Level sets for functions of three variables

As we discussed last Friday, it’s hard to draw a graph of a function of three variables. So we must visualize its level sets.

Recall that the level sets of the function

\[ P(x, y, z) = x + y + 10z \]

are parallel planes that are almost horizontal.

**Example.** Sketch the level sets of the function

\[ f(x, y, z) = x^2 + y^2 - z^2. \]
Limits and continuity

In order to be able to do calculus for multivariable functions, we need to be able to talk about limits.

**Informal definition.** We say that

\[ \lim_{(x,y) \to (a,b)} f(x,y) = L \]

if \( f(x,y) \to L \) as \( (x,y) \to (a,b) \) along any path in the xy-plane.

Here are two examples to illustrate some of the issues that arise.

**Example.** Consider

\[ \lim_{(x,y) \to (0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}. \]

**Example.** Consider

\[ \lim_{(x,y) \to (0,0)} \frac{2xy}{x^2 + y^2}. \]
Partial derivatives

Consider a function of two variables $f(x, y)$. How do we talk about its rate of change at a given point?

**Definition.** The partial derivative of $f(x, y)$ in the $x$-direction at the point $(a, b)$ is defined by

$$\frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}.$$ 

In other words we vary $x$ but keep $y$ constant as we take the limit.

**Example.** Consider $f(x, y) = 9 - x^2 - y^2$. Let’s calculate

$$\frac{\partial f}{\partial x}(1, 2)$$

directly from this definition.

There is another, more efficient way to calculate this partial derivative.
Let’s try a more complicated example.

**Example.** Consider \( g(x, y) = y \ln(xy) + y \).

The partial derivative with respect to \( y \) is defined in a similar fashion.

**Definition.** The partial derivative of \( f(x, y) \) in the \( y \)-direction at the point \((a, b)\) is defined by

\[
\frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}.
\]

We keep \( x \) constant and vary \( y \) as we take the limit.

**Example.** Consider \( g(x, y) = y \ln(xy) + y \) again and calculate \( \partial g/\partial y \) this time.
Example. Consider the function \( f(x, y) = 9 - x^2 - y^2 \) at the point \((1, 2)\). In what direction, the \( x \)-direction or the \( y \)-direction, does \( f(x, y) \) decrease most rapidly?