1. (12 points) Calculate:

(a) the gradient of the function $f(x, y) = e^{xy}\sin 2x$ at the point $(\pi/6, 0)$.

$$\frac{\partial f}{\partial x} = ye^{xy}\sin 2x + e^{xy}(\cos 2x)(2)$$

$$\frac{\partial f}{\partial y} = xe^{xy}\sin 2x$$

$$\nabla f(\pi/6, 0) = \vec{i} + \left(\frac{\pi}{6}\right)\left(\frac{\sqrt{3}}{2}\right)\vec{j}$$

$$= \vec{i} + \frac{\sqrt{3}\pi}{12}\vec{j}$$

(b) the projection $\text{proj}_{\mathbf{b}}\mathbf{a}$ of the vector $\mathbf{a} = \mathbf{i}+4\mathbf{j}-\mathbf{k}$ in the direction of $\mathbf{b} = 2\mathbf{i}-3\mathbf{j}+2\mathbf{k}$.

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\left(\frac{\vec{b}}{|\vec{b}|}\right) = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\vec{b}$$

$$= \frac{-12}{17}(2\vec{i} - 3\vec{j} + 2\vec{k})$$

$$= -\frac{24}{17}\vec{i} + \frac{36}{17}\vec{j} - \frac{24}{17}\vec{k}$$

(c) the curl of the vector field $\mathbf{F}(x, y, z) = x^3yz^2\mathbf{i} + 3x^3yz^2\mathbf{j} + 2x^2yz^2\mathbf{k}$.

$$\text{curl } F = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3yz^2 & 3x^3yz^2 & 2x^2yz^2 \end{vmatrix}$$

$$= (2x^2z^2 - 6x^3yz)\vec{i}$$
$$-(4xyz^2 - 2x^3yz)\vec{j}$$
$$+(9x^2yz^2 - x^3z^2)\vec{k}$$

2. (12 points) Which of the following five lines are parallel? Which are equal? **In order to receive any credit for your answer, you must provide brief justifications for your assertions.**

$$\ell_1 : x = 1 + t, \quad y = t, \quad z = 2 - 5t \qquad (1,0,2)$$
$$\ell_2 : x + 1 = y - 2 = 1 - z \qquad (-1,2,1)$$
$$\ell_3 : x = 1 + t, \quad y = 4 + t, \quad z = 1 - t$$
$$\ell_4 : \mathbf{r}(t) = (2 + 2t)\mathbf{i} + (1 + 2t)\mathbf{j} - (3 + 10t)\mathbf{k} \qquad (2,1,-3)$$
$$\ell_5 : x = 2 + t, \quad y = 3 + t, \quad z = 2 + t$$

Direction vectors:

$$\vec{D}_1 = \vec{i} + \vec{j} - 5\vec{k}$$
$$\vec{D}_2 = \vec{i} + \vec{j} - \vec{k}$$
$$\vec{D}_3 = \vec{i} + \vec{j} - \vec{k}$$
$$\vec{D}_4 = 2\vec{i} + 2\vec{j} - 10\vec{k}$$
$$\vec{D}_5 = \vec{i} + \vec{j} + \vec{k}$$

$\vec{D}_1$ and $\vec{D}_4$ are parallel

$\vec{D}_2$ and $\vec{D}_3$ are parallel

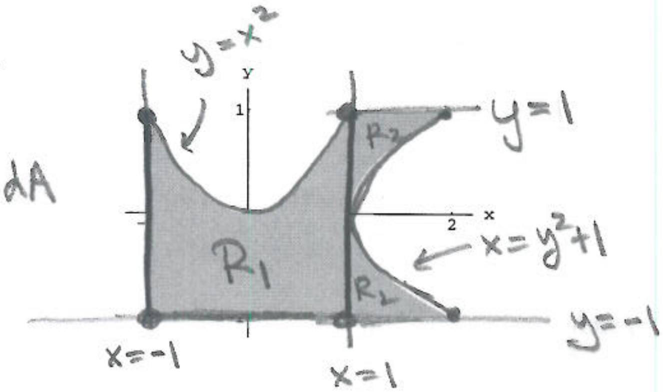
The point $(2,1,-3)$ satisfies ℓ_1 and ℓ_4, so those lines are equal.

