More on vector-valued functions and parameterized curves

Last class we were talking about the derivative of a vector-valued function

$$\mathbf{f}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}.$$

Definition. Let $\mathbf{f}(t)$ be a vector-valued function. Then

$$\mathbf{f}'(t) = \lim_{h \to 0} \frac{\mathbf{f}(t+h) - \mathbf{f}(t)}{h}$$

As with any vector quantity, it is important to understand the meaning of its direction and length.

Since we have this interesting vector associated to $\mathbf{f}(t)$, how do we compute it?

Theorem. Let $\mathbf{f}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. Then $\mathbf{f}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$.

Example. Consider a curve that is very similar to the circular helix. Let

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + t\mathbf{k}.$$

Then

$$\mathbf{r}'(t) = (-\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + \mathbf{k}.$$

So the derivative at $t = \pi/3$ is

$$-\frac{\sqrt{3}}{2}\,\mathbf{i} + \mathbf{j} + \mathbf{k}.$$

This vector is tangent to the elliptical helix at $t = \pi/3$.

Example. Find the equation of the tangent line to the curve

$$\mathbf{r}(t) = e^t\,\mathbf{i} + 2\sin t\,\mathbf{j} + (t^2 - 2)\,\mathbf{k}$$

at the point $(1, 0, -2)$.

Arc length

At this point, we need to learn how to calculate the length of a given curve or the distance travelled if the curve corresponds to motion through space. Given a curve and a partition of that curve, we can approximate its length by adding the lengths of line segments and then taking a limit.

Consider a vector-valued function $\mathbf{f}(t)$ and two points $\mathbf{f}(a)$ and $\mathbf{f}(b)$ on the curve. For a fixed natural number n, divide the interval between a and b into n subintervals of equal width $\Delta t = (b - a)/n$. That is,

$$a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$$

where $t_{k+1} = t_k + \Delta t$. Then define $P_k = \mathbf{f}(t_k)$ and

$$L_n = \sum_{k=1}^{n} \operatorname{dist}(P_{k-1}, P_k).$$

Finally, the arc length of $\mathbf{f}(t)$ between $\mathbf{f}(a)$ and $\mathbf{f}(b)$ is defined to be

$$\lim_{n\to\infty} L_n$$

.

We can prove the following theorem which gives an effective way to compute arc length.

Theorem. Arc length $= \int_a^b |\mathbf{f}'(t)|\, dt$.

Example. Find the arc length of the curve

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \tfrac{2}{3}t^{3/2}\,\mathbf{k}$$

from $t = 0$ to $t = 4\pi$.

If the vector-valued function $\mathbf{f}(t)$ represents motion in space such as the path of an airplane, then the arc length would be the distance travelled, and its derivative would be its speed. If we differentiate our formula for arc length, we are able to conclude that

$$|\mathbf{f}'(t)| = \text{speed}.$$

When we divided by h in our limit, we obtained a vector having two nice properties:

1. Its direction is tangent to the curve.
2. Its length is the speed of the motion.

Example. Given $\mathbf{r}(t)$ from the tangent line example and the vector-valued function

$$\mathbf{s}(u) = u\,\mathbf{i} + (u^2 - 1)\,\mathbf{j} + (u - 3)\mathbf{k},$$

do the curves that they trace out intersect? If so, at what angle?