A little more on the method of Lagrange multipliers

Last class I gave an overly complicated solution to the constrained max/min problem for the function $f(x,y) = \frac{1}{4}(y^2 - x^2)$ subject to the constraint $C(x,y) = x^2 + y^2 = 1$.

We have

$$\nabla f(x,y) = -\frac{x}{2}\,\mathbf{i} + \frac{y}{2}\,\mathbf{j}, \quad \text{and} \quad \nabla C(x,y) = 2x\,\mathbf{i} + 2y\,\mathbf{j}.$$

The Lagrange multiplier equation $\nabla f = \lambda \nabla C$ yields the two scalar equations

$$\begin{cases} -\dfrac{x}{2} = 2\lambda x \\ \dfrac{y}{2} = 2\lambda y, \end{cases}$$

which are equivalent to

$$\begin{cases} -x = 4\lambda x \\ \quad y = 4\lambda y. \end{cases}$$

I solved these two equations correctly on Monday, but there is a better solution.

If we multiply the first equation by y and the second equation by x, we get

$$-xy = xy$$

because both sides are equal to $4\lambda xy$. From this equation, we see that either $x = 0$ or $y = 0$.

Example. Find the point on the plane $x + y + 2z = 1$ that is closest to the origin.

Try this one on your own. What is $C(x,y,z)$? For the function $f(x,y,z)$, you should minimize the distance of a point (x,y,z) to the origin. In fact, the algebra is easier if you minimize the square of this distance. (Why is this OK and why is the algebra easier?)

Multiple integrals

We start our discussion of integration by defining the integral of a function $f(x, y)$ of two variables. However, before we get into the details, it helps if we review what we know about the integral

$$\int_a^b f(x)\,dx$$

of a function of one variable. There are three interpretations of $\int_a^b f(x)\,dx$ that I would like you to keep in mind.

You can review these interpretations of the integral on pp. 354–356 and pp. 467–468 of the textbook.

These three interpretations have analogues for the integral

$$\iint\limits_R f(x, y)\, dA$$

of a function of two variables.

The definition of the double integral is also analogous to the definition in the one-variable case. For $\int_a^b f(x)\, dx$, we partition the interval $[a, b]$ into subintervals (usually of equal width).

In each subinterval $[x_{i-1}, x_i]$, we pick a "test" number x_i^*, and we consider the Riemann sum

$$\sum_{i=1}^{n} f(x_i^*)\, \Delta x.$$

If the function $f(x)$ is reasonable (for example, continuous) on the inteval $[a, b]$, then the Riemann sums converge to a unique number as $n \to \infty$. That is,

$$\lim_{n\to\infty} \left(\sum_{i=1}^{n} f(x_i^*)\, \Delta x \right) = \int_a^b f(x)\, dx.$$

To define the integral of $f(x, y)$ over a rectangle R, we also define Riemann sums.

Theorem. Suppose the function $f(x, y)$ is continuous on the rectangle R. Then its Riemann sums converge to a unique number as both m and n tend to infinity. That is,

$$\lim_{m,n\to\infty} \left(\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*) \, \Delta x \, \Delta y \right) = \iint_R f(x, y) \, dA.$$

Example. The animation that I made to illustrate this limiting process involves the function

$$f(x, y) = 2 - \tfrac{1}{2}(x-2)^2 - \tfrac{1}{2}(y-2)^2$$

and the rectangle

$$R = \left\{ (x, y) \mid \tfrac{1}{2} \le x \le 3,\ 1 \le y \le 3 \right\}.$$

By the end of next class, you will be able to calculate this integral exactly, and you will get

$$\iint\limits_R f(x, y)\, dA = \frac{185}{24} \approx 7.70833.$$

Often in applications, one has data rather than a formula for the function. In that case, Riemann sums can be used to approximate the value of the integral.

The Midpoint Rule

The Midpoint Rule uses the midpoints of the subrectangles as test points.

Example.
A 20-ft-by-30-ft swimming pool is filled with water. The depth is measured at 5-ft intervals, starting at one corner of the pool, and the values are recorded in the table below. Use the Midpoint Rule to estimate the volume of water in the pool.

	0	5	10	15	20	25	30
0	2	3	4	6	7	8	8
5	2	3	4	7	8	10	8
10	2	4	6	8	10	12	10
15	2	3	4	5	6	8	7
20	2	2	2	2	3	4	4

Properties of the integral

Theorem. The integral satisfies some basic but important properties.

1. $\displaystyle\iint\limits_R f(x,y)+g(x,y)\,dA = \iint\limits_R f(x,y)\,dA + \iint\limits_R g(x,y)\,dA$
2. $\displaystyle\iint\limits_R c\,f(x,y)\,dA = c\iint\limits_R f(x,y)\,dA$ if c is a constant
3. If $f(x,y) \geq g(x,y)$ for all (x,y) in R, then
$$\iint\limits_R f(x,y)\,dA \geq \iint\limits_R g(x,y)\,dA.$$