The point $(-1,2,1)$ lies on ℓ_2 but not on ℓ_3, so ℓ_2 and ℓ_3 are parallel.

ℓ_5 is not parallel to any of the other four lines.

3. (12 points) Consider the following region R in the xy-plane. It is bounded by the curves $y = x^2$ and $x = y^2 + 1$ and the lines $y = -1$, $y = 1$, and $x = -1$. Calculate

$$\iint_R 6xy^2\, dA.$$



$$\iint_R 6xy^2\, dA = \iint_{R_1} 6xy^2\, dA + \iint_{R_2} 6xy^2\, dA$$

$$\iint_{R_1} 6xy^2\, dA = \int_{-1}^{1}\int_{-1}^{x^2} 6xy^2\, dy\, dx$$

$$= \int_{-1}^{1} 2x\Big[y^3\Big]_{y=-1}^{y=x^2} dx = \int_{-1}^{1} 2x(x^6+1)\, dx$$

$$= \left[\frac{x^8}{4} + x^2\right]_{x=-1}^{x=1} = \left(\tfrac{1}{4}+1\right) - \left(\tfrac{1}{4}+1\right) = 0$$

$$\iint_{R_2} 6xy^2\, dA = \int_{-1}^{1}\int_{1}^{y^2+1} 6xy^2\, dx\, dy$$

$$= \int_{-1}^{1} 3y^2\Big[x^2\Big]_1^{y^2+1} dy = \int_{-1}^{1} 3y^2(y^4+2y^2+\cancel{1}-\cancel{1})\, dy$$

$$= \int_{-1}^{1} 3y^6 + 6y^4\, dy = \left[\frac{3}{7}y^7 + \frac{6}{5}y^5\right]_{y=-1}^{y=1}$$

$$= \left(\frac{3}{7}+\frac{6}{5}\right) - \left(-\frac{3}{7}-\frac{6}{5}\right) = \frac{57}{35} + \frac{57}{35}$$

$$= \frac{114}{35}$$

4. (12 points) Consider the curves

$$\mathbf{r}_1(t) = e^t\mathbf{i} + t\mathbf{j} + (t-1)\mathbf{k} \qquad \text{and} \qquad \mathbf{r}_2(t) = t\mathbf{i} + (t-1)\mathbf{j} - t^2\mathbf{k}.$$

Determine their point(s) of intersection and the angle(s) at which they intersect.

Write 2nd curve as $\vec{r}_2(s) = s\vec{i} + (s-1)\vec{j} - s^2\vec{k}$

$$s = e^t$$

$$\left.\begin{array}{l} s-1 = t \\ -s^2 = t-1 \end{array}\right\} \quad s-2 = -s^2 \Rightarrow s^2+s-2=0$$

$$(s+2)(s-1)=0$$

$$s=-2 \text{ or } s=1$$

$s=-2 \Rightarrow t=-3$, but $-2 \neq e^{-3} \Rightarrow$ not a point of intersection

$s=1 \Rightarrow t=0$ and $e^0=1 \Rightarrow$ intersection

point of intersection is $(1,0,-1)$

$$\vec{r}_1'(t) = e^t\vec{i} + \vec{j} + \vec{k}$$

$$\vec{r}_1'(0) = \vec{i} + \vec{j} + \vec{k}$$

$$\vec{r}_2'(t) = \vec{i} + \vec{j} - 2t\vec{k}$$

$$\vec{r}_2'(1) = \vec{i} + \vec{j} - 2\vec{k}$$

θ = angle of intersection

$$\cos\theta = \frac{\vec{r}_1'(0)\cdot\vec{r}_2'(1)}{|\vec{r}_1'(0)||\vec{r}_2'(1)|}$$

$$= 0$$

$\Rightarrow$ angle of intersection is $\frac{\pi}{2}$ radians or 90°.

