \cdot \mathbf{B} = \mathbf{A} \cdot r\mathbf{B}$
4. $\mathbf{A} \cdot (\mathbf{B}+\mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$
5. $|\mathbf{A} \cdot \mathbf{B}| \leq |\mathbf{A}||\mathbf{B}|$ (which follows from $\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$)

Now let's apply what we have learned to a problem involving lines.

Example. Let ℓ_1 be line through $(1,0,0)$ and $(1,2,2)$. Let ℓ_2 be the line through $(1,1,1)$ and $(1+\sqrt{2},2,2)$. Do these lines intersect? What is the angle of intersection?