Last class we defined the cross product. Now we have two ways to think about it.

**Geometric definition.** Given two vectors \( \mathbf{A} \) and \( \mathbf{B} \), we define a new vector \( \mathbf{A} \times \mathbf{B} \) by

\[
\mathbf{A} \times \mathbf{B} = |\mathbf{A}||\mathbf{B}|(\sin \theta) \mathbf{N}
\]

where \( \mathbf{N} \) is the unit vector obtained by applying the right-hand rule to \( \mathbf{A} \) and \( \mathbf{B} \).

**Algebraic definition.** The cross product

\[
(a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}) \times (b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}) = (a_2 b_3 - a_3 b_2) \mathbf{i} + (a_3 b_1 - a_1 b_3) \mathbf{j} + (a_1 b_2 - a_2 b_1) \mathbf{k}.
\]

There is a handy way of remembering this formula that uses the determinant of a \( 3 \times 3 \) matrix.

We can calculate the determinant

\[
\begin{vmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
\beta_1 & \beta_2 & \beta_3 \\
\gamma_1 & \gamma_2 & \gamma_3
\end{vmatrix}
\]

in one of two ways.

1. Expansion by minors: The determinant

\[
\begin{vmatrix}
a & b \\
c & d
\end{vmatrix} = ad - bc,
\]

and the determinant

\[
\begin{vmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
\beta_1 & \beta_2 & \beta_3 \\
\gamma_1 & \gamma_2 & \gamma_3
\end{vmatrix} = \alpha_1 \begin{vmatrix}\beta_2 & \beta_3 \\ \gamma_2 & \gamma_3 \end{vmatrix} - \alpha_2 \begin{vmatrix}\beta_1 & \beta_3 \\ \gamma_1 & \gamma_3 \end{vmatrix} + \alpha_3 \begin{vmatrix}\beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{vmatrix}.
\]

2. Sum of six terms, each a product of three numbers: We can also compute this determinant by writing it as

\[
\begin{vmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
\beta_1 & \beta_2 & \beta_3 \\
\gamma_1 & \gamma_2 & \gamma_3
\end{vmatrix} = (\alpha_1 \beta_2 \gamma_3) + (\alpha_2 \beta_3 \gamma_1) + (\alpha_3 \beta_1 \gamma_2) - (\alpha_3 \beta_2 \gamma_1) - (\alpha_1 \beta_3 \gamma_2) - (\alpha_2 \beta_1 \gamma_3)
\]

Then we get
Using this formula, we can write

\[
\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.
\]

**Example.** We use the determinant formula to calculate \((4\mathbf{i} + \mathbf{j}) \times (7\mathbf{i} + 2\mathbf{k})\).

Applications to Lines and Planes

We have discussed finding the equation of a plane if we are given a point \(P\) on the plane and a normal vector \(\mathbf{N}\). Now that we can compute the cross product of two vectors, we are able to find the equation of a plane determined in more familiar ways, e.g., determined by three noncollinear points or by two intersecting lines. We use the cross product to determine \(\mathbf{N}\).

**Example.** Find an equation for the plane that contains the three points \(P_1 = (1, 1, 1)\), \(P_2 = (2, -2, 2)\), and \(P_3 = (0, 2, 1)\).
Example. Find the equation of the plane that contains the two lines

\[ x = y = z \quad \text{and} \quad \frac{x - 1}{2} = \frac{y - 1}{3} = z - 1. \]

Why do these lines intersect? Where do they intersect? What are their direction vectors \( \mathbf{D}_1 \) and \( \mathbf{D}_2 \)?
Example. Let \( \ell_1 \) be line through \((1,0,0)\) and \((1,2,2)\). Let \( \ell_2 \) be the line through \((1,1,1)\) and \((1 + \sqrt{2},2,2)\). Do these lines intersect? What is the angle of intersection?