5. (12 points) Find the point(s) on the cone $z^2 = x^2 + y^2$ that are closest to the point $(3,1,0)$.

constrained min: s = distance squared to $(3,1,0)$

$$s = (x-3)^2 + (y-1)^2 + z^2$$

constraint: $c(x,y,z) = x^2 + y^2 - z^2 = 0$.

$$\nabla s = 2(x-3)\vec{i} + 2(y-1)\vec{j} + 2z\vec{k}$$

$$\nabla c = 2x\vec{i} + 2y\vec{j} - 2z\vec{k}$$

$$\nabla s = \lambda \nabla c \Rightarrow \cancel{2}(x-3) = \lambda(\cancel{2}x)$$
$$\cancel{2}(y-1) = \lambda(\cancel{2}y)$$
$$\cancel{2}z = \lambda(-\cancel{2}z)$$

Note that the point $(0,0,0)$ on cone is special because $\nabla c(0,0,0) = \vec{0}$.

Away from that point, $z \neq 0$ and the third equation implies $\lambda = -1$.

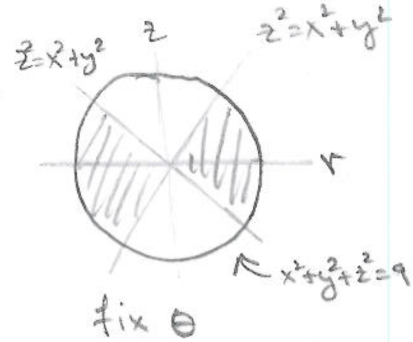
$$\lambda = -1 \Rightarrow x - 3 = -x \Rightarrow x = \frac{3}{2}$$
$$y - 1 = -y \Rightarrow y = 1/2$$

$$z^2 = x^2 + y^2 = \frac{9}{4} + \frac{1}{4} = \frac{10}{4} \Rightarrow z = \pm \frac{\sqrt{10}}{2}$$

The two points $\left(\frac{3}{2}, \frac{1}{2}, \pm\frac{\sqrt{10}}{2}\right)$ are $\sqrt{5}$ away from $(3,1,0)$. They are definitely closer than $(0,0,0)$, which is $\sqrt{10}$ away from $(3,1,0)$.

6. (12 points) Find the volume of the solid region R lying inside the sphere $x^2+y^2+z^2=9$ and outside the cone $z^2=x^2+y^2$. To be precise,

$$R=\left\{(x,y,z)\,|\,x^2+y^2+z^2\le 9 \text{ and } z^2\le x^2+y^2\right\}.$$



$$\text{Volume} = \iiint_R 1\, dV.$$

Use spherical coordinates:

$$\int_0^{2\pi}\int_0^3\int_{\pi/4}^{3\pi/4} \rho^2 \sin\phi \, d\phi\, d\rho\, d\theta =$$

$$\int_0^{2\pi}\int_0^3 \rho^2 \Big[-\cos\phi\Big]_{\phi=\pi/4}^{\phi=\frac{3\pi}{4}} d\rho\, d\theta =$$

$$\int_0^{2\pi}\int_0^3 \rho^2\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right) d\rho\, d\theta =$$

$$\sqrt{2}\int_0^{2\pi}\int_0^3 \rho^2\, d\rho\, d\theta = \sqrt{2}\int_0^{2\pi}\left[\frac{\rho^3}{3}\right]_{\rho=0}^{\rho=3} d\theta$$

$$= 9\sqrt{2}\int_0^{2\pi} d\theta = (2\pi)(9\sqrt{2})$$

$$= 18\sqrt{2}\,\pi$$

7. (12 points) **Note that there is a second part to this problem on the next page.**

