Overview of the integrals involved in vector analysis

Vector analysis involves two new types of integrals—line integrals and flux integrals. A line integral is a special case of a path integral, and a flux integral is a special case of a surface integral.

	new integral	its application to vector fields
one-dimensional	path integral	line integral
two-dimensional	surface integral	flux integral

Path integrals

Your textbook calls these integrals "line integrals along a curve" and sometimes you will also see the term "line integrals with respect to arc length."

Recall that all of our integrals so far involved summing up a function.

Definition. A path integral of a function $f(x, y)$ along a curve C in the xy-plane is an integral of the form

$$\int_C f(x, y)\, ds,$$

where ds represents the differential of arc length (see the September 25 handout).

How do we compute such an integral?

If we parameterize the curve using a vector-valued function $\mathbf{r}(t)$ with $a \leq t \leq b$, then

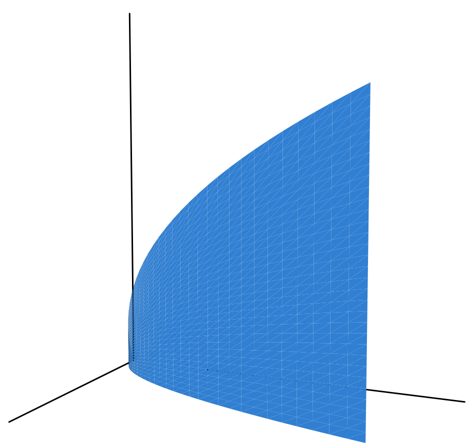
$$\int_C f(x, y)\, ds = \int_a^b f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt.$$

There is a similar definition for functions of three variables and curves in space.

Example. Consider a curved fence that sits on the ground along a parabolic path of the form

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}.$$

Suppose that its height is $h(x, y) = x + \sqrt{y}$. What is the surface area for the part of the fence that sits along the parabolic arc from $(0, 0)$ to $(2, 4)$?



Path integrals are independent of parameterization. In other words, if both $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ trace out the same curve C, then

$$\int_C f(x, y)\, ds$$

can be calculated using either $\mathbf{r}_1(t)$ or $\mathbf{r}_2(t)$.

Example. Consider the path integral

$$\int_C x\, ds$$

where C is the line segment from $(0,0)$ to $(1,1)$. There are many ways to parameterize C. For example, consider the three parameterizations

$$\begin{aligned}\mathbf{r}_1(t) &= t\mathbf{i} + t\mathbf{j}\\ \mathbf{r}_2(t) &= t^2\mathbf{i} + t^2\mathbf{j}\\ \mathbf{r}_3(t) &= (1-t^2)\mathbf{i} + (1-t^2)\mathbf{j},\end{aligned}$$

where $0 \leq t \leq 1$ in all three cases.

Example. Consider a semicircular piece of wire of radius R. Find its center of mass.