Last class we discussed the dot product, but we did not have enough time to cover everything that you should know about this important operation. You should study Section 9.3 in the textbook, and you may want to review the parts of handout from last class that were not covered in class.

An important application of the dot product: equations of planes

Suppose we are given a point $P = (x_0, y_0, z_0)$ on a plane and a vector $\mathbf{N}$ normal to the plane.

Then any other point $Q = (x, y, z)$ on the plane satisfies the equation

$$\mathbf{PQ} \cdot \mathbf{N} = 0.$$

Example. Find an equation for the plane that contains the point $(3, -2, 5)$ and is perpendicular to the line

$$\frac{x-2}{3} = \frac{1-y}{6} = \frac{z+2}{2}.$$

The cross product

For vectors in space, there is another "multiplication" that corresponds to the notion of torque in physics. This operation is called the cross product of two vectors.

Definition. Given two vectors $\mathbf{A}$ and $\mathbf{B}$, we define a new vector $\mathbf{A} \times \mathbf{B}$ by

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}|(\sin\theta)\,\mathbf{N}$$

where $\mathbf{N}$ is the unit vector obtained by applying the right-hand rule to $\mathbf{A}$ and $\mathbf{B}$.

Examples.

1. $\mathbf{i} \times \mathbf{j} =$
2. $\mathbf{j} \times \mathbf{k} =$
3. $\mathbf{k} \times \mathbf{i} =$

In fact, you can remember the cross products of these vectors using a "triangle."

Properties of Cross Product

1. $\mathbf{B} \times \mathbf{A} = -\mathbf{A} \times \mathbf{B}$ because direction$(\mathbf{B} \times \mathbf{A}) = -$direction$(\mathbf{A} \times \mathbf{B})$
2. If $\mathbf{A}, \mathbf{B} \neq \mathbf{0}$, then $\mathbf{A} \times \mathbf{B} = \mathbf{0} \iff \mathbf{A}$ and $\mathbf{B}$ are parallel, i.e., direction$(\mathbf{A}) = \pm$ direction$(\mathbf{B})$.
3. $r(\mathbf{A} \times \mathbf{B}) = (r\mathbf{A}) \times \mathbf{B} = \mathbf{A} \times (r\mathbf{B})$
4. $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C})$
5. $(\mathbf{A} \times \mathbf{B}) \perp \mathbf{A}$ and $(\mathbf{A} \times \mathbf{B}) \perp \mathbf{B}$
6. $|\mathbf{A} \times \mathbf{B}|^2 = |\mathbf{A}|^2|\mathbf{B}|^2 - (\mathbf{A} \cdot \mathbf{B})^2$.

Before we derive a formula for $\mathbf{A} \times \mathbf{B}$ in terms of the coordinates of $\mathbf{A}$ and $\mathbf{B}$, here's one last geometric observation.

It is often useful to have a formula for the cross product in terms of its coordinates. However, sometimes we are better off using only the properties mentioned above.

Example. Compute $(4\mathbf{i} + \mathbf{j}) \times (7\mathbf{i} + 2\mathbf{k})$.

To get a general formula, we can repeat the same type of calculation on

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}).$$

The result is

$$(a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

There is a handy way of remembering this formula by using the determinant of a 3×3 matrix. Recall that we can calculate the determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

by writing it as

$$\begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix}$$

Then we get

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (aei) + (bfg) + (cdh)$$
$$-(ceg) - (afh) - (bdi)$$

Using this formula, we can write

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Example. We use the determinant formula to calculate $(4\mathbf{i} + \mathbf{j}) \times (7\mathbf{i} + 2\mathbf{k})$.