(a) Fix a radius R. Using the surface area formula we discussed in this course, derive the surface area of a hemisphere of the form

$$x^2 + y^2 + z^2 = R^2 \quad \text{with} \quad z \geq 0.$$

hemisphere is graph $z = g(x,y) = \sqrt{R^2 - x^2 - y^2}$

$$\frac{\partial g}{\partial x} = \frac{1}{2}(R^2 - x^2 - y^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{R^2 - x^2 - y^2}}$$

$$\frac{\partial g}{\partial y} = \frac{-y}{\sqrt{R^2 - x^2 - y^2}}$$

$$dS = \sqrt{\frac{x^2}{R^2 - x^2 - y^2} + \frac{y^2}{R^2 - x^2 - y^2} + 1}\; dA$$

$\leftarrow$ area in xy-plane

$$= \sqrt{\frac{R^2}{R^2 - x^2 - y^2}}\; dA = \frac{R}{\sqrt{R^2 - x^2 - y^2}}\, dA$$

$$\text{surface area} = \iint_D \frac{R}{\sqrt{R^2 - x^2 - y^2}}\, dA$$

D is disk of radius R centered at $(0,0)$ in xy-plane

use polar: $\displaystyle\int_0^{2\pi}\int_0^R \frac{R}{\sqrt{R^2 - r^2}}\, r\, dr\, d\theta$

$u = R^2 - r^2$
$du = -2r\,dr$
$r = 0 \Rightarrow u = R^2$
$r = R \Rightarrow u = 0$

$$= \int_0^{2\pi}\int_0^{R^2} \frac{R}{\sqrt{u}}\left(\frac{1}{2}\right) du\, d\theta$$

$$= \frac{R}{2}\int_0^{2\pi} 2\left[\sqrt{u}\right]_{u=0}^{u=R^2} d\theta = R\int_0^{2\pi} R\, d\theta$$

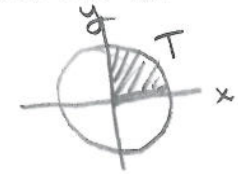
$$= R^2(2\pi) = 2\pi R^2.$$

Problem 7 (continued):

(b) With the aid of your result in part (a), calculate the x-coordinate $\overline{x}$ of the center of mass of that portion of the sphere $x^2 + y^2 + z^2 = R^2$ that lies in the first octant (x, y, and z all positive). You can use the integral

$$\int \frac{u^2}{\sqrt{a^2 - u^2}}\, du = -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$$

without justification. Also, recall that $\overline{x}$ is the surface integral of x over the surface divided by the surface area of the surface.



$$\iint_S x\, dS = \iint_T x \frac{R}{\sqrt{R^2 - x^2 - y^2}}\, dA$$

T ← xy-plane

$$\underset{\text{polar}}{=} \int_0^{\pi/2} \int_0^R (r\cos\theta) \frac{R}{\sqrt{R^2 - r^2}}\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} R\cos\theta \left(\int_0^R \frac{r^2}{\sqrt{R^2 - r^2}}\, dr \right) d\theta$$

Using the integral formula above

$$\int_0^R \frac{r^2}{\sqrt{R^2 - r^2}}\, dr = \left[-\frac{r}{2}\sqrt{R^2 - r^2} + \frac{R^2}{2}\sin^{-1}\frac{r}{R} \right]_{r=0}^{r=R}$$

$$= \left(\frac{R^2}{2}\right)\left(\frac{\pi}{2}\right) \Longrightarrow$$

$$\iint_S x\, dS = \left(\frac{\pi R^3}{4}\right) \int_0^{\pi/2} \cos\theta\, d\theta = \frac{\pi R^3}{4}.$$

$$\overline{x} = \left(\frac{\cancel{\pi} R^{\cancel{3}}}{4}\right) \frac{2}{\cancel{\pi} \cancel{R^2}} = \frac{R}{2}.$$

8. (16 points) **Note: This problem has multiple parts on pages 9–12**

(a) Compute and classify the critical points of the function $f_1(x, y) = x^2 - 2x + y^2 - 2y$.

