More on line integrals

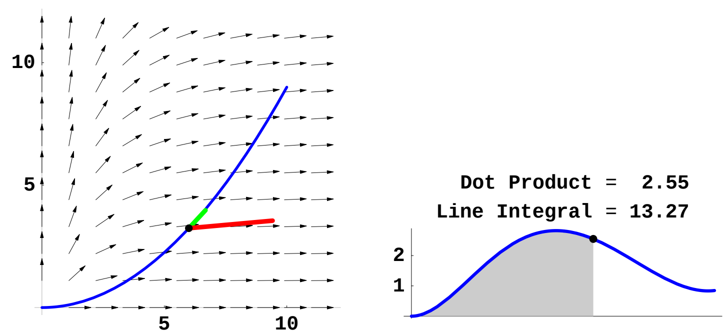
Let's begin with a short summary of what we discussed about line integrals last time.

Definition. Consider a vector field $\mathbf{F}(x, y)$ and a curve C. The line integral of $\mathbf{F}(x, y)$ along C is the path integral

$$\int_C (\mathbf{F} \cdot \mathbf{T})\, ds,$$

where $\mathbf{T}$ is the unit tangent vector along the curve C.

I updated the animation on the web site so that the dot product number makes sense.



Last class we talked about how we go about calculating line integrals: Suppose that the curve C is parameterized by a vector-valued function $\mathbf{r}(t)$ from $t = a$ to $t = b$. Then

$$\int_C (\mathbf{F} \cdot \mathbf{T})\, ds = \int_a^b (\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t))\, dt \equiv \int_C \mathbf{F} \cdot d\mathbf{r} \equiv \int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}.$$

Some classical notation: In the planar case, the vector field is often written as

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}.$$

Suppose that the curve C is parameterized as $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. Then

$$\mathbf{r}'(t) = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j},$$

and

$$\int_a^b (\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t))\, dt = \int_a^b \left(P(x, y)\frac{dx}{dt} + Q(x, y)\frac{dy}{dt} \right) dt.$$

For this reason, line integrals are often written as

$$\int_C P(x, y)\, dx + Q(x, y)\, dy.$$

In space, we have

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}.$$

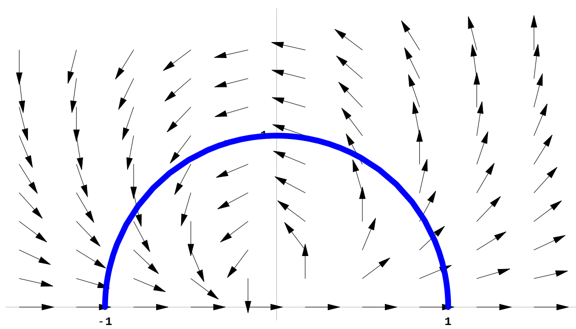
Then the line integral is written as

$$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz.$$

Example. Let's revisit Example 3 from last class using the classical notation. Recall that

$$\mathbf{F}(x,y) = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}$$

and C consists of the top half of the positively-oriented unit circle centered at the origin.



In the classical notation, we write this line integral as

$$\int_C (x^2 - y^2)\,dx + 2xy\,dy.$$

Rather than parameterize the curve C using a vector-valued function $\mathbf{r}(t)$, we use the two scalar functions

$$x(t) = \cos t \qquad \text{and} \qquad y(t) = \sin t.$$

where $0 \le t \le \pi$. Then $dx = -\sin t\,dt$ and $dy = \cos t\,dt$. We get

$$\int_C (x^2 - y^2)\,dx + 2xy\,dy = \int_0^\pi (\cos^2 t - \sin^2 t)(-\sin t)\,dt + 2(\cos t)(\sin t)(\cos t)\,dt$$

$$= \int_0^\pi (\cos^2 t)(\sin t) + (\sin^3 t)\,dt$$

$$= \int_0^\pi (\cos^2 t)(\sin t)\,dt + \int_0^\pi (\sin^3 t)\,dt = \tfrac{2}{3} + \tfrac{4}{3} = 2.$$

