Vector-valued functions and parameterized curves

First, let's quickly review parameterized curves in the plane (see Section 1.7 of your text).

Definition. A *parametric curve* in the xy-plane is a pair of scalar functions

$$x = f(t)$$
$$y = g(t)$$

We trace out the curve by plotting all points of the form

$$\text{trace } = \{(f(t), g(t)) \,|\, \text{for all } t \text{ in the domains of } f \text{ and } g\}.$$

Example.

$$x(t) = 3t + 1$$
$$y(t) = 2t + 2$$

t	x	y
-1	-2	0
0	1	2
1	4	4
2	7	6

We can solve for t to get a nonparametric representation of the curve.

$$x = 3t + 1$$

$$x - 1 = 3t$$

$$\frac{x-1}{3} = t$$

Therefore, we have

$$y = 2\left(\frac{x-1}{3}\right) + 2$$

$$= \frac{2}{3}x - \frac{2}{3} + 2$$

$$= \frac{2}{3}x + \frac{4}{3}.$$

Remark. Any parameterized equation of the form

$$x = at + b$$
$$y = ct + d$$

is a line. We have already seen how all lines in the plane can be parameterized in this way.

Curves in space can be described in essentially the same manner as curves in the plane. Their parametric representation is given by three scalar-valued functions

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t).$$

Vector-valued functions

When we study curves in the plane or in space, it is often useful to employ vector techniques, and we do so by using vector-valued functions.

Given a parameterized curve in space of the form

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t),$$

we can combine these three functions to make one vector-valued function

$$\mathbf{P}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

The vector $\mathbf{P}(t)$ is often thought of as a position vector that varies with the parameter t.

Example 1. Let $\mathbf{L}(t) = (4 - t)\,\mathbf{i} + (5t - 1)\,\mathbf{j} + (3 + \frac{1}{2}t)\,\mathbf{k}$.

Example 2. Let $\mathbf{H}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k}$.

Example 3. Let $\mathbf{K}(t) = (2 + \cos \frac{3}{2}t) \cos t\,\mathbf{i} + (2 + \cos \frac{3}{2}t) \sin t\,\mathbf{j} + \sin \frac{3}{2}t\,\mathbf{k}$.

In one-dimensional calculus, we define the derivative of a scalar-valued function as the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

It is the limit of the change in $f(x)$ divided by the change in x. We can do the same for vector-valued functions.

Definition. Let $\mathbf{f}(t)$ be a vector-valued function. Then

$$\mathbf{f}'(t) = \lim_{h \to 0} \frac{\mathbf{f}(t+h) - \mathbf{f}(t)}{h}$$

What does this limit represent? First, let's consider the definition in terms of motion in space.

We see that the secants limit on a tangent vector. We divide by h to stop the vectors from shrinking to zero. As we shall see on Wednesday, there is another good reason for dividing by h.

Since we have this interesting vector associated to $\mathbf{f}(t)$, how do we compute it?

Theorem. Let $\mathbf{f}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{f}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

Example. Consider a curve that is very similar to the circular helix. Let

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + t\mathbf{k}.$$

Then

$$\mathbf{r}'(t) = (-\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + \mathbf{k}.$$

So the derivative at $t = \pi/3$ is

$$-\frac{\sqrt{3}}{2}\mathbf{i} + \mathbf{j} + \mathbf{k}.$$

This vector is tangent to the elliptical helix at $t = \pi/3$.

Example. Find the equation of the tangent line to the curve

$$\mathbf{r}(t) = e^t\,\mathbf{i} + 2\sin t\,\mathbf{j} + (t^2 - 2)\,\mathbf{k}.$$

at the point $(1, 0, -2)$.