$\frac{\partial f_1}{\partial x} = 2x - 2 = 0 \Rightarrow x = 1$ one critical point $(1,1)$

$\frac{\partial f_1}{\partial y} = 2y - 2 = 0 \Rightarrow y = 1$

$H(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow H(1,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow D = 4$, $\frac{\partial^2 f}{\partial x^2} > 0$ local min

(b) Compute and classify the critical points of the function $f_2(x, y) = 3x - x^3 + y^2$.

$\frac{\partial f_2}{\partial x} = 3 - 3x^2 = 0 \Rightarrow 1 - x^2 = 0 \Rightarrow x = \pm 1$

$\frac{\partial f_2}{\partial y} = 2y = 0 \Rightarrow y = 0$ two critical points $(\pm 1, 0)$

$H(x,y) = \begin{pmatrix} -6x & 0 \\ 0 & 2 \end{pmatrix}$

$H(1,0) = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow D = -12 \Rightarrow$ saddle

$H(-1,0) = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow D = 12 \Rightarrow$ local min, $\frac{\partial^2 f}{\partial x^2} > 0$

Problem 8 (continued):

(c) Compute and classify the critical points of the function $f_3(x, y) = x^3 - 3x + y^2$.

Almost same as (b).

$\frac{\partial f_3}{\partial x} = 3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$\frac{\partial f_3}{\partial y} = 2y = 0 \Rightarrow y = 0$ two critical points $(\pm 1, 0)$

$H(x,y) = \begin{pmatrix} 6x & 0 \\ 0 & 2 \end{pmatrix}$

$H(1,0) = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow D = 12$ and $\frac{\partial^2 f}{\partial x^2} > 0 \Rightarrow$ local min

$H(-1,0) = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow D = -12 \Rightarrow$ saddle

(d) Compute and classify the critical points of the function $f_4(x, y) = xy - x - y$.

$\frac{\partial f_4}{\partial x} = y - 1 = 0 \Rightarrow y = 1$ one critical point $(1,1)$

$\frac{\partial f_4}{\partial y} = x - 1 = 0 \Rightarrow x = 1$

$H(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow D = -1 \Rightarrow$ saddle

Problem 8 (continued):

(e) Compute and classify the critical points of the function $f_5(x,y) = x^4 + y^4 - 4xy$.

$\frac{\partial f_5}{\partial x} = 4x^3 - 4y = 0 \Rightarrow x^3 - y = 0 \Rightarrow y = x^3$

$\frac{\partial f_5}{\partial y} = 4y^3 - 4x = 0 \Rightarrow y^3 - x = 0 \Rightarrow x = y^3$

$x = y^3 \Rightarrow x = (x^3)^3 = x^9 \Rightarrow x^9 - x = 0$,

$x^9 - x = x(x^8-1) = x(x^4-1)(x^4+1) = x(x^2-1)(x^2+1)(x^4+1)$

$= x(x-1)(x+1)(x^2+1)(x^4+1)$

$\Rightarrow x = 0, \pm 1$

three critical points $(1,1)$, $(0,0)$, $(-1,-1)$

$H(x,y) = \begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$ $D(1,1) = D(-1,-1) = 144-16 > 0$ $\frac{\partial^2 f_5}{\partial x^2} > 0$ Local min

$D(0,0) = -16$ saddle

(f) Compute and classify the critical points of the function $f_6(x,y) = x^4 + y^4 + 4xy$.

Almost same as (e).