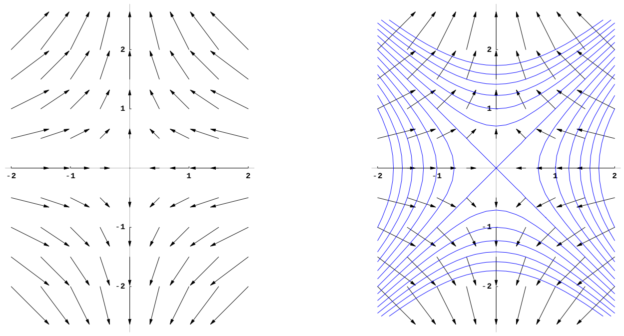
Line integrals for gradient vector fields

If a vector field is a gradient vector field, then its line integrals are easier to compute than the general case. Let's start with some examples of gradient vector fields. (For definitions, see the November 17 handout.)

Example 1. (October 20 and November 17 handouts) Consider the vector field

$$\mathbf{F}(x,y) = \left(-\frac{x}{2}\right)\mathbf{i} + \left(\frac{y}{2}\right)\mathbf{j}.$$

Then $\mathbf{F}(x,y) = \nabla f(x,y)$, where $f(x,y) = \frac{1}{4}(y^2 - x^2)$.



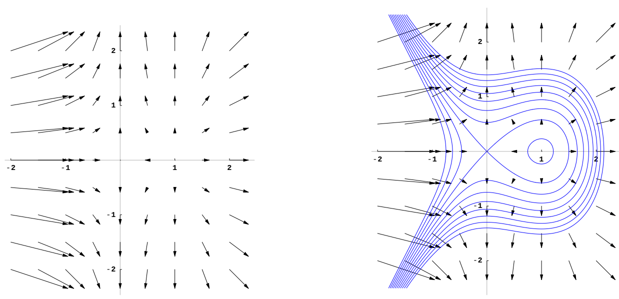
The figure on the left is the gradient vector field alone while the figure on the right has the field superimposed on the level sets of $f(x,y)$.

Example 2. Consider the vector field

$$\mathbf{F}(x,y) = \left(x^2 - x\right)\mathbf{i} + y\mathbf{j}.$$

Then $\mathbf{F}(x,y) = \nabla f(x,y)$, where

$$f(x,y) = \frac{x^3}{3} - \frac{x^2}{2} + \frac{y^2}{2}.$$

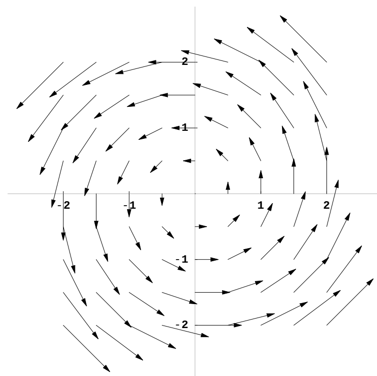


The figure on the left is the gradient vector field alone while the figure on the right has the field superimposed on the level sets of $f(x,y)$.

Example 3. Consider the merry-go-round vector field

$$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}.$$

We will soon see that this vector field is not a gradient vector field.



Why are gradient vector fields so special?

Theorem. (Fundamental Theorem of Calculus for line integrals) If $\mathbf{F} = \nabla f$, then

$$\int_{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Note that this theorem implies that the value of the line integral in this case depends only on the endpoints of the curve and not on the curve itself. This is called *path independence.*

Example. Consider the line integral

$$\int_C (2x + y)\, dx + (x + e^y)\, dy$$

where C is any curve from $(1, 0)$ to $(0, 1)$.

Consequences of the Fundamental Theorem for line integrals:

If $\mathbf{F} = \nabla f$, then we have

1. path independence, and
2. the line integral $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every *closed* curve.

Theorem. If $\mathbf{F}$ satisfies either item 1 or item 2, then $\mathbf{F} = \nabla f$ for some function f.

Terminology: If $\mathbf{F}$ is a force field and any one of these three equivalent situations occur, then f is potential energy, and we have conservation of energy for this vector field. In this case, we say that the vector field is *conservative*.

Example. Consider the gravitational force field

$$\mathbf{G}(\mathbf{x}) = -\frac{MG}{|\mathbf{x}|^3}\mathbf{x}$$

that we discussed on November 15 and 17. Then $\mathbf{G}(\mathbf{x}) = \nabla g$, where

$$g(\mathbf{x}) = \frac{MG}{|\mathbf{x}|}.$$

We say that g is a potential function for G.