$\frac{\partial f_6}{\partial x} = 4x^3 + 4y = 0 \Rightarrow y = -x^3$

$\frac{\partial f_6}{\partial y} = 4y^3 + 4x = 0 \Rightarrow x = -y^3 \Rightarrow x = -(-x^3)^3$
$= x^9$

$x^9 - x = 0 \Leftrightarrow x(x^8-1) = 0 \Leftrightarrow x(x^4-1)(x^4+1) = 0$

$\Leftrightarrow x(x^2-1)(x^2+1)(x^4+1) = 0$

$\Leftrightarrow x(x-1)(x+1)(x^2+1)(x^4+1) = 0$

$\Leftrightarrow x = 0, \pm 1$

three critical points $(1,-1)$, $(0,0)$, $(-1,1)$

$H(x,y) = \begin{pmatrix} 12x^2 & 4 \\ 4 & 12y^2 \end{pmatrix}$

$H(1,-1) = H(-1,1) = \begin{pmatrix} 12 & 4 \\ 4 & 12 \end{pmatrix}$

$H(0,0) = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$

$D = -16$ saddle

$D = 144 - 16 > 0$

$\frac{\partial^2 f_6}{\partial x^2} > 0$

Local min

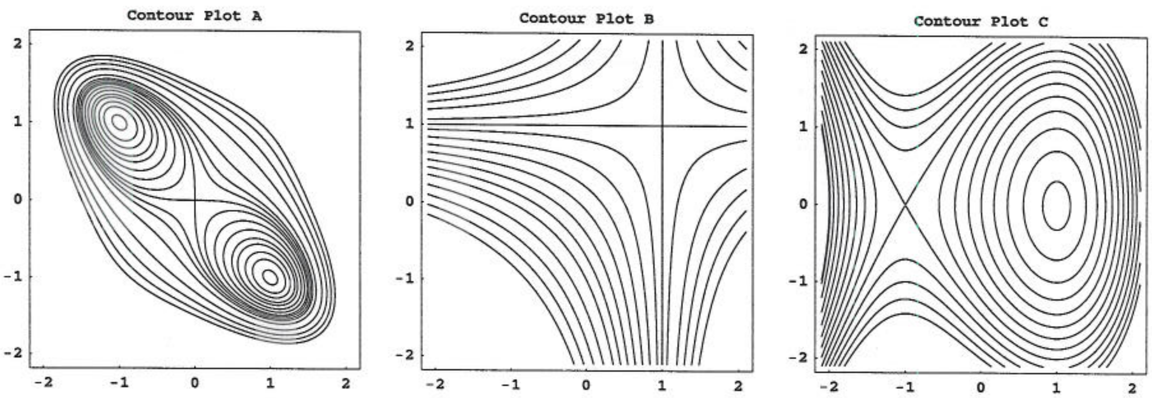
Problem 8 (continued):

(g) The six functions considered in parts (a)–(f) are:

1. $f_1(x,y) = x^2 - 2x + y^2 - 2y$ 2. $f_2(x,y) = 3x - x^3 + y^2$ 3. $f_3(x,y) = x^3 - 3x + y^2$

4. $f_4(x,y) = xy - x - y$ 5. $f_5(x,y) = x^4 + y^4 - 4xy$ 6. $f_6(x,y) = x^4 + y^4 + 4xy$

Here are three contour plots:



Match each contour plot with its corresponding function $f(x,y)$ from the choices above. Using your results from parts (a)–(f), provide a brief justification for your selection. **You will not receive any credit for your answer unless you provide a valid justification.**

A. The function for contour plot A is 6. My reason for choosing this answer is:

Three critical points: $(-1,1)$, $(0,0)$, $(1,-1)$.
The only function with three critical points at these places is f_6.

B. The function for contour plot B is 4. My reason for choosing this answer is:

One critical point at $(1,1)$, and it is a saddle. The only function with one critical point that is a saddle is f_4.

C. The function for contour plot C is 3. My reason for choosing this answer is:

Two critical points: $(-1,0)$ and $(1,0)$.
f_2 and f_3 satisfy this situation.
f_3 has a saddle at $(-1,0)$ and f_